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LAPLACE-STIELTHJES TYPE INTEGRALS: THE BOREL RELATION AND THE h-MEASURE OF AN EXCEPTIONAL SET.

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In the class of functions $F\colon \mathbb{R}\to\mathbb{R}_+$ of the form $F(x)=\int_0^{+\infty} f(u)e^{xu}\nu(du)$, where ν is a non-negative measure on $\mathbb{R}_+$ with unbounded support $\operatorname{supp}\nu$, $a(x)$ is an arbitrary non-negative ν-measurable function on $\mathbb{R}_+$, conditions are established under which the asymptotic relation $\ln F(x)\leqslant(1+o(1))\ln\mu_*(x,F)$ holds for $x\to+\infty$ outside some set E of finite h-measure, i.e., $\int_E dh(x)<+\infty$; here $\mu_*(x,F)=\sup\{a(u)e^{xu}\colon u\in\operatorname{supp}\nu\}$..

Key words: Laplace-Stieltjes integral, Borel relation, h-measure, exceptional set.

For every non-costant entire functions of the form $f(z)=\sum_{n=0}^{+\infty}a_nz^n$ of finite order $\rho_f<+\infty$, and $\mu_f(r)=\max\{|a_n|r^n\colon n\geq 0\}$ and $M_f(r)=\max\{|f(z)|\colon\ |z|=r\}$, it is well known that (see [3, 5], [2, Part IV, Ch. 1, § 3, Problem 54]) the (Borel) relation

$$\ln M_f(r)\sim\ln\mu_f(r) \tag{1}$$

holds as $r\to+\infty$ ($r\in[0,+\infty)\setminus E$), where $\varepsilon>0$ is arbitrary given and the set E has finite logarithmic mesure, i.e. $\ln-\text{meas}\, E:=\int_{E\cap[1,+\infty)}d\ln r<+\infty$. Statements about analogues of this Borel relation have been repeatedly proven for the entire Dirichlet series. In [1] it is proved that, in order that for every integer Dirichlet series $F\in\mathcal{D}(\lambda)$ of the form

$$F(z)=\sum_{n=0}^{+\infty}F_n\exp\{z\lambda_n\},\quad 0=\lambda_0<\lambda_n<\lambda_{n+1}\uparrow+\infty\ (1\leqslant n\uparrow+\infty),$$

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$$\ln M(x,F) \sim \ln \mu(x,F) \tag{2}$$

relation holds as $x \to +\infty$ outside some set of finite Lebesgue measure, it is necessary and sufficient that the following condition

$$\sum_{n=1}^{+\infty} \frac{1}{n\lambda_n} < +\infty, \tag{3}$$

holds; here $M(x,F) = \sup\{|F(x+iy)| \colon y \in \mathbb{R}\}$, $\mu(x,F) = \max\{|F_n|e^{x\lambda_n} \colon n \geqslant 0\}$. Similar results was also obtained for integrals of the form

$$F(x) = \int_{\mathbb{R}_+} a(t)e^{xt}\nu(dt), \tag{4}$$

where ν is non-negative measure on $\mathbb{R}_+ = [0,+\infty)$ with unbounded support, $a(t)$ is arbitrary non-negative ν-measurable function on $\mathbb{R}_+$, $S = \operatorname{supp}\nu \cap \{x \in \mathbb{R}_+ \colon\ f(x) > 0\}$. By $\mathcal{I}(\nu)$ denote the class of functions $F\colon \mathbb{R} \to \mathbb{R}_+$ represented by integrals of the form (4). For $F \in \mathcal{I}(\nu)$ and $x \in \mathbb{R}$ let us denote

$$\mu_*(x,F) = \operatorname{ess\,sup}\{f(u)e^{xu} \colon u \in S\}.$$

In particular, in paper [9] the following theorem was proved.

Theorem 1 ([9]). *If condition*

$$\int_0^{+\infty} t^{-2} \ln \nu_0(t)\, dt < +\infty \tag{5}$$

holds with $\nu_0(t) = \nu([0,t])$, *then for every function* $F \in \mathcal{I}(\nu)$ *there exists a set* E *of finite Lebesgue measure such that the asymptotic relation*

$$\ln F(x) = (1+o(1)) \ln \mu_*(x) \tag{6}$$

holds as $x \to +\infty$ $(x \notin E)$.

Note that the condition (3) on the sequence of the exponents (λ_k) is satisfied if and only if the condition (5) is satisfied with $\nu_0(t) = n(t) = \sum_{\lambda_n \leqslant t} 1$ ([7,9]).

We denote by $\mathcal{L}$ the class of positive continuous functions $\Phi\colon \mathbb{R}_+ := [0,+\infty) \to \mathbb{R}_+$ such that $\Phi(x) \nearrow +\infty$ $(0 \leqslant x \to +\infty)$, and by $\mathcal{L}^+$ the class of positive continuous differentiable functions $h(x)\colon [0,+\infty) \to [0,+\infty)$ such that $\int_0^{+\infty} dh(x) = +\infty$ and $1 \leqslant h'(x) \nearrow +\infty$ $(0 \leqslant x \nearrow +\infty)$. Denote by $\mathcal{L}_1$ the class of function $\psi \in \mathcal{L}$ such that $\int_{x_0}^{+\infty} dt/\psi(t) < +\infty$, $\psi(t) \geqslant t$ for $t \geqslant t_0 > 0$, and by $\mathcal{L}_2$ the class of the functions $\psi \in \mathcal{L}_1$ such that the inverse function ψ^{-1} to function ψ satisfies the condition $\psi^{-1}(uv) \leqslant u\psi^{-1}(v)$ $(u,v \geqslant 1)$. Let $\Phi \in \mathcal{L}$. Let us introduce the following class of Dirichlet series

$$\mathcal{D}(\lambda,\Phi) = \{F \in \mathcal{D}(\lambda) \colon\ \ln \mu(x,F) \geqslant x\Phi(x)\ (x \geqslant x_0)\}.$$

For a Lebesgue measurable set $E \subset [0,+\infty)$ we call

$$m_h E \equiv \int_E dh(x)$$

its h-measure, at $h(x) \equiv x$ the h-measure of a set E is its Lebesgue measure.

In the article [10] the following theorem was proved.

Theorem 2 ([10])**.** *Let $h \in \mathcal{L}^+$, $\Phi \in \mathcal{L}$. If a function $F \in \mathcal{D}(\lambda, \Phi)$ and a number $b > 0$ such that*

$$\sum_{n=1}^{+\infty} \frac{h'(\varphi(\lambda_n) + b)}{n\lambda_n} < +\infty, \tag{7}$$

then asymptotic relation (2) holds as $x \to +\infty$ outside some set E of finite h-measure ($m_h E < +\infty$), where the function φ is an inverse function to the function Φ.

In this article we will prove a similar statement for the functions from the class $\mathcal{I}(\nu)$. The proof method differs from the classical Wiman-Valiron type method was used in the article [10].

We say that $F \in \mathcal{I}(\nu, \Phi)$ if $F \in \mathcal{I}(\nu)$ and

$$\ln F(x) \geqslant x\Phi(x) \quad (x \geqslant x_0).$$

We prove the following theorem.

Theorem 3. *Let $h \in \mathcal{L}^+$, $\Phi \in \mathcal{L}$, $\Phi_0(x) = x\Phi(x)$. If a function $F \in \mathcal{I}(\nu, \Phi)$ and there exists a function $\psi \in \mathcal{L}_2$ that the condition*

$$h'(\varphi_0(t)) \ln \nu_0(t) = o\Big(\psi^{-1}(t)\Big) \quad (t \to +\infty) \tag{8}$$

satistfy with $\nu_0(t) = \nu([0, t])$, then asymptotic relation (6) holds as $x \to +\infty$ outside some set E of finite h-measure ($m_h E < +\infty$), where the function φ_0 is an inverse function to the function Φ_0.

Proof of Theorem 3. We need the following lemma.

Lemma 1 ([9], Lemma 1)**.** *Let $h \in \mathcal{L}_+$, $\psi \in \mathcal{L}_1$, and $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable non-negative and non-decreasing function. Then h-measure of the set*

$$E = \{x \geqslant 0\colon g'(x) \geqslant h'(x)\psi(g(x))\}$$

has finite h-measure.

In the following proof, we reason similarly to the proof of the corresponding theorem in [9] (see also [7]). We assume that $F'(x)$ denotes the right-hand derivative. Denote $g(x) = \ln F(x)$. Let

$$P_x(dt) = \frac{a(t)e^{xt}}{F(x)} \nu(dt)$$

be probability measure. Then for the mean of random variable $\xi = t$ we get

$$\mathbf{M}\xi = \int_0^{+\infty} \xi P_x(dt) = \int_0^{+\infty} a(t)e^{xt} \frac{t}{F(x)} \nu(dt) = g'(x).$$

So, by Markoff inequality $P_x\{\xi \geqslant a\} \leqslant M\xi/a$ for $a = cM\xi = cg'(x)$ we obtain

$$\int_{t \geqslant cg'(x)} a(t)e^{tx}\nu(dt) = \int_{\xi \geqslant cg'(x)} a(t)e^{tx}\nu(dt) = P_x\{\xi \geqslant a\} \leqslant \frac{1}{c}.$$

Let for fixed $x > 0$

$$G := G_x = \Big\{t > 0\colon t \leqslant 2g'(x)\Big\}.$$

As in article [9] we have

$$F(x) \le 2\int_G a(t)e^{xt}\nu(dt) \tag{9}$$

for all $x \geqslant x_0$. Indeed,

$$\int_{\mathbb{R}_+\setminus G} a(t)e^{xt}d\nu(t) \leqslant \frac{1}{2g'(x)}\int_{\mathbb{R}_+\setminus G} a(t)e^{xt}t\cdot d\nu(t) \leqslant$$
$$\leqslant \frac{1}{2g'(x)}\int_{\mathbb{R}_+} a(t)f'(xt)\cdot td\nu(t) = \frac{F(x)}{2}.$$

Therefore,

$$F(x) = \int_G a(t)e^{xt}\nu(dt) + \int_{\mathbb{R}_+\setminus G} a(t)e^{xt}d\nu(t) \leqslant$$
$$\leqslant \int_G a(t)e^{xt}\nu(dt) + \frac{F(x)}{2},$$

and the inequality (9) follows. Then, from inequality (9) one has

$$F(x) \leqslant 2\int_{G_1} a(t)f(xt)\nu(dt) \leqslant 2\mu(x,F)\cdot\nu_0(2g'(x)) \quad (x \geqslant x_0). \tag{10}$$

Therefore, by Lemma 1 with $1/2\psi(t)$ instead $\psi(t)$ for all $x \notin E$ we have $g'(x) \leqslant 1/2h'(x)\psi(g(x))$ and

$$F(x) \leqslant 2\mu_*(x,F)\nu_0(h'(x)\psi(g(x))), \tag{11}$$

moreover,

$$m_h(x) = \int_E dh(x) \leqslant \int_E \frac{g'(x)}{\psi(g(x))}dx \leqslant \int_0^{+\infty}\frac{du}{\psi(u)} < +\infty.$$

Note, that $g(x) \geqslant \Phi_0(x)$, $h'(x) \geqslant 1$ $(x \geqslant x_0)$, $\psi(t) \geqslant t$ $(t \geqslant t_0)$, thus, $h'(x)\psi(g(x)) \geqslant g(x)$ and

$$\varphi_0\Big(h'(x)\psi(g(x))\Big) \geqslant \varphi_0(g(x)) \geqslant \varphi_0(\Phi_0(x)) = x.$$

Finally, it remains to use the condition (8) with the function $\psi \in \mathcal{L}_2$

$$\ln\nu_0(h'(x)\psi(g(x))) = o\Big(\psi^{-1}\Big(h'(x)\psi(g(x))\Big)/h'\Big(h'(x)\psi(g(x))\Big)\Big) \leqslant$$
$$\leqslant o\Big(\psi^{-1}\Big(h'(x)\psi(g(x))\Big)/h'(x)\Big) \leqslant o\Big(\psi^{-1}\Big(\psi(g(x))\Big)\Big) = o(g(x))$$

as $x \to +\infty$ $(x \notin E)$. Therefore,

$$\ln F(x) \leqslant \ln\mu_*(x,F) + \ln 2 + o(\ln F(x))$$

as $x \to +\infty$ $(x \notin E)$ and the statement of Theorem 3 follows from this. ☐

Remark 1. It is easy to verify that condition $\int_{x_0}^{+\infty} dx/\psi(x)$ is equivalent to condition $\int_{t_0}^{+\infty} t^{-2}\psi^{-1}(t)dt < +\infty$. On the other hand, for given non-decreasing function $\alpha(t)$ the condition $\int_{t_0}^{+\infty} t^{-2}\alpha(t)dt < +\infty$ is equivalent to the condition that there exists a function $\psi \in \mathcal{L}_1$ such that $\alpha(t) = o(\psi^{-1}(t))$ $(t \to +\infty)$.

The authors do not know whether it is possible to choose the function ψ so that the condition of belonging to the class $\mathcal{L}_2$ is fulfilled.

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References

1. Skaskiv O.B. *Behavior of the maximal term of a Dirichlet series that defines an entire function*// Math. Notes – 1985. – V.37, №1. – P. 686–693. https://doi.org/10.1007/BF01652509
2. Pólya G., Szegő G. Aufgaben und Lehrsätze aus der Analysis. – Berlin: Springer, V.2, 1925.
3. Wiman A. Über den Zusammenhang zwischen dem Maximalbetrage einer analytischen Function und dem grössten Gliebe der zugehörigen Taylorischen Reiche// Acta Math. – 1914. – V.37. – P. 305–326.
4. Valiron G. *Sur les fonctions entieres d'ordre fini et d'ordre null et en particulier les fonctions a correspondance reguliere*// Ann. Fac. Sci. Toulouse Math. (6). – 1914. – V.5. – P. 117–257.
5. Valiron G. *Sur le maximum du module des fonctions entieres*// C.r. Acad. Sci. – 1918. – V.166. – P. 605–608.
6. Sheremeta M.M. Entire Dirichlet series. – Kyiv: ISDO, 1993. – 168 p. (in Ukrainian)
7. Skaskiv O.B., Bandura A.I. Asymptotic estimates of positive integrals and entire functions. – Lviv–Ivano-Frankivsk, – LNU-INFTUNG, Goliney, 2015, 108 p. (in Ukrainian)
8. Sunyer-Balaguer F. *Generalizacion del metodo de Wiman-Valiron a una classe de series de Dirichlet*// Publ. semin. mat. Fac. cient. Zaragoza. – 1962. – V.3. – P. 43–47.
9. Skaskiv O.B. *On certain relations between the maximum modulus and the maximal term of an entire Dirichlet series*// Math. Notes. – 1999. – V.66, no.2. – P. 223–232. https://doi.org/10.1007/BF02674881
10. Kuryliak A.O., Salo T.M. *Borel type asymptotic relation for entire Dirichlet series and h-measure of an exceptional sets*// Mat. Stud. – 2026. – V.65, no.1. – P. 15–21. https://doi.org/10.30970/ms.65.1.15-21

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ІНТЕГРАЛИ ТИПУ ЛАПЛАСА-СТІЛТ’ЄСА: СПІВВІДНОШЕННЯ БОРЕЛЯ і h-МІРА ВИНЯТКОВОЇ МНОЖИНИ.

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В класі функцій $F\colon \mathbb{R}\to\mathbb{R}_+$ вигляду $F(x)=\int_0^{+\infty} f(u)e^{xu}\nu(du)$, де ν — невід’ємна міра на $\mathbb{R}_+$ з необмеженим носієм $\operatorname{supp}\nu$, $a(x)$ — довільна невід’ємна ν-вимірна функція на $\mathbb{R}_+$, встановлені умови, за яких асимптотичне співвідношення $\ln F(x)\leqslant(1+o(1))\ln\mu_*(x,F)$ виконується при $x\to+\infty$ зовні деякої множини E скінченної h-міри, тобто, $\int_E dh(x)<+\infty$; тут $\mu_*(x,F)=\sup\{a(u)e^{xu}\colon u\in\operatorname{supp}\nu\}$.

Ключові слова: інтеграл Лапласа-Стілт’єса, співвідношення Бореля, h-міра, виняткова множина.