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ELEMENTARY NOTES ON α-PSEUDOSTARLIKE DIRICHLET SERIES

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A Dirichlet series $F(s) = e^{s\lambda_1} + \sum_{k=2}^{\infty} f_k \exp\{s\lambda_k\}$, $0 < \lambda_k \uparrow \infty$, that converges absolutely in the half-plane $\Pi_0 = \{s\colon \mathrm{Re}s < 0\}$, is said to be α-pseudostarlike, $\alpha \in \mathbb{R}$, if

$$\mathrm{Re}\left\{(1-\alpha)F'(s)/F(s) + \alpha F''(s)/F'(s)\right\} > 0, \quad s \in \Pi_0.$$

It is proved that each α-pseudostarlike function is pseudostarlike, that is $\mathrm{Re}\{F'(s)/F(s)\} > 0$ for all $s \in \Pi_0$. Moreover, if $\alpha \geq 1$ then a α-pseudostarlike function is pseudoconvex, that is $\mathrm{Re}\{F''(s)/F'(s)\} > 0$ for all $s \in \Pi_0$. A similar result is obtained for series of the form $F(s) = e^{-s\lambda_1} + \sum_{k=2}^{\infty} f_k \exp\{s\lambda_k\}$.

Key words: Dirichlet series, α-pseudostarlike function.

1. **Introduction**

For $\alpha \in \mathbb{R}$ P.T. Mocanu ([1]) called a function

$$f(z) = z + \sum_{n=2}^{\infty} f_n z^n \tag{1}$$

α-starlike if $\frac{f(z)f'(z)}{z} \neq 0$ in $\mathbb{D} = \{z\colon |z| < 1\}$ and

$$\mathrm{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 0, \quad z \in \mathbb{D}.$$

In [2] it is proved that each α-starlike function (1) is univalent and starlike in $\mathbb{D}$, that is $\mathrm{Re}\frac{zf'(z)}{f(z)} > 0$ for all $z \in \mathbb{D}$. Many mathematicians devoted their works to the study of the properties of α-starlike functions and their generalizations (we point out here only articles [3], [4], [5], [6], [7], [8]).

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As in [9] and [10, p. 135] by SD_0 we denote a class of Dirichlet series

$$F(s) = e^{s\lambda_1} + \sum_{k=2}^{\infty} f_k \exp\{s\lambda_k\}, \quad s = \sigma + it, \tag{2}$$

absolutely convergent in the half-plane $\Pi_0 = \{s\colon \mathrm{Re}s < 0\}$, where $0 < \lambda_k \uparrow +\infty$. Each function $F \in SD_0$ is non-univalent in Π_0, but if

$$\sum_{k=2}^{\infty} \lambda_k |f_k| \le \lambda_1$$

then ([9], [10, p. 135]) function (2) is conformal in Π_0. A conformal in Π_0 function (2) is said to be pseudostarlike if $\mathrm{Re}\{F'(s)/F(s)\} > 0$ for all $s \in \Pi_0$, and is said to be pseudoconvex if $\mathrm{Re}\{F''(s)/F'(s)\} > 0$ for all $s \in \Pi_0$ ([9], [10, p. 137]). By the way, if

$$\sum_{k=2}^{\infty} \lambda_k |f_k| \le \lambda_1$$

then function (2) is pseudostarlike, and if

$$\sum_{k=2}^{\infty} \lambda_k^2 |f_k| \le \lambda_1^2$$

then function (2) is pseudoconvex.

Combining α-starlikeness with pseudostarliness and pseudoconvexity, we arrive at the following definition. For some real number α a conformal in Π_0 function (2) is said to be α-pseudostarlike if for all $s \in \Pi_0$

$$\mathrm{Re}\left\{(1-\alpha)\frac{F'(s)}{F(s)} + \alpha\frac{F''(s)}{F'(s)}\right\} > 0. \tag{3}$$

Here we will study the properties of such functions.

2. α-Pseudostarlike Dirichlet series

It is clear that if $\alpha = 0$ then α-pseudostarlike function is pseudostarlike, and if $\alpha = 1$ then α-pseudostarlike function is pseudoconvex. There is a more general statement.

Theorem 1. *Each α-pseudostarlike function F is pseudostarlike. Moreover, if $\alpha \ge 1$ then F is pseudoconvex.*

Proof. If we set $\Phi(s) = \frac{F'(s)}{F(s)}$ in (3) then we obtain

$$\mathrm{Re}\left\{\Phi(s) + \alpha\frac{d\ln\Phi(s)}{ds}\right\} > 0$$

and, thus,

$$\mathrm{Re}\left\{\Phi(s) - i\alpha\frac{\partial\ln\Phi(s)}{\partial t}\right\} > 0, \quad s = \sigma + it. \tag{4}$$

Remark also that

$$\frac{F'(s)}{F(s)}=\frac{\lambda_1 e^{s\lambda_1}+\sum\limits_{k=2}^{\infty} f_k\lambda_k\exp\{s\lambda_k\}}{e^{s\lambda_1}+\sum\limits_{k=2}^{\infty} f_k\exp\{s\lambda_k\}}=\frac{\lambda_1+\sum\limits_{k=2}^{\infty} f_k\lambda_n\exp\{s(\lambda_k-\lambda_1)\}}{1+\sum\limits_{k=2}^{\infty} f_k\exp\{s(\lambda_k-\lambda_1)\}}\to\lambda_1 \qquad (5)$$

as $\sigma\to-\infty$, that is there exists $\sigma^*>-\infty$ such that $\mathrm{Re}\Phi(s)>0$ in half-plane $\{s\colon \mathrm{Re}s<\sigma^*\}$.

We need to show that $\mathrm{Re}\Phi(s)>0$ for all $s\in\Pi_0$. Assume from the contrary that this inequality does not hold, that is in view of (5) there exits a point $s_0=\sigma_0+it_0$ in Π_0 such that $\mathrm{Re}\Phi(s)\geq 0$ for $\sigma\leq\sigma_0$ and $\mathrm{Re}\Phi(s_0)=0$. Then $\arg\Phi(\sigma_0+it)$ has either a maximum or a minimum for $t=t_0$, whence it follows that $\frac{\partial\arg\Phi(\sigma_0+it)}{\partial t}=0$ for $t=t_0$. Therefore,

$$\mathrm{Re}\left\{\Phi(s)-i\alpha\frac{\partial\ln\Phi(s)}{\partial t}\right\}=0$$

for $s=s_0$, which is impossible in view of (4). Taking into account (5) the pseudostarlikeness is proved.

If $\alpha\geq 1$ then $(1-\alpha)\mathrm{Re}\frac{F'(s)}{F(s)}\leq 0$ and (3) implies $\mathrm{Re}\frac{F''(s)}{F'(s)}>0$, that is F is pseudoconvex. The proof of Theorem 1 is complete.

Remark 1. If F is pseudostarlike and pseudoconvex then F is α-pseudostarlike for each $\alpha\in[0,1]$.

Remark 2. If either $0\leq\beta\leq\alpha$ or $\alpha\leq\beta\leq 0$ and F is α-pseudostarlike then F is β-pseudostarlike.

Indeed, if $0\leq\beta\leq\alpha$ and F is α-pseudostarlike then $\mathrm{Re}\Phi(s)>0$,

$$\mathrm{Re}\left\{\Phi(s)+\alpha\frac{d\ln\Phi(s)}{ds}\right\}>0$$

and, thus,

$$\mathrm{Re}\frac{d\ln\Phi(s)}{ds}>-\frac{\mathrm{Re}\Phi(s)}{\alpha}\geq-\frac{\mathrm{Re}\Phi(s)}{\beta},$$

whence

$$\mathrm{Re}\left\{\Phi(s)+\beta\frac{d\ln\Phi(s)}{ds}\right\}>0.$$

If $\alpha\leq\beta\leq 0$ then similarly

$$\mathrm{Re}\frac{d\ln\Phi(s)}{ds}<\frac{\mathrm{Re}\Phi(s)}{|\alpha|}\leq\frac{\mathrm{Re}\Phi(s)}{|\beta|}$$

and again

$$\mathrm{Re}\left\{\Phi(s)+\beta\frac{d\ln\Phi(s)}{ds}\right\}>0.$$

3. $\alpha\Sigma$-Pseudostarlike Dirichlet series.

Now let Dirichlet series

$$F(s) = e^{-sh} + \sum_{k=1}^{\infty} f_k \exp\{s\lambda_k\}, \quad s = \sigma + it, \tag{6}$$

absolutely convergent in the half-plane Π_0, where $h > 0$ and $h < \lambda_k \uparrow +\infty$. Dirichlet series (6) is called ([11]) Σ-pseudostarlike if $\mathrm{Re}\frac{F'(s)}{F(s)} < 0$ in Π_0 and is called Σ-pseudoconvex if $\mathrm{Re}\frac{F''(s)}{F'(s)} < 0$ in Π_0. Therefore, as above, we can call the function (6) $\alpha\Sigma$-pseudostarlike if

$$\mathrm{Re}\left\{(1-\alpha)\frac{F'(s)}{F(s)} + \alpha\frac{F''(s)}{F'(s)}\right\} < 0 \quad s \in \Pi_0. \tag{7}$$

Theorem 2. *Each $\alpha\Sigma$-pseudostarlike function is Σ-pseudostarlike. Moreover, if $\alpha \geq 1$ then this function is Σ-pseudoconvex.*

Proof. As above we set $\Phi(s) = \dfrac{F'(s)}{F(s)}$. Then (7) implies

$$\mathrm{Re}\left\{\Phi(s) - i\alpha\frac{\partial \ln \Phi(s)}{\partial t}\right\} < 0, \quad s = \sigma + it. \tag{8}$$

Also we have

$$\frac{F'(s)}{F(s)} = \frac{-he^{-hs} + \sum\limits_{k=1}^{\infty} f_k\lambda_k \exp\{s\lambda_k\}}{e^{-hs} + \sum\limits_{k=1}^{\infty} f_k \exp\{s\lambda_k\}} = \frac{-h + \sum\limits_{k=1}^{\infty} f_k\lambda_k \exp\{s(\lambda_k + h)\}}{1 + \sum\limits_{k=2}^{\infty} f_k \exp\{s(\lambda_k + h)\}} \to -h \tag{9}$$

as $\sigma \to 0$, that is there exists $\sigma^* < 0$ such that $\mathrm{Re}\Phi(s) < 0$ if $\sigma^* < \mathrm{Re}s < 0$.

We need to show that $\mathrm{Re}\Phi(s) < 0$ for all $s \in \Pi_0$. Assume from the contrary that this inequality does not hold, that is in view of (9) there exits a point $s_0 = \sigma_0 + it_0$ in Π_0 such that $\mathrm{Re}\Phi(s) \leq 0$ for $\sigma_0 \leq \sigma < 0$ and $\mathrm{Re}\Phi(s_0) = 0$. Then, as above, we obtain $\frac{\partial \arg \Phi(\sigma_0+it)}{\partial t} = 0$. Therefore,

$$\mathrm{Re}\left\{\Phi(s) - i\alpha\frac{\partial \ln \Phi(s)}{\partial t}\right\} = 0$$

for $s = s_0$, which is impossible in view of (8). Thus, $\mathrm{Re}\Phi(s) < 0$ for all $s \in \Pi_0$ and function (6) is Σ-pseudostarlike.

If $\alpha \geq 1$ then $(1-\alpha)\mathrm{Re}\frac{F'(s)}{F(s)} \geq 0$ and (7) implies $\mathrm{Re}\frac{F''(s)}{F'(s)} < 0$, that is F is Σ-pseudoconvex. The proof of Theorem 2 is complete.

Remark 3. As in Remark 2 it is easy to show that if either $0 \leq \beta \leq \alpha$ or $\alpha \leq \beta \leq 0$ and F is $\alpha\Sigma$-pseudostarlike then F is $\beta\Sigma$-pseudostarlike.

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ЕЛЕМЕНТАРНІ ЗАУВАЖЕННЯ ДО α-ПСЕВДОЗІРКОВИХ РЯДІВ ДІРІХЛЕ

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Абсолютно збіжний у півплощині $\Pi_0 = \{s\colon \mathrm{Re} s < 0\}$ ряд Діріхле $F(s) = e^{s\lambda_1} + \sum_{k=2}^{\infty} f_k \exp\{s\lambda_k\}$, $0 < \lambda_k \uparrow \infty$, називається α-псевдозірковим, $\alpha \in \mathbb{R}$, якщо

$$\mathrm{Re}\left\{(1-\alpha)F'(s)/F(s) + \alpha F''(s)/F'(s)\right\} > 0, \quad s \in \Pi_0.$$

Доведено, що кожна α-псевдозіркова функція є псевдозірковою, тобто $\mathrm{Re}\{F'(s)/F(s)\} > 0$ для всіх $s \in \Pi_0$. Крім цього, якщо $\alpha \geq 1$, то α-псевдозіркова функція є псевдоопуклою, тобто $\mathrm{Re}\{F''(s)/F'(s)\} > 0$ для всіх $s \in \Pi_0$. Подібний результат отримано для рядів вигляду $F(s) = e^{-s\lambda_1} + \sum_{k=2}^{\infty} f_k \exp\{s\lambda_k\}$.

Ключові слова: ряд Діріхле, α-псевдозіркова функція.