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FIRST ORDER COEFFICIENTS DETERMINING IN A TIME-FRACTIONAL DIFFUSION EQUATION

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We study the inverse problem on determining a classical solution of the first boundary-value problem for a time-fractional diffusion equation and time-dependent continuous coefficients at derivatives of the first order in the equation. We use space-integral overdetermination conditions. We find sufficient conditions for a time-local unique solvability of the inverse problem.

Key words: fractional derivative, inverse problem, integral condition, Green's vector-function

1. **Introduction**

Equations with fractional derivatives and inverse problems to them are appearing in different branches of science and engineering, and the range of the applicability of the generated models is increase considerable. The conditions for classical solvability of the Cauchy and boundary value problems to equations with a time and a time-space fractional diffusion equations were obtained in [1]–[8] and other papers. The inverse initial and boundary value problems with different unknown functions or parameters, in particular, unknown minor or senior coefficients for such equations were investigated in many papers.

In this paper, for the equation

$$ {}^cD_t^\beta u - \Delta u - \sum_{j=1}^{n} r_j u_{x_j} - r(t)u = F_0(x,t), \quad x \in \Omega,\ t \in (0,T] \qquad (1)$$

2020 Mathematics Subject Classification: 35S10 with the Caputo fractional derivative of order $\beta \in (0,1)$ we study the inverse problem

$$u(x,t)=0,\ \ x\in\partial\Omega,\ \ t\in[0,T], \tag{2}$$

$$u(x,0)=F_1(x),\ \ x\in\bar{\Omega}, \tag{3}$$

$$\int\limits_{\Omega} u(x,t)\varphi_j'(x_j)dx=\Phi_j(t),\ \ t\in[0,T],\ \ j\in\{1,\dots,n\} \tag{4}$$

of determining $(u, r_1, \dots, r_n)$ where $F_0, F_1, r, \varphi_j, \Phi_j$ are given functions, Ω is a region in $\mathbb{R}^n$, $n\geq 2$, bounded by a closed surface of class C^1.

Very often, when studying inverse problems, the value of the sought solution of the problem at the final moment of time is additionally set, for example, in [9]–[13], some additional boundary data is set, for example, in [14]. Integral overdetermination conditions were used, for example, in [15]–[19], in [16] with the scalar product (u,φ_0) in abstract Hilbert space, in [18] in spaces of Bessel potentials.

In the case of two unknown functions inverse problems for a time-fractional diffusion equation were studied, for example, in [10, 13] under final time data, in [20, 21] under a time and space-integral overdetermination conditions, in [22] under two space-integral overdetermination conditions, in [23] under two time-integral overdetermination conditions. Inverse problem of kind (1)–(4) for a time- or time-space fractional diffusion equations was not studied.

2. Notations, definitions and auxiliary results

Let $Q=\Omega\times(0,T]$. We denote by $f*g$ the convolution of f and g, use the function

$$f_\lambda(t)=\frac{\theta(t)t^{\lambda-1}}{\Gamma(\lambda)}\ \text{for}\ \lambda>0\quad\text{and}\quad f_\lambda(t)=f'_{1+\lambda}(t)\ \text{for}\ \lambda\leqslant 0$$

where $\Gamma(\lambda)$ is the Gamma-function, $\theta(t)$ is the Heaviside function.

Note that $f_\lambda * f_\mu = f_{\lambda+\mu}$.

The Riemann-Liouville derivative $v^{(\beta)}(t)$ of order $\beta>0$ is defined by the formula

$$v^{(\beta)}(t)=f_{-\beta}(t)*v(t).$$

The Caputo (Caputo-Djrbashian, regularized) fractional derivative of order $\beta\in(m-1,m)$ is defined by

$$^cD^\beta v(t)=\frac{1}{\Gamma(m-\beta)}\int_0^t (t-\tau)^{m-\beta-1}\frac{d^m}{d\tau^m}v(\tau)d\tau,$$

and therefore

$$^cD^\beta v(t)=v^{(\beta)}(t)-\sum_{j=0}^{m-1} f_{j+1-\beta}(t)v^{(j)}(0). \tag{5}$$

We shall use the notations

$$(v,\varphi):=\int\limits_{\Omega} v(x)\varphi(x)dx,\ \ (Lv)(x,t)=D_t^\beta v(x,t)-(\Delta v)(x,t),\quad (x,t)\in Q$$

and the functional space $C_{2,\beta}(\bar{Q})=\{v\in C(\bar{Q})\colon {}^cD_t^\beta v, \Delta v\in C(Q)\}$.

The vector-function $(u, r_1, \dots, r_n) \in C_{2,\beta}(\bar{Q}) \times [C[0,T]]^n$ is called a classical solution of the problem (1)–(4) if it satisfies the equation (1) in Q and the conditions (2), (3), (4).

From (2)–(4) the compatibility conditions follow

$$F_1(x) = 0,\ x \in \partial\Omega, \quad (F_1, \varphi_j) = \Phi_j(0), \quad j \in 1, \dots, n. \tag{6}$$

The vector-function $\big(G_0(x,t), G_1(x,t)\big)$ is called a Green's vector-function of the problem (2), (3) to the equation

$$(Lu)(x,t) = F(x,t), \quad (x,t) \in Q \tag{7}$$

if under rather regular F, F_1 the function

$$u(x,t) = \int_0^t d\tau \int_\Omega G_0(x-y, t-\tau) F(y,\tau) dy + \int_\Omega G_1(x-y,t) F_1(y) dy, \quad (x,t) \in \overline{Q} \tag{8}$$

is a classical solution of this problem.

Such Green's vector-function exists ([1, 3, 6]): $G_0(x,t)$ is the fundamental solution of the equation (1), $G_1(x,t) = f_{1-\beta} * G_0(x,t)$, for multi-indeces γ, $|\gamma| \in \{0,1\}$ the following bounds were found

$$\begin{aligned}
|D_x^\gamma G_0(x,t)| &\le C t^{-\beta\frac{n+|\gamma|}{2}+\beta-1} e^{-c(|x|t^{-\frac{\beta}{2}})^{\frac{2}{2-\beta}}} \Psi_{n+|\gamma|-2}(|x|t^{-\frac{\beta}{2}}), \\
|D^\gamma G_1(x,t)| &\le C t^{-\beta\frac{n+|\gamma|}{2}} e^{-c(|x|t^{-\frac{\beta}{2}})^{\frac{2}{2-\beta}}} \Psi_{n+|\gamma|-2}(|x|t^{-\frac{\beta}{2}}), \\
|{}^cD_t^\beta G_0(x,t)| &\le C t^{-\frac{n\beta}{2}-1} e^{-c(|x|t^{-\frac{\beta}{2}})^{\frac{2}{2-\beta}}} \Psi_{n-2}(|x|t^{-\frac{\beta}{2}}), \\
|{}^cD_t^\beta G_1(x,t)| &\le C t^{-\frac{n\beta}{2}-\beta} e^{-c(|x|t^{-\frac{\beta}{2}})^{\frac{2}{2-\beta}}} \Psi_{n}(|x|t^{-\frac{\beta}{2}}),
\end{aligned} \tag{9}$$

where

$$\Psi_m(z) = \begin{cases} 1, & m < 0; \\ 1 + |\ln|z||, & m = 0; \\ |z|^{-m}, & m > 0, \end{cases} \quad \text{for } |z| < 1, \quad \Psi_m(z) = \Psi_m(1) \ \text{ for } |z| > 1,$$

and $c < (2-\beta)(\frac{\beta^\beta}{4})^{\frac{1}{2-\beta}}$. Hereinafter c, C, C_k, c_k, d_k $(k \in \mathbb{N})$ are positive constants.

We denote

$$(G_0\varphi)(x,t) = \int_\Omega G_0(x-y,t)\varphi(y)dy,$$

$$(G_1\varphi)(x,t) = \int_\Omega G_1(x-y,t)\varphi(y)dy,$$

$$(\mathcal{G}_0\varphi)(x,t) = \int_0^t d\tau \int_\Omega G_0(x-y,t-\tau)\varphi(y)dy, \ \ (x,t) \in Q.$$

Lemma 1. *Assume that $\varphi \in C(\bar{\Omega})$, $\varphi_j(x_j)$, $\varphi_j''(x_j)$ are continuous and bounded functions. Then*

$$(G_0\varphi)(\cdot,t), \frac{\partial}{\partial y_j}(G_0\varphi)(\cdot,t) \in C(\bar{\Omega}),\ t \in (0,T],\ \mathcal{G}_0\varphi, \frac{\partial}{\partial y_j}\mathcal{G}_0\varphi, G_1\varphi,$$

$$
{}^cD_t^\beta(\mathcal{G}_0\varphi,\varphi_j),\ {}^cD_t^\beta\frac{\partial}{\partial y_j}(\mathcal{G}_0\varphi,\varphi_j),\ {}^cD_t^\beta(G_1\varphi,\varphi_j)\in C(\bar{Q}),\ j\in\{1,\dots,n\},
$$

and it exists positive constant C^* such that for all $t\in(0,T]$

$$
\|(G_0\varphi)(\cdot,t)\|_{C(\bar\Omega)}\le C^*t^{\beta-1}\,\|\varphi\|_{C(\bar\Omega)},\ \ \Big\|\frac{\partial}{\partial y_j}(G_0\varphi)(\cdot,t)\Big\|_{C(\bar\Omega)}\le C^*t^{\frac{\beta}{2}-1}\,\|\varphi\|_{C(\bar\Omega)},
$$

$$
\|(\mathcal{G}_0\,\varphi)(\cdot,t)\|_{C(\bar\Omega)}\le C^*t^{\beta}\,\|\varphi\|_{C(\bar\Omega)},\ \ \Big\|\Big(\frac{\partial}{\partial y_j}\mathcal{G}_0\varphi\Big)(\cdot,t)\Big\|_{C(\bar\Omega)}\le C^*t^{\beta/2}\,\|\varphi\|_{C(\bar\Omega)},
$$

$$
\|{}^cD_t^\beta\ D^\gamma(\mathcal{G}_0\,\varphi,\varphi_j)\|_{C(\bar\Omega)}\le C^*t^{\frac{\beta}{2}(n-|\gamma|)}\,\|\varphi\|_{C(\bar\Omega)},\ \ |\gamma|\in\{0,1\},
$$

$$
\|G_1\,\varphi\|_{C(\bar Q)}\le C^*\|\varphi\|_{C(\bar\Omega)},\ \ \|{}^cD_t^\beta(G_1\varphi,\varphi_j)\|\le C^*t^{\frac{\beta}{2}(n-2)}\|\varphi\|_{C(\bar\Omega)},\ \ j\in\{1,\dots,n\}.
$$

Proof. From the estimates (9) at $n\ge3$ for multy-indeces γ, $|\gamma|\in\{0,1\}$

$$
\Big|\int_\Omega D_x^\gamma G_0(x-y,t)\varphi(y)dy\Big|\le
$$

$$
\le\int\limits_{\{y\in\Omega:\ |x-y|^2<t^\beta\}}|D_x^\gamma G_0(x-y,t)\varphi(y)|dy+\int\limits_{\{y\in\Omega:\ |x-y|^2>t^\beta\}}|D_x^\gamma G_0(x-y,t)\varphi(y)|dy\le
$$

$$
\le Ct^{-\frac{\beta(n+|\gamma|)}{2}+\beta-1}\Bigg[\int\limits_{\{y\in\Omega:\ |x-y|<t^{\frac{\beta}{2}}\}}|x-y|t^{-\frac{\beta}{2}}]^{2-n-|\gamma|}|\varphi(y)|dy+
$$

$$
+\int\limits_{\{y\in\Omega:\ |x-y|>t^{\frac{\beta}{2}}\}}e^{-c[|x-y|t^{-\frac{\beta}{2}}]^{\frac{2}{2-\beta}}}|\varphi(y)|dy\Bigg],
$$

and moving to the spherical coordinate system, we will have

$$
\Big|\int_\Omega D_x^\gamma G_0(x-y,t)\varphi(y)dy\Big|\le
$$

$$
\le C_1\Bigg[\frac{1}{t}\int\limits_0^{t^{\beta/2}}r^{1-|\gamma|}dr+\int\limits_{t^{\beta/2}}^{\infty}r^{n-1}t^{-\frac{\beta(n+|\gamma|)}{2}}e^{-c[rt^{-\frac{\beta}{2}}]^{\frac{2}{2-\beta}}}dr\Bigg]\,\|\varphi\|_{C(\bar\Omega)}\le
$$

$$
\le C_2t^{\beta-1-\frac{\beta|\gamma|}{2}}\,\|\varphi\|_{C(\bar\Omega)},\quad y\in\bar\Omega,\ \ 0<t\le T,
$$

$\beta-1-\frac{\beta|\gamma|}{2}>-1$ for $|\gamma|\in\{0,1\}$. Similarly we get that

$$
\Big|{}^cD_t^\beta\int\limits_0^t d\tau\int\limits_\Omega\ \big(G_0(x-y,\tau),\varphi_j(x_j)\big)\varphi(y)dy\Big|=
$$

$$
=\Big|{}^cD_t^\beta\int\limits_0^t d\tau\int\limits_\Omega\ \Big(\int\limits_\Omega\ G_0(x-y,\tau)\varphi_j(x_j)dx\Big)\varphi(y)dy\Big|\le
$$

$$\le C_3 \int\limits_0^t \left[\tau^{-1-\beta} \int\limits_0^{t^{\beta/2}} r^{n+1} dr + \tau^{-1-\frac{n\beta}{2}} \int\limits_{t^{\beta/2}}^{\infty} r^{2n-1} e^{-c[r\tau^{-\frac{\beta}{2}}]^{\frac{2}{2-\beta}}} dr \right] d\tau \, \|\varphi\|_{C(\bar\Omega)} \le$$

$$\le C_4 \int\limits_0^t \tau^{\frac{n\beta}{2}-1} \, \|\varphi\|_{C(\bar\Omega)} \le C^* t^{\frac{n\beta}{2}} \, \|\varphi\|_{C(\bar\Omega)},$$

$$\left| {}^cD_t^\beta \int\limits_0^t d\tau \int\limits_\Omega \frac{\partial}{\partial y_j} \big(G_0(x-y,\tau), \varphi_j(x_j)\big) \varphi(y) dy \right| \le C^* t^{\frac{\beta}{2}(n-1)} \, \|\varphi\|_{C(\bar\Omega)},$$

$$\left| \int\limits_\Omega G_1(x-y,t)\varphi(y)dy \right| \le C^* \|\varphi\|_{C(\bar\Omega)}, \qquad (x,t) \in \bar Q,$$

$$\left| {}^cD_t^\beta \int\limits_\Omega \big(G_1(x-y,t), \varphi_j(x_j)\big)\varphi(y)dy \right| =$$

$$= \left| {}^cD_t^\beta \int\limits_\Omega \left(\int\limits_\Omega G_1(x-y,t)\varphi_j(x_j)dx \right) \varphi(y)dy \right| \le$$

$$\le C_5 \left[t^{-\beta} \int\limits_0^{t^{\beta/2}} r^{n-1} dr + t^{-\beta(\frac{n}{2}+1)} \int\limits_{t^{\beta/2}}^{\infty} r^{2n-1} e^{-c[rt^{-\frac{\beta}{2}}]^{\frac{2}{2-\beta}}} dr \right] \|\varphi\|_{C(\bar\Omega)} \le$$

$$\le C^* t^{\frac{\beta}{2}(n-2)} \, \|\varphi\|_{C(\bar\Omega)}, \;\; j \in \{1, \ldots, n\}.$$

□

It follows from [2] the following result.

Theorem 1. *Let F_1 is bounded and satisfies Hölder continuity condition in Ω, F_0 is bounded and satisfies Hölder continuity condition with respect to space variables for all $t \in [0,T]$. Then there exists the unique classical solution of the problem (7), (2), (3). It is given by (8).*

We define the weighted norms [9]

$$\|v\|_\sigma = \sup_{t\in(0,T]} \; e^{-\sigma t} \, \|v(\cdot,t)\|_{C(\bar\Omega)} \;\text{ for }\; v \in C(\bar Q).$$

We have the equivalence of norms $\|v\|_\sigma$ and $\|v\|_{C(\bar Q)}$:

$$e^{-\sigma T} \|v\|_{C(\bar Q)} \le \|v\|_\sigma \le \|v\|_{C(\bar Q)}.$$

Lemma 2. *If $\varphi \in C(\bar Q)$, $\varphi_j(x_j)$, $\varphi_j''(x_j)$ ($j \in \{1,\ldots,n\}$) are continuous and bounded functions in Ω, then there exist positive numbers $C_\sigma, C_{\sigma,k}$ ($k \in \{0,1,2,3,4\}$) such that the following bounds hold*

$$\|\mathcal{G}_0\varphi\|_\sigma \le C_\sigma \|\varphi\|_\sigma, \;\; \left\| \frac{\partial}{\partial y_j} \mathcal{G}_0 \varphi \right\|_\sigma \le C_{\sigma,0} \|\varphi\|_\sigma, \;\; \|G_1\varphi\|_\sigma \le C_{\sigma,1} \|\varphi\|_\sigma,$$

$$\|{}^cD_t^\beta\ (\mathcal{G}_0\varphi,\varphi_j)\|_\sigma \le C_{\sigma,2}\|\varphi\|_\sigma,\ \ \Big\|{}^cD_t^\beta\ \frac{\partial}{\partial y_j}(\mathcal{G}_0\varphi,\varphi_j)\Big\|_\sigma \le C_{\sigma,3}\|\varphi\|_\sigma,$$

$$\|{}^cD_t^\beta\ (G_1\varphi,\varphi_j)\|_\sigma \le C_{\sigma,4}\|\varphi\|_\sigma,\ \ j\in\{1\dots,n\},$$

moreover, $C_\sigma\to 0$, $C_{\sigma,k}\to 0$ *(*$k\in\{0,2,3\}$*) if* $\sigma\to+\infty$.

Proof. By using Lemma 1 we get

$$\|\mathcal{G}_0\varphi\|_\sigma = \sup_{t\in(0,T]} e^{-\sigma t}\Big\|\int_0^t\Big(\int_\Omega G_0(\cdot-y,t-\tau)\varphi(y,\tau)dy\Big)d\tau\Big\|_{C(\bar Q)} =$$

$$= \sup_{t\in(0,T]}\Big\|\int_0^t e^{-\sigma(t-\tau)}\Big(\int_\Omega G_0(\cdot-y,t-\tau)e^{-\sigma\tau}\varphi(y,\tau)dy\Big)d\tau\Big\|_{C(\bar Q)} \le$$

$$\le C^*\sup_{t\in(0,T]}\int_0^t e^{-\sigma(t-\tau)}\ (t-\tau)^{\beta-1}d\tau\ \sup_{\tau\in(0,T]} e^{-\sigma\tau}\|\varphi(\cdot,\tau)\|_{C(\bar Q)} =$$

$$= C^*\sup_{t\in(0,T]}\int_0^t e^{-\sigma\tau}\ \tau^{\beta-1}d\tau\ \|\varphi\|_\sigma = C_\sigma\|\varphi\|_\sigma,$$

$$\Big\|\frac{\partial}{\partial y_j}\mathcal{G}_0\varphi\Big\|_\sigma = \sup_{t\in(0,T]}\Big\|\int_0^t e^{-\sigma(t-\tau)}\Big(\int_\Omega \frac{\partial}{\partial y_j}G_0(\cdot-y,t-\tau)e^{-\sigma\tau}\ \varphi(y,\tau)dy\Big)d\tau\Big\|_{C(\bar Q)} \le$$

$$\le C^*\sup_{t\in(0,T]}\int_0^t e^{-\sigma\tau}\ \tau^{\frac{\beta}{2}-1}d\tau\ \|\varphi\|_\sigma,$$

$$\|{}^cD_t^\beta\ (\mathcal{G}_0\varphi,\varphi_j)\|_\sigma =$$

$$= C\sup_{t\in(0,T],x\in\bar\Omega}\Big|{}^cD_t^\beta\int_0^t e^{-\sigma(t-\tau)}d\tau\int_\Omega\Big(\int_\Omega G_0(x-y,t-\tau)e^{-\sigma\tau}\ \varphi(y,\tau)dy\Big)\varphi_j(x_j)dx\Big| \le$$

$$\le C^*\sup_{t\in(0,T]}\int_0^t e^{-\sigma\tau}\ \tau^{\frac{n\beta}{2}-1}d\tau\ \|\varphi\|_\sigma,$$

$$\Big\|{}^cD_t^\beta\ \frac{\partial}{\partial y_j}(\mathcal{G}_0\varphi,\varphi_j)\Big\|_\sigma =$$

$$= C\sup_{t\in(0,T],x\in\bar\Omega}\Big|{}^cD_t^\beta\int_0^t e^{-\sigma(t-\tau)}d\tau\int_\Omega\Big(\int_\Omega \frac{\partial}{\partial y_j}G_0(x-y,t-\tau)e^{-\sigma\tau}\ \varphi(y,\tau)dy\Big)\varphi_j(x_j)dx\Big| \le$$

$$\le C^*\sup_{t\in(0,T]}\int_0^t e^{-\sigma\tau}\ \tau^{\frac{\beta}{2}(n-1)-1}d\tau\ \|\varphi\|_\sigma,$$

$$\|G_1\varphi\|_\sigma = \sup_{t\in(0,T]} \left\| \int\limits_\Omega G_1(\cdot - y, t)e^{-\sigma t}\varphi(y,t)dy \right\|_{C(\bar{Q})} \leq C^* \, \|\varphi\|_\sigma,$$

$$\|{}^c D_t^\beta \, (G_1\varphi, \varphi_j)\|_\sigma =$$

$$= \sup_{t\in(0,T]} \left\| {}^c D_t^\beta \int\limits_\Omega \left(\int\limits_\Omega G_1(\cdot - y, t)e^{-\sigma t}\varphi(y,t)dy \right) \varphi_j(x_j)dx \right\|_{C(\bar{Q})} \leq C^* \, T^{\frac{\beta}{2}(n-2)} \|\varphi\|_\sigma.$$

We see that

$$C_\sigma := C^* \sup_{t\in(0,T]} \int\limits_0^t e^{-\sigma\tau} \, \tau^{\beta-1} d\tau \leq C^* \frac{\Gamma(\beta)}{\sigma^\beta} \to 0, \; \sigma\to\infty,$$

$$C_{\sigma,0} := C^* \sup_{t\in(0,T]} \int\limits_0^t e^{-\sigma\tau} \, \tau^{\frac{\beta}{2}-1} d\tau \leq C^* \frac{\Gamma(\beta/2)}{\sigma^{\beta/2}} \to 0, \; \sigma\to\infty, \; C_{\sigma,1} = C^*,$$

$$C_{\sigma,2} := C^* \sup_{t\in(0,T]} \int\limits_0^t e^{-\sigma\tau} \, \tau^{\frac{n\beta}{2}-1} d\tau \leq C^* \frac{\Gamma(\frac{n\beta}{2})}{\sigma^{\frac{n\beta}{2}}} \to 0, \; \sigma\to\infty,$$

$$C_{\sigma,3} := C^* \sup_{t\in(0,T]} \int\limits_0^t e^{-\sigma\tau} \, \tau^{\frac{\beta}{2}(n-1)-1} d\tau \leq C^* \frac{\Gamma(\frac{\beta}{2}(n-1))}{\sigma^{\frac{\beta}{2}(n-1)}} \to 0, \; \sigma\to\infty,$$

$$C_{\sigma,4} = C^* \, T^{\frac{\beta}{2}(n-2)}.$$

□

3. **Solution of the inverse problem**

Lemma 3. *Assume that F_1 is a bounded and Hölder function in Ω, F_0 is a bounded and Hölder function in spatial variables for all $t\in[0,T]$, $\varphi_j(x_j)$, $\varphi_j''(x_j)$ are continuous and bounded functions, $r, \Phi_j \in C[0,T]$, $|\Phi_j(t)| \geq \psi_0 > 0$, $t\in[0,T]$, $j\in\{1,\ldots,n\}$, and the condition (6) holds.*

The vector-function $(u, r_1, \ldots, r_n) \in C_{2,\beta}(\bar{Q}) \times \big(C[0,T]\big)^n$ is the solution of the problem (1)–(4) if and only if u satisfies the nonlinear integro-differential equation

$$u(x,t) = u_0(x,t) + \int\limits_0^t d\tau \int\limits_\Omega G_0(x-y, t-\tau)\, r(\tau) u(y,\tau)dy - \tag{10}$$

$$-\int\limits_0^t \sum_{k=1}^n \frac{d\tau}{\Phi_k(\tau)} \int\limits_\Omega \frac{\partial}{\partial y_k} G_0(x-y,t-\tau) \, \Big[r(\tau)\big(u(z,\tau), \varphi_k(z_k)\big) + \big(F_0(z,\tau), \varphi_k(z_k)\big) -$$

$$- {}^c D_\tau^\beta \big(u(z,\tau), \varphi_k(z_k)\big) + \big(u(z,\tau), \varphi_k''(z_k)\big) \Big] \, u(y,\tau)dy$$

in a subspace $M(\bar{Q})$ of functions $u \in C(\bar{Q})$ with continuous ${}^cD_t^\beta\big(u(z,t),\varphi_k(z_k)\big)$, $k \in \{1,\dots,n\}$ where

$$u_0(x,t)=\int_0^t d\tau\int_\Omega G_0(x-y,t-\tau)\,F_0(y,\tau)+\int_\Omega G_1(x-y,t)F_1(y)dy,\ \ (x,t)\in\bar{Q},$$

and

$$\begin{aligned}r_k(t)=\frac{1}{\Phi_k(t)}\Big[&r(t)\big(u(z,t),\varphi_k(z_k)\big)+\big(F_0(z,t),\varphi_k(z_k)\big)-\\ &-{}^cD_t^\beta\big(u(z,t),\varphi_k(z_k)\big)+\big(u(z,t),\varphi_k''(z_k)\big)\Big],\ \ k\in 1,\dots,n.\end{aligned}\tag{11}$$

Proof. From the equation (1), using the conditions (4), we get (11). For each $u \in C_{2,\beta}(\bar{Q})$ we have $r_1,\dots,r_n \in C[0,T]$.

From properties of the components of the Green's vector-function, in particular, Hölder condition's property of them and their derivatives of the first order, by Theorem 1, we get that $u \in C_{2,\beta}(\bar{Q})$ is the solution of the boundary-value problem (1)–(3) iff it is the solution of the equation

$$\begin{aligned}u(x,t)=\int_0^t d\tau\int_\Omega &G_0(x-y,t-\tau)\Big[F_0(y,\tau)+r(\tau)u(y,\tau)+\\ +\sum_{j=1}^n r_j(\tau)\frac{\partial}{\partial y_j}u(y,\tau)\Big]dy&+\int_\Omega G_1(x-y,t)F_1(y)dy,\ \ (x,t)\in\bar{Q}.\end{aligned}\tag{12}$$

Substituting $r_1,\dots,r_n$ from (11) in (12), for the solution of the problem (1)–(4) we get the equation (10). By assumptions and Theorem 1, $u_0 \in C_{2,\beta}(\bar{Q})$.

On the contrary, let $u \in M(\bar{Q})$ be the solution of the equation (10). Because (12) coincides with (10), if the expressions (11) are substituted into (12) instead of $r_j(t)$, according to Theorem 1, the function u satisfies the problem (1)–(3). We also show that u satisfies the conditions (4) if $r_1,\dots,r_n$ are defined according to (11). □

Theorem 2. *In assumptions of Lemma 3 there exists $T_1 > 0$ and for all $T \in (0,T_1]$ the unique solution $(u,r_1,\dots,r_n) \in C_{2,\beta}(\bar{Q}) \times \big(C[0,T]\big)^n$ of the problem (1)–(4), u satisfies the equation (10), $r_j(t)$ ($j \in \{1,\dots,n\}$) are defined by (11).*

Proof. Let $w_j(t)=w_j(t,u):={}^c D_t^\beta\big(u(z,t),\varphi_j(z_j)\big)$, $j\in\{1,\dots,n\}$.

According to Lemma 3, it is sufficient to prove a time-local solvability of the equation (10) in $M(\bar{Q})$. From the equation (10) we find

$$\begin{aligned}w_j(t,u)=w_j(t,u_0)+{}^c D_t^\beta\int_0^t r(\tau)d\tau\int_\Omega\big(G_0(x-y,t-\tau),\varphi_j(x_j)\big)\,u(y,\tau)dy-\\ -\sum_{k=1}^n {}^cD_t^\beta\int_0^t d\tau\int_\Omega\frac{\partial}{\partial y_k}\big(G_0(x-y,t-\tau),\varphi_j(x_j)\big)\,\frac{1}{\Phi_k(\tau)}\Big[r(\tau)\big(u(z,\tau),\varphi_k(z_k)\big)+\\ +\big(F_0(z,\tau),\varphi_k(z_k)\big)+\big(u(z,\tau),\varphi_k''(z_k)\big)-w_k(\tau,u)\Big]u(y,\tau)dy.\end{aligned}$$

On $M(\bar{Q})$ we consider the operator $P=(P_1,P_2)$:

$$(P_1v)(x,t)=u_0(x,t)+\int\limits_0^t d\tau\int\limits_\Omega G_0(x-y,t-\tau)\,r(\tau)v(y,\tau)dy-$$

$$-\int\limits_0^t\sum_{k=1}^n\frac{d\tau}{\Phi_k(\tau)}\int\limits_\Omega\frac{\partial}{\partial y_k}G_0(x-y,t-\tau)\,\Big[r(\tau)\big(v(z,\tau),\varphi_k(z_k)\big)+\big(F_0(z,\tau),\varphi_k(z_k)\big)-$$

$$-w_k(\tau,v)+\big(v(z,\tau),\varphi_k''(z_k)\big)\Big]v(y,\tau)dy,$$

and $P_2v=(P_{2,1}v,\dots,P_{2,n}v)$,

$$(P_{2,j}v)(x,t)=w_j(t,u_0)+$$

$$+{}^cD_t^\beta\int\limits_0^t d\tau\int\limits_\Omega\big(G_0(x-y,t-\tau),\varphi_j(x_j)\big)\,r(\tau)\big(v(z,\tau),\varphi_j(z_j)\big)\,dy-$$

$$-\sum_{k=1}^n{}^cD_t^\beta\int\limits_0^t d\tau\int\limits_\Omega\frac{\partial}{\partial y_k}\big(G_0(x-y,t-\tau),\varphi_j(x_k)\big)\,\frac{1}{\Phi_k(\tau)}\Big[r(\tau)\big(v(z,\tau),\varphi_k(z_k)\big)+$$

$$+\big(F_0(z,\tau),\varphi_k(z_k)\big)+\big(v(z,\tau),\varphi_k''(z_k)\big)-w_k(\tau,v)\Big]v(y,\tau)dy,$$

$$(x,t)\in\bar{Q},\ v\in M(\bar{Q}),\ \ j\in\{1,\dots,n\}.$$

For $v\in M(\bar{Q})$ with $Q=\Omega\times(0,T_0]$ let

$$\|v\|:=\|v\|_{\sigma,M}=\max[\|v\|_\sigma,\max_{j\in\{1,\dots,n\}}\|w_j\|_\sigma],$$

$$\|Pv\|=\|Pv\|_{\sigma,M}=\max[\|P_1v\|_\sigma,\max_{j\in\{1,\dots,n\}}\|P_{2,j}v\|_\sigma],$$

$$M_R(\bar{Q})=\{v\in M(\bar{Q}):\ \|v\|=\|v\|_{\sigma,M}\le R\}.$$

By assumptions and Theorem 1, $u_0\in C_{2,\beta}(\bar{Q})$, $\frac{\partial}{\partial x_j}u_0(x,t)\in C(\bar{Q})$. Moreover, by Lemma 2, it exists positive constant $d_0=d_0(T_0)$ such that $\|u_0\|=\|u_0\|_{\sigma,M}\le d_0$.

By using Lemma 2, for each $v\in M_R(\bar{Q})$ with $R\ge\max\{1,\|F_0\|_\sigma\}$ we obtain

$$\|P_1v\|_\sigma=\sup_{t\in(0,T]}e^{-\sigma t}\|(P_1v)(\cdot,t)\|_\sigma\le$$

$$\le\|u_0\|_\sigma+\sup_{t\in(0,T]}\Bigg\{\Bigg|\int\limits_0^t d\tau\int\limits_\Omega e^{-(t-\tau)\sigma}G_0(x-y,t-\tau)r(\tau)e^{-\tau\sigma}v(y,\tau)dy\Bigg|+$$

$$+\Bigg|\sum_{j=1}^n\int\limits_0^t\frac{1}{\Phi_j(\tau)}d\tau\int\limits_\Omega e^{-(t-2\tau)\sigma}\frac{\partial}{\partial y_j}G_0(x-y,t-\tau)\Big[\big(e^{-\tau\sigma}v(z,\tau),\varphi_k''(z_k)+r(\tau)\varphi_k(z_k)\big)+$$

$$+\big(e^{-\tau\sigma}F_0(z,\tau),\varphi_k(z_k)\big)-e^{-\tau\sigma}w_k(\tau,v)\Big]e^{-\tau\sigma}v(y,\tau)dy\Bigg|\Bigg\}\le$$

$$\le d_0 + C_\sigma \max_{\tau\in[0,T]} |r(\tau)|\cdot\|v\|_\sigma + \frac{\max\{C_{\sigma,0},C_{\sigma,2}\}e^{T\sigma}}{\psi_0}\sum_{k=1}^{n}\Bigg[\big(\|F_0\|_\sigma +$$

$$+\max_{\tau\in[0,T]}|r(\tau)|\cdot\|v\|_\sigma\big)\int\limits_\Omega |\varphi_k(z_k)|dz + \|v\|_\sigma \int\limits_\Omega |\varphi_k''(z_k)|dz + \|w_k\|_\sigma\Bigg]\|v\|_\sigma \le$$

$$\le d_0 + d_1 C_\sigma\, R + d_2\max\{C_{\sigma,0},C_{\sigma,2}\}e^{T\sigma}\,R^2 \le d_0 + d_3\max\{C_\sigma, C_{\sigma,0},C_{\sigma,2}\}\,e^{T\sigma}\,R^2,$$

$$\|P_{2,j}v\|_\sigma = \sup_{t\in(0,T]} e^{-\sigma t}\|(P_{2,j}v)(\cdot,t)\|_\sigma \le \sup_{t\in(0,T]}\Big\{\|w_j(t,u_0)\|_\sigma +$$

$$+\Big|{}^cD_t^\beta \int\limits_0^t d\tau \int\limits_\Omega e^{-(t-\tau)\sigma}G_0(x-y,t-\tau)r(\tau)\big(\,e^{-\tau\sigma}v(z,\tau),\varphi_j(z_j)\big)\,dy +$$

$$+\sum_{k=1}^{n}{}^cD_t^\beta\int\limits_0^t d\tau\int\limits_\Omega e^{-(t-2\tau)\sigma}\Big(\frac{\partial}{\partial y_k}G_0(x-y,t-\tau),\varphi_j(x_k)\Big)\times$$

$$\times\frac{1}{\Phi_k(\tau)}\Big[r(\tau)\big(e^{-\tau\sigma}v(z,\tau),\varphi_k(z_k)\big)+$$

$$+\big(e^{-\tau\sigma}F_0(z,t),\varphi_k(z_k)\big)+\big(e^{-\tau\sigma}v(z,\tau),\varphi_k''(z_k)\big)-w_k(t,e^{-\tau\sigma}v)\Big]\,e^{-\tau\sigma}v(y,\tau)dy\Big|\Big\}\le$$

$$\le d_0 + d_3\max\{C_\sigma,C_{\sigma,0},C_{\sigma,2},C_{\sigma,3}\}\,e^{T\sigma}\,R^2,\ \ j\in\{1,\dots,n\}.$$

Therefore, for each $v\in M_{\sigma,R}$,

$$\|Pv\| \le d_0 + d_3\max\{C_\sigma,C_{\sigma,0},C_{\sigma,2},C_{\sigma,3}\}e^{T\sigma}\,R^2.$$

We choose $R\ge\max[1,\|F_0\|_\sigma,2d_0]$. Then $d_0\le\frac{R}{2}$. Using that $C_\sigma\to 0$, $C_{\sigma,j}\to 0$ ($j\in\{0,2,3\}$) if $\sigma\to\infty$ we may choose σ_0 such that

$$d_3\widehat{C} = d_3\max\{C_{\sigma_0},C_{\sigma_0,0},C_{\sigma_0,2},C_{\sigma_0,3}\} < \frac{1}{8R}.$$

Then we choose $T_1\in(0,T_0]$ such that $e^{\sigma_0 T_1}<4$ and for all $T\in(0,T_1]$ obtain

$$\|Pv\| < \frac{R}{2}+\frac{R}{2} = R.$$

Similarly, for each $v_1,v_2\in M_R$, $T\in(0,T_1]$ we get

$$\|Pv_1-Pv_2\| < d_3\widehat{C}e^{T\sigma_0}\,R\,\|v_1-v_2\| < \frac{1}{2}\,\|v_1-v_2\|.$$

By the Banach theorem on a fixed point and the equivalence of norms $\|v\|_{C(\bar Q)}$ and $\|v\|_\sigma$ we obtain the unique solvability of the equation (10) in $M(\bar Q)$ for any $T\in(0,T_1]$.

Let $(u_1,r_{1,1},\dots,r_{n,1}),(u_2,r_{1,2},\dots,r_{n,2})\in C_{2,\beta}(\bar Q)\times\big(C[0,T]\big)^n$ be two solutions of the problem (1)–(4). Then for $u=u_1-u_2$, $r_j=r_{j,1}-r_{j,2}$ ($j\in\{1\dots,n\}$), we obtain the problem

$$
\begin{aligned}
(Lu)(x,t) = r(t)u(x,t) + \sum_{j=1}^{n}[r_{j,2}(t)\frac{\partial}{\partial x_j}u(x,t) + r_j(t)\frac{\partial}{\partial x_j}u_2(x,t)], \quad (x,t)\in Q,\\
u(x,t)=0,\ \ x\in\partial\Omega,\ t\in[0,T],\\
u(x,0)=0,\quad x\in\bar{\Omega},\\
\big(u(z,t),\varphi_j'(z_j)\big)=0,\ \ t\in[0,T],\ \ j\in\{1\dots,n\}.
\end{aligned}
\tag{13}
$$

As above, we get that u satisfies the equation

$$
\begin{aligned}
u(x,t) = \int_0^t d\tau \int_\Omega G_0(x-y,t-\tau)r(\tau)u(y,\tau)dy-\\
-\sum_{j=1}^{n}\int_0^t d\tau\int_\Omega \frac{\partial}{\partial y_j}G_0(x-y,t-\tau)\big[r_{j,2}(t)u(y,\tau)+r_j(t)u_2(y,\tau)\big]dy,\ \ (x,t)\in\bar{Q}.
\end{aligned}
\tag{14}
$$

By using the overdetermination conditions, from the equation of the problem (13), we find

$$
r_j(t) = \frac{1}{\Phi_j(t)}\Big[r(t)\big(u(z,t),\varphi_j(z_j)\big)+\big(u(z,t),\varphi_j''(z_j)\big)-{}^c D_\tau^\beta\big(u(z,\tau),\varphi_j(z_j)\big)\Big], \tag{15}
$$

$j\in\{1,\dots,n\}$. In assumptions, $r_j\in C[0,T],\ j\in\{1,\dots,n\}$. After substituting (15) in (14) the last one takes the form

$$
\begin{aligned}
u(x,t) = \int_0^t d\tau\int_\Omega G_0(x-y,t-\tau)r(\tau)u(y,\tau)dy-\\
-\int_0^t d\tau\int_\Omega \frac{\partial}{\partial y_k}G_0(x-y,t-\tau)\sum_{k=1}^{n}\frac{1}{\Phi_k(\tau)}\Big\{r_{k2}(t)u(y,\tau)+\Big[r(\tau)\big(u(z,\tau),\varphi_k(z_k)\big)-\\
-{}^c D_\tau^\beta\big(u(z,\tau),\varphi_k(z_k)\big)+\big(u(z,\tau),\varphi_k''(z_k)\big)\Big]u_2(y,\tau)\Big\}dy,\ \ (x,t)\in\bar{Q}.
\end{aligned}
$$

We obtain the linear homogeneous integro-differential equation of the second kind that, by proven above, has the unique solution $u=0$ in $M(\bar{Q})$. Then (15) implies $r_j(t)=0$, $t\in[0,T]$, $j\in 1,\dots,n$. □

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ВИЗНАЧЕННЯ КОЕФІЦІЄНТІВ ПЕРШОГО ПОРЯДКУ В РІВНЯННІ ДИФУЗІЇ З ДРОБОВОЮ ПОХІДНОЮ ЗА ЧАСОМ

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Досліджуємо обернену задачу про визначення класичного розв'язку першої крайової задачі для рівняння дифузії з дробовою похідною за часом та залежних від часу неперервних коефіцієнтів при похідних першого порядку у рівнянні. Використовуємо інтегральні за просторовими змінними умови перевизначення. Знаходимо достатні умови локальної за часом однозначної розв'язності оберненої задачі.

Ключові слова: дробова похідна, обернена задача, інтегральна умова, вектор-функція Гріна