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ON SOME ANALOGUE OF THE SHAH EQUATION FOR ANALYTIC IN THE UNIT DISK FUNCTIONS

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For a function of the form $f(z) = F(1/(1-z))$, analytic in the unit disk, where F is an entire transcendental function, satisfying the differential equation $(1-z)^4w'' + (\xi_1(1-z)^3 + \xi_0(1-z)^2)w' + (\psi_2(1-z)^2 + \psi_1(1-z) + \psi_0)w = 0$, the boundedness of the l-index, starlikeness, convexity, close-to-convexity and growth are investigated.

Key words: differential equation, analytical function, l-index, growth, starlikeness, convexity, close-to-convexity.

1. **Introduction.**

An analytic univalent in $\mathbb{D} = \{z\colon |z| < 1\}$ function

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \tag{1}$$

is said to be convex if $f(\mathbb{D})$ is a convex domain. It is well known ([1, p.203]) that the condition $\mathrm{Re}\,\{1 + zf''(z)/f'(z)\} > 0$ $(z \in \mathbb{D})$ is necessary and sufficient for the convexity of f. By W. Kaplan ([2]) the function f is said to be close-to-convex in $\mathbb{D}$ (see also [1, p. 583]) if there exists a convex in $\mathbb{D}$ function Φ such that $\mathrm{Re}\,(f'(z)/\Phi'(z)) > 0\,(z \in \mathbb{D})$. Every close-to-convex in $\mathbb{D}$ function f is univalent in $\mathbb{D}$ and, therefore, $f'(0) \neq 0$. Hence it follows that the function f is close-to-convex in $\mathbb{D}$ if and only if the function

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n, \quad g_n = f_n/f_1, \tag{2}$$

is close-to-convex in $\mathbb{D}$. We remark also, that the function (2) is said to be starlike in $\mathbb{D}$, if $g(\mathbb{D})$ is starlike domain regarding the origin. The condition $\mathrm{Re}\,\{1 + zg'(z)/g(z)\} > 0$

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($z \in \mathbb{D}$) is necessary and sufficient for the starlikeness of g. It is clear, that every starlike function is close-to-convex.

Let l be a positive continuous function on $[0, 1)$ such that $l(r) > \beta/(1-r)$, $\beta = \text{const} > 1$ for all $r \in [0, 1)$. An analytic in $\mathbb{D}$ function f is said [3] to be of bounded l-index if there exists $N \in \mathbb{Z}_+$ such that for all $n \in \mathbb{Z}_+$ and $z \in \mathbb{D}$

$$\frac{|f^{(n)}(z)|}{n!l^n(|z|)} \le \max\left\{\frac{|f^{(k)}(z)|}{k!l^k(|z|)} : 0 \le k \le N\right\}.$$

The least such integers is called the l-index of f and is denoted by $N(l, f; \mathbb{D})$.

S.M. Shah [4] indicated conditions on real parameters β_0, β_1, γ_0, γ_1, γ_2 of the differential equation

$$z^2 w'' + (\beta_0 z^2 + \beta_1 z)w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = 0, \tag{3}$$

under which there exists an entire transcendental solution (1) such that f and all its derivatives are close-to-convex in $\mathbb{D}$. In [5], three analogues of the Shah equation are considered, which have analytical solutions in the disk $\mathbb{D}$. Among them there is the equation

$$(1-z)^3 w'' + a(1-z)w' + bw = 0. \tag{4}$$

It has been proven ([5]) that if $a < 0$, $b < 0$ and $a + b \ge -3$ then equation (4) has an analytic in $\mathbb{D}$ solution $f(z) = F(1/(1-z))$, where the function

$$F(t) = t + \sum_{n=2}^{\infty} F_n t^n$$

is entire transcendental and close-to-convex in $\mathbb{D}$. This solution is a function of bounded l-index $N(l, f; \mathbb{D}) \le 1$ with $l(|z|) = 4/(1-|z|)^2$. Remark that there was formulated a conjecture ([6]) that for an entire function f the function $H(z) = f(1/(1-z)^n)$, $n \in \mathbb{N}$, is of bounded l-index in $\mathbb{D}$ with $l(|z|) = \beta/(1-|z|)^{n+1}, \beta > 1$, if and only if f is of bounded index. Later it was completely proved in [7].

In this article we will focus on the differential equation

$$(1-z)^4 w'' + (\xi_1(1-z)^3 + \xi_0(1-z)^2)w' + (\psi_2(1-z)^2 + \psi_1(1-z) + \psi_0)w = 0. \tag{5}$$

2. l-Index boundedness.

Suppose that an analytic in $\mathbb{D}$ function f is a solution of equation (5). Then

$$(1-z)^4 f''(z) + (\xi_1(1-z)^3 + \xi_0(1-z)^2)f'(z) + (\psi_2(1-z)^2 + \psi_1(1-z) + \psi_0)f(z) \equiv 0. \tag{6}$$

Put

$$l(|z|) = \frac{A}{(1-|z|)^2}, \quad A \ge 6.$$

Then from (6) we obtain

$$\frac{|f''(z)|}{2!l^2(|z|)} \le \frac{1}{2l(|z|)}\left(\frac{|\xi_1|}{|1-z|} + \frac{|\xi_0|}{|1-z|^2}\right)\frac{|f'(z)|}{1!l(|z|)} +$$

$$+\frac{1}{2l^2(|z|)}\left(\frac{|\psi_2|}{|1-z|^2} + \frac{|\psi_1|}{|1-z|^3} + \frac{|\psi_0|}{|1-z|^4}\right)|f(z)| \le$$

$$\le \frac{(1-|z|)^2}{2A}\frac{|\xi_1|+|\xi_0|}{(1-|z|)^2}\frac{|f'(z)|}{1!l(|z|)}+\frac{(1-|z|)^4}{2A^2}\frac{|\psi_2|+|\psi_1|+|\psi_0|}{(1-|z|)^4}|f(z)|\le$$
$$\le\left(\frac{|\xi_1|+|\xi_0|}{2A}+\frac{|\psi_2|+|\psi_1|+|\psi_0|}{2A^2}\right)\max\left\{\frac{|f'(z)|}{1!l(|z|)},\,|f(z)|\right\}\le$$
$$\le\frac{6(|\xi_1|+|\xi_0|)+|\psi_2|+|\psi_1|+|\psi_0|}{12A}\max\left\{\frac{|f'(z)|}{1!l(|z|)},\,|f(z)|\right\}\le\max\left\{\frac{|f'(z)|}{1!l(|z|)},\,|f(z)|\right\}, \tag{7}$$

provided

$$A\ge\frac{|\xi_1|+|\xi_0|}{2}+\frac{|\psi_2|+|\psi_1|+|\psi_0|}{12}. \tag{8}$$

Differentiating (6) we obtain

$$(1-z)^4f'''(z)+((\xi_1-4)(1-z)^3+\xi_0(1-z)^2)f''(z)+$$
$$+((\psi_2-3\xi_1)(1-z)^2+(\psi_1-2\xi_0)(1-z)+\psi_0)f'(z)-(2\psi_2(1-z)+\psi_1)f(z)\equiv 0,$$

whence as above

$$\frac{|f'''(z)|}{3!l^3(|z|)}\le\frac{1}{3l(|z|)}\left(\frac{|\xi_1-4|}{1-|z|}+\frac{|\xi_0|}{(1-|z|)^2}\right)\frac{|f''(z)|}{2!l^2(|z|)}+$$
$$+\frac{1}{6l^2(|z|)}\left(\frac{|\psi_2-3\xi_1|}{(1-|z|)^2}+\frac{|\psi_1-2\xi_0|}{(1-|z|)^3}+\frac{|\psi_0|}{(1-|z|)^4}\right)\frac{|f'(z)|}{1!l(|z|)}+$$
$$+\frac{1}{6l^3(|z|)}\left(\frac{2|\psi_2|}{(1-|z|)^3}+\frac{|\psi_1|}{(1-|z|)^4}\right)|f(z)|\le$$
$$\le\frac{1}{3l(|z|)}\frac{|\xi_1-4|+|\xi_0|}{(1-|z|)^2}\frac{|f''(z)|}{2!l^2(|z|)}+\frac{1}{6l^2(|z|)}\frac{|\psi_2-3\xi_1|+|\psi_1-2\xi_0|+|\psi_0|}{(1-|z|)^4}\frac{|f'(z)|}{1!l(|z|)}+$$
$$+\frac{1}{6l^3(|z|)}\frac{2|\psi_2|+|\psi_1|}{(1-|z|)^4}|f(z)|=$$
$$=\frac{|\xi_1-4|+|\xi_0|}{3A}\frac{|f''(z)|}{2!l^2(|z|)}+\frac{|\psi_2-3\xi_1|+|\psi_1-2\xi_0|+|\psi_0|}{6A^2}\frac{|f'(z)|}{1!l(|z|)}+$$
$$+\frac{(2|\psi_2|+|\psi_1|)(1-|z|)^2}{6A^3}|f(z)|\le$$
$$\le\frac{|\xi_1-4|+|\xi_0|}{3A}\frac{|f''(z)|}{2!l^2(|z|)}+\frac{|\psi_2-3\xi_1|+|\psi_1-2\xi_0|+|\psi_0|}{36A}\frac{|f'(z)|}{1!l(|z|)}+\frac{2|\psi_2|+|\psi_1|}{216A}|f(z)|\le$$
$$\le\left(\frac{|\xi_1|+|\xi_0|}{2}+\frac{|\psi_2|+|\psi_1|+|\psi_0|}{12}+\frac{4}{3}\right)\frac{1}{A}\max\left\{\frac{|f''(z)|}{2!l^2(|z|)},\,\frac{|f'(z)|}{1!l(|z|)},\,|f(z)|\right\}\le$$
$$\le\max\left\{\frac{|f''(z)|}{2!l^2(|z|)},\,\frac{|f'(z)|}{1!l(|z|)},\,|f(z)|\right\}, \tag{9}$$

provided

$$A\ge\frac{|\xi_1|+|\xi_0|}{2}+\frac{|\psi_2|+|\psi_1|+|\psi_0|}{12}+\frac{4}{3}. \tag{10}$$

Now let $j\ge 2$. Since

$$((1-z)^4f''(z))^{(j)}=(1-z)^4f^{(j+2)}(z)-4j(1-z)^3f^{(j+1)}(z)+6j(j-1)(1-z)^2f^{(j)}(z)-$$
$$-4j(j-1)(j-2)(1-z)f^{(j-1)}(z)+j(j-1)(j-2)(j-3)f^{(j-2)}(z)$$

and

$$((\xi_1(1-z)^3+\xi_0(1-z)^2)f'(z))^{(j)}=(\xi_1(1-z)^3+\xi_0(1-z)^2)f^{(j+1)}(z)-$$
$$-j(3\xi_1(1-z)^2+2\xi_0(1-z))f^{(j)}(z)+j(j-1)(3\xi_1(1-z)+\xi_0)f^{(j-1)}(z)-$$
$$-j(j-1)(j-2)\xi_1 f^{(j-2)}(z)$$

and

$$((\psi_2(1-z)^2+\psi_1(1-z)+\psi_0)f(z))^{(j)}=(\psi_2(1-z)^2+\psi_1(1-z)+\psi_0)f^{(j)}(z)-$$
$$-j(2\psi_2(1-z)+\psi_1)f^{(j-1)}(z)+j(j-1)\psi_2 f^{(j-2)}(z),$$

from (6) we obtain

$$(1-z)^4 f^{(j+2)}(z)-((4j-\xi_1)(1-z)^3-\xi_0(1-z)^2)f^{(j+1)}(z)+$$
$$+((6j(j-1)-3j\xi_1+\psi_2)(1-z)^2-(2j\xi_0-\psi_1)(1-z)+\psi_0)f^{(j)}(z)+$$
$$+j(-4(j-1)(j-2)+3(j-1)\xi_1-2\psi_2)(1-z)+(j-1)\xi_0-\psi_1)f^{(j-1)}(z)+$$
$$+j(j-1)((j-2)(j-3)-(j-2)\xi_1+\psi_2)f^{(j-2)}(z)=0,$$

whence as above we get

$$\frac{|f^{(j+2)}(z)|}{(j+2)!l^{j+2}(|z|)}\le\frac{4j+|\xi_1|+|\xi_0|}{(j+2)l(|z|)(1-|z|)^2}\frac{|f^{(j+1)}(z)|}{(j+1)!l^{j+1}(|z|)}+$$
$$+\frac{6j(j-1)+3j|\xi_1|+|\psi_2|+2j|\xi_0|+|\psi_1|+|\psi_0|}{(j+2)(j+1)l^2(|z|)(1-|z|)^4}\frac{|f^{(j)}(z)|}{j!l^j(|z|)}+$$
$$+\frac{4(j-1)(j-2)+3(j-1)|\xi_1|+2|\psi_2|+(j-1)|\xi_0|+|\psi_1|}{(j+2)(j+1)l^3(|z|)(1-|z|)^4}\frac{|f^{(j-1)}(z)|}{(j-1)!l^{j-1}(|z|)}+$$
$$+\frac{j(j-1)((j-2)(j-3)+(j-2)|\xi_1|+|\psi_2|)}{(j+2)(j+1)j(j-1)l^4(|z|)(1-|z|)^4}\frac{|f^{(j-2)}(z)|}{(j-2)!l^{j-2}(|z|)}\le$$
$$\le\frac{16+|\xi_1|+|\xi_0|}{4A}\frac{|f^{(j+1)}(z)|}{(j+1)!l^{j+1}(|z|)}+$$
$$+\frac{72+2(3|\xi_1|+2|\xi_0|)+|\psi_0|+|\psi_1|+|\psi_2|}{12A^2}\frac{|f^{(j)}(z)|}{j!l^j(|z|)}+$$
$$+\frac{48+12(3|\xi_1|+|\xi_0|)/10+2|\psi_2|+|\psi_1|}{12A^3}\frac{|f^{(j-1)}(z)|}{(j-1)!l^{j-1}(|z|)}+$$
$$+\frac{84+6|\xi_1|+7|\psi_2|}{84A^4}\frac{|f^{(j-2)}(z)|}{(j-2)!l^{j-2}(|z|)}\le$$
$$\le\left(\frac{|\xi_1|+|\xi_0|}{2}+\frac{|\psi_2|+|\psi_1|+|\psi_0|}{12}+6\right)\frac{1}{A}\times$$
$$\times\max\left\{\frac{|f^{(j+1)}(z)|}{(j+1)!l^{j+1}(|z|)},\frac{|f^{(j)}(z)|}{j!l^j(|z|)},\frac{|f^{(j-1)}(z)|}{(j-1)!l^{j-1}(|z|)},\frac{|f^{(j-2)}(z)|}{(j-2)!l^{j-2}(|z|)}\right\}\le$$
$$\le\max\left\{\frac{|f^{(j+1)}(z)|}{(j+1)!l^{j+1}(|z|)},\frac{|f^{(j)}(z)|}{j!l^j(|z|)},\frac{|f^{(j-1)}(z)|}{(j-1)!l^{j-1}(|z|)},\frac{|f^{(j-2)}(z)|}{(j-2)!l^{j-2}(|z|)}\right\}\qquad(11)$$

provided

$$A\ge\frac{|\xi_1|+|\xi_0|}{2}+\frac{|\psi_2|+|\psi_1|+|\psi_0|}{12}+6.\qquad(12)$$

If we put

$$A = \frac{|\xi_1| + |\xi_0|}{2} + \frac{|\psi_2| + |\psi_1| + |\psi_0|}{12} + 6$$

then (8), (10) and (12) hold and from (7), (9) and (11) we obtain the following theorem.

Theorem 1. *If an analytic in $\mathbb{D}$ function f is a solution of equation* (5) *with complex parameters $\xi_1, \xi_0, \psi_2, \psi_1, \psi_0$, then it is of the bounded l-index $N(l, f; \mathbb{D}) \le 1$ with $l(|z|) = \dfrac{A}{(1-|z|)^2}$, $A = \dfrac{|\xi_1| + |\xi_0|}{2} + \dfrac{|\psi_2| + |\psi_1| + |\psi_0|}{12} + 6$.*

3. **Growth and geometric properties.**

Let $M_f(r) = \max\{|f(z)| \colon |z| = r\}$. If f is a function of the bounded l-index then [3, p. 71] $\ln M_f(r) = O\left(\int_0^r l(t)dt\right)$ as $r \uparrow 1$. Therefore, if an analytic in $\mathbb{D}$ function f is a solution of equation (5) then $\ln M_f(r) = O\left(1/(1-r)\right)$ as $r \uparrow 1$. The last relation can be refined by considering special cases of equation (5).

As in [5], we will look for a solution of the equation (5) in the form

$$f(z) = F\left(\frac{1}{1-z}\right), \tag{13}$$

where F is an entire transcendental function and $F'(0) \ne 0$. The class of such functions we denote by $\mathfrak{S}$. If $f \in \mathfrak{S}$ then

$$f'(z) = F'\left(\frac{1}{1-z}\right)\frac{1}{(1-z)^2},\ f''(z) = F''\left(\frac{1}{1-z}\right)\frac{1}{(1-z)^4} + F'\left(\frac{1}{1-z}\right)\frac{2}{(1-z)^3} \tag{14}$$

Substituting (13) and (14) into (5) we get

$$F''\left(\frac{1}{1-z}\right) + 2(1-z)F'\left(\frac{1}{1-z}\right) + (\xi_1(1-z) + \xi_0)F'\left(\frac{1}{1-z}\right) +$$

$$+(\psi_2(1-z)^2 + \psi_1(1-z) + \psi_0)F\left(\frac{1}{1-z}\right) \equiv 0.$$

If we put $t = \frac{1}{1-z}$ then from hence we obtain

$$F''(t) + 2F'(t)/t + (\xi_1/t + \xi_0)F'(t) + (\psi_2/t^2 + \psi_1/t + \psi_0)F(t) \equiv 0,$$

i. e. $t^2F''(t) + ((\xi_1+2)t + \xi_0 t^2)F'(t) + (\psi_2 + \psi_1 t + \psi_0 t^2)F(t) \equiv 0$ and, thus, F is solution of the differential equation

$$t^2 w'' + (\xi_0 t^2 + (\xi_1 + 2)t)w' + (\psi_0 t^2 + \psi_1 t + \psi_2)w = 0,$$

which coincides with the Shah equation (3) if $\beta_0 = \xi_0$, $\beta_1 = \xi_1 + 2$ and $\gamma_j = \psi_j$ for $j = 0, 1, 2$. Therefore, using previously proven results on the properties of solutions of equation (3), we can obtain the corresponding statements for solutions of equation (5). For example, in [4] (see also [8, p. 62]) the following statement has been proven.

Lemma 1. *Let $\beta_1 > 0$, $\gamma_0 = 0$, $-1 \le \beta_0 < 0$. If either $\gamma_2 = 0$ and $-\beta_1 \le \gamma_1 < 0$, or $\beta_1 + \gamma_2 = 0$ and $-\beta_1/2 \le \gamma_1 < 0$ then differential equation* (3) *has an entire solution*

$$\varphi(z) = \sum_{n=0}^{\infty} \varphi_n z^n$$

such that $\varphi, \varphi', \varphi'', \ldots$ *are all close-to-convex in* $\mathbb{D}$ *and* $\ln M_\varphi(r) \sim |\beta_0| r$ *as* $r \to +\infty$.

It's easy to check that Lemma 1 implies the following result.

Theorem 2. *Let* $\xi_1 > -2$, $\psi_0 = 0$, $-1 \le \xi_0 < 0$. *If either* $\psi_2 = 0$ *and* $-(\xi_1+2) \le \psi_1 < 0$, *or* $\xi_1 + 2 + \psi_2 = 0$ *and* $-(\xi_1/2+1) \le \psi_1 < 0$ *then differential equation* (5) *has solution* f *of the form* (13) *that belong to the class* $\mathfrak{S}$, *where the function*

$$F(t) = \sum_{k=0}^{\infty} F_k t^k$$

is close-to-convex in $\mathbb{D}$ *along with all its derivatives and* $\ln M_F(\varrho) \sim |\xi_0| \varrho$ *as* $\varrho \to +\infty$.

Likewise the results obtained in articles [9], [10], [11], [12], [13] and summarized in [8, pp. 61-83] for equation (3) can be transferred to the case of equation (5).

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ПРО ОДИН АНАЛОГ РІВНЯННЯ ШАХА ДЛЯ АНАЛІТИЧНИХ В ОДИНИЧНОМУ КРУЗІ ФУНКЦІЙ

Мирослав ШЕРЕМЕТА, Юрій ТРУХАН

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Для аналітичної в одиничному крузі функції вигляду $f(z) = F(\frac{1}{1-z})$, де F — ціла трансцендентна функція, що задовольняє диференціальне рівняння $(1-z)^4 w'' + (\xi_1(1-z)^3 + \xi_0(1-z)^2)w' + (\psi_2(1-z)^2 + \psi_1(1-z) + \psi_0)w = 0$ досліджуються обмеженість l-індекса, зірковість, опуклість, близькість до опуклості та можливе зростання.

Ключові слова: диференціальне рівняння, аналітична функція, l-індекс, зірковість, опуклість, близькість до опуклості.