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THE DARBOUX PROBLEM FOR A COUNTABLE HYPERBOLIC 1D SYSTEM OF EQUATIONS

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A theorem on the existence and uniqueness of the global generalized solution to the boundary value problem for a linear countable system of first-order hyperbolic equations in a curvilinear sector is proved.

Key words: hyperbolic system, countable system, method of characteristics, Volterra integral equations, successive approximations, Darboux problem.

1. **Introduction**

The development of the theory of countable systems of ordinary differential equations began with the works [1, 2], in which theorems on the existence and uniqueness of the solution of the Cauchy problem for the system

$$\frac{dx_i}{dt} = F_i(t, x_1, x_2, \dots), \quad i \in \mathbb{N}. \tag{1}$$

were proved.

Finite-dimensional systems of ordinary differential equations

$$\frac{dx_i}{dt} = f_i(t, x_1, \dots, x_n), \quad i = 1, \dots, n, \quad n \geqslant 1, \tag{2}$$

particularly those involving a high dimension of n, were usually considered as truncations of system (1). Many works are devoted to such systems (see, for example, the bibliography in [3]).

The study of countable systems of ordinary differential equations was also carried out in [4–11].

A more detailed analysis of the development of the theory of countable systems of ordinary differential equations in the space of bounded numerical sequences, as well as a review of the relevant literature, is given in the monograph ([6]).

Theoretical and practical aspects of scientific research required the study of countable systems of partial differential equations. In [12] a countable system of quasilinear equations of the form

$$\sum_{k=1}^{\infty} a_k(x_1, x_2, \dots; z_1, z_2, \dots)\frac{\partial z_i}{\partial x_k} = f_i(x_1, x_2, \dots; z_1, z_2, \dots), \quad i \in \mathbb{N} \tag{3}$$

and its finite truncation

$$\sum_{l=1}^{n} a_l(x_1, x_2, \dots, x_n, 0, 0, \dots; z_1, z_2, \dots, z_m, 0, 0, \dots)\frac{\partial z_\nu}{\partial x_l} =$$
$$= f_\nu(x_1, x_2, \dots, x_n, 0, 0, \dots; z_1, z_2, \dots, z_m, 0, 0, \dots), \quad \nu = 1, 2, \dots, m, \tag{4}$$

were considered, and it was shown that

$$\lim_{m\to\infty} \nu_{km} = z_k,$$

where, $\nu_{km}, m \in \mathbb{N}$ are the solutions of the truncated system (4), and z_k is a solution of system (3).

The problem of truncating countable hyperbolic systems of partial differential equations with initial and boundary conditions was studied in [12–15]. In [15] the theory of countable systems of partial differential equations and a truncation method were applied to the investigation of infinite systems of wave equations by means of the averaging method. The key point of this technique is the assumption of the possibility of expanding a certain class of functions into Fourier series with respect to the eigenfunctions of the corresponding boundary value problem.

In [16] countable systems of stochastic differential equations modeling the motion of interacting particles in a random medium were studied.

In many cases, searching for solutions of applied problems for partial differential equations in the form of Fourier series leads to countable systems of integral equations for determining the Fourier coefficients ([17–19]).

Various versions of problems for countable partial differential equations were considered, for example, in [20–23]. To the best of the authors' knowledge, there are no recent publications addressing problems for countable hyperbolic systems of equations.

The subject of this paper is the boundary value problem for a countable hyperbolic system of first-order equations in a domain with an initial line degenerated to a point. Such problems for hyperbolic equations and systems are used in many applied settings and are known as Darboux problems ([24, 25]).

The proofs in this paper are carried out by the method of characteristics, which makes it possible to reduce the original problem to a system of countable linear integral equations. The global solvability of this system is established by the Banach fixed-point theorem using weighted norms and the techniques developed in [26].

2. **Statement of the problem**

Let G be a curvilinear sector in the upper half-plane $t > 0$ of the plane xOt, bounded by the curves given by the equations

$$x = a(t) \quad \text{i} \quad x = b(t), \quad a(0) = b(0) = 0, \quad b(t) > a(t),$$

for all $t > 0$. The functions a and $b \in C^1(\mathbb{R}_+)$, $(\mathbb{R}_+ = [0, \infty))$.

In G, we consider a countable hyperbolic system of linear first-order differential equations

$$\frac{\partial u_i}{\partial t} + \lambda_i(x,t)\frac{\partial u_i}{\partial x} = \sum_{j=1}^{\infty} c_{ij}(x,t)u_j(x,t) + f_i(x,t), \quad i \in \mathbb{N}. \tag{5}$$

Let $I^+ = \{2k-1\colon k \in \mathbb{N}\}$, a $I^- = \mathbb{N} \setminus I^+$ and assume that λ_i are ordered at each point of G as follows:

$$\lambda_i(a(t),t) - a^{'}(t) > 0, \quad i \in \overline{1, 2k-1},$$
$$\lambda_i(b(t),t) - b^{'}(t) < 0, \quad i \in \overline{2, 2k}.$$

For system (5), for all $t \in \mathbb{R}_+$, we prescribe the boundary conditions

$$u_i(a(t),t) = \sum_{j \in I^-} \alpha_{ij}(t)u_j(a(t),t) + h_i(t), \quad i \in I^+, \tag{6}$$

$$u_i(b(t),t) = \sum_{j \in I^+} \beta_{ij}(t)u_j(b(t),t) + r_i(t), \quad i \in I^-. \tag{7}$$

We consider problem (5)–(7) in the space $C^{(\infty)}(\overline{G})$, whose elements are countable sets of continuous, uniformly bounded functions [14] defined on $\overline{G}$, with the norm for the vector $u(x,t) = (u_1(x,t), u_2(x,t), \dots)$ given by

$$\|u\| = \sup_{i \in \mathbb{N},\, (x,t) \in \overline{G}} \max \{|u_i(x,t)|\}.$$

Let $\varphi_i(\tau; x, t)$ denote the solution of the Cauchy problem

$$\frac{d\xi}{d\tau} = \lambda_i(\xi, \tau), \quad i \in \mathbb{N}, \quad \xi|_{\tau=t} = x. \tag{8}$$

The solution of problem (8) is called the characteristic of system (5).

Let $L_i(x,t)$ be the integral curve defined by $\xi = \varphi_i(\tau; x, t)$ emanating from $(x,t) \in \overline{G}$, and let $t_i(x,t)$ be the ordinate of the intersection point of the i-th characteristic with $x = a(t)$ or $x = b(t)$ in the direction of decreasing t.

Integrating each equation of system (5) along the corresponding characteristics (see Fig. 1) we obtain the system of integro-functional equations

$$u_i(x,t) = \omega_i[u](x,t) + \int\limits_{t_i(x,t)}^{t} \Big(\sum_{j=1}^{\infty} c_{ij}u_j + f_i\Big)[(\varphi_i(\tau; x,t), \tau)]d\tau, \quad i \in \mathbb{N}, \tag{9}$$

where

$$\omega_i[u](x,t) = \begin{cases} u_i\big(a(t_i(x,t), t_i(x,t)\big), & \text{if } \varphi_i(t_i(x,t); x,t) = a(t_i(x,t)); \\ u_i\big(b(t_i(x,t), t_i(x,t)\big), & \text{if } \varphi_i(t_i(x,t); x,t) = b(t_i(x,t)). \end{cases} \tag{10}$$

We denote by $\mathfrak{M}$ the space of bounded real numerical sequences ([6]).

Definition 1. A continuous function $u\colon \overline{G} \to \mathfrak{M}$ that satisfies the system of integro-functional equations (9)–(10) is called a generalized solution of problem (5)–(7).

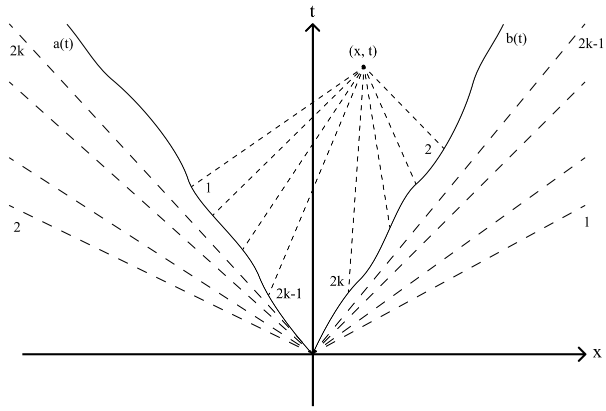

FIG. 1. Distribution of the characteristics of system (5) in G

For an arbitrary point $(x,t) \in G$, $s \in \mathbb{N}$, and according to the distribution of characteristics from (9)–(10), we obtain

$$u_{2k-1}(x,t) = u_{2k-1}(a(t_{2k-1}(x,t)), t_{2k-1}(x,t)) +$$
$$+ \int\limits_{t_{2k-1}(x,t)}^{t} \Big(\sum_{j=1}^{\infty} c_{2k-1,j} u_j + f_{2k-1}\Big)[(\varphi_{2k-1}(\tau; x,t),\tau)]d\tau, \quad k = 1,2,\dots,s, \tag{11}$$
$$u_{2k}(x,t) = u_{2k}(b(t_{2k}(x,t)), t_{2k}(x,t)) +$$
$$+ \int\limits_{t_{2k}(x,t)}^{t} \Big(\sum_{j=1}^{\infty} c_{2k,j} u_j + f_{2k}\Big)[(\varphi_{2k}(\tau; x,t),\tau)]d\tau, \quad k = s+1, s+2, \dots.$$

Considering the first system of (11) and using the boundary conditions on the left side of G, we obtain

$$u_{2k-1}(x,t) = \sum_{j=1}^{\infty} \alpha_{2k-1,2j}(t_{2k-1}(x,t)) u_{2j}(a(t_{2k-1}(x,t)), t_{2k-1}(x,t)) +$$
$$+ \int\limits_{t_{2k-1}(x,t)}^{t} \Big(\sum_{j=1}^{\infty} c_{2k-1,j} u_j + f_{2k-1}\Big)[(\varphi_{2k-1}(\tau; x,t),\tau)]d\tau + h_{2k-1}(t_{2k-1}(x,t)),$$
$$k = 1,2,\dots,s. \tag{12}$$

In turn, the second system of (11), combined with the boundary conditions on the right side of G, yields

$$u_{2k}(x,t) = \sum_{j=1}^{\infty} \beta_{2k,2j-1}(t_{2k}(x,t)) u_{2j-1}(b(t_{2k}(x,t)), t_{2k}(x,t)) +$$

$$+\int\limits_{t_{2k}(x,t)}^{t}\Big(\sum_{j=1}^{\infty}c_{2k,j}u_j+f_{2k}\Big)[(\varphi_{2k}(\tau;x,t),\tau)]d\tau+r_{2k}(t_{2k}(x,t)),$$
$$k=s+1,s+2,\dots. \tag{13}$$

Thus, in the domain G, the original problem is reduced to the system of integro-functional equations (12)–(13).

3. **Existence and uniqueness of problem solution**

Theorem 1. *Suppose the input data of problem* (5)–(7) *satisfy the following conditions for all* $i,j\in\mathbb{N}$*:*

(1) $\lambda_i\in C^{(\infty)}(\overline{G})\bigcap \mathrm{Lip}_x(\overline{G})$;

(2) *the functions*

$$c_{ij}(x,t),\ c_i\equiv\sum_{j=1}^{\infty}|c_{ij}(x,t)|,\ f_i(x,t),\ (x,t)\in\overline{G},$$
$$\alpha_{2i-1,2j}(t),\ \alpha_{2i-1}\equiv\sum_{j=1}^{\infty}|\alpha_{2i-1,2j}(t)|,\ h_{2i-1}(t),\ t\in\mathbb{R}_+,$$
$$\beta_{2i,2j-1}(t),\ \beta_{2i}\equiv\sum_{j=1}^{\infty}|\beta_{2i,2j-1}(t)|,\ r_{2i}(t),\ t\in\mathbb{R}_+$$

are continuous;

(3) *the following inequalities hold:*

$$c_i(x,t)\leqslant c(x,t),\ \alpha_{2i-1}(t)\leqslant\alpha(t),\ \beta_{2i}(t)\leqslant\beta(t),$$

where $c\in C(\overline{G}),\ \alpha,\beta\in C(\mathbb{R}_+)$;

(4) *the zero-order compatibility conditions hold*

$$u_{2i-1}(0,0)=\sum_{j=1}^{\infty}\alpha_{2i-1,2j}(0)u_{2j}(0,0)+h_{2i-1}(0),\ i\in\mathbb{N},$$
$$u_{2i}(0,0)=\sum_{j=1}^{\infty}\beta_{2i,2j-1}(0)u_{2j-1}(0,0)+r_{2i}(0),\ i\in\mathbb{N},$$

where

$$|A(0)B(0)|<1\quad\textit{and}\quad\det A(0)\neq 0,$$

with

$$A=\max_{t\in\mathbb{R}_+}|\alpha(t)|,\quad B=\max_{t\in\mathbb{R}_+}|\beta(t)|.$$

Then there exists a unique generalized solution of problem (5)–(7)*.*

Proof. Let

$$C=\max_{(x,t)\in\overline{G}}|c(x,t)|.$$

We prove the existence and uniqueness of a solution to the system of integro-functional equations (12)–(13) by the method of successive approximations. First, we establish existence. The zero-order approximations are defined as

$$u_{2k-1}^{(0)}(x,t) = h_{2k-1}(t_{2k-1}(x,t)), \quad k = 1,2,\ldots,s,$$
$$u_{2k}^{(0)}(x,t) = r_{2k}(t_{2k}(x,t)), \quad k = s+1, s+2,\ldots. \tag{14}$$

Then for $m = 1,2,\ldots$ we obtain the successive approximations

$$u_{2k-1}^{(m)}(x,t) = h_{2k-1}(t_{2k-1}(x,t)) +$$
$$+ \sum_{j=1}^{\infty} \alpha_{2k-1,2j}(t_{2k-1}(x,t)) u_{2j}^{(m-1)}(a(t_{2k-1}(x,t)), t_{2k-1}(x,t)) +$$
$$+ \int\limits_{t_{2k-1}(x,t)}^{t} \left(\sum_{j=1}^{\infty} c_{2k-1,j} u_j^{(m-1)} + f_{2k-1} \right) [(\varphi_{2k-1}(\tau;x,t),\tau)] d\tau, \quad k = 1,2,\ldots,s, \tag{15}$$
$$u_{2k}^{(m)}(x,t) = r_{2k}(t_{2k}(x,t)) +$$
$$+ \sum_{j=1}^{\infty} \beta_{2k,2j-1}(t_{2k}(x,t)) u_{2j-1}^{(m-1)}(b(t_{2k}(x,t)), t_{2k}(x,t)) +$$
$$+ \int\limits_{t_{2k}(x,t)}^{t} \Big(\sum_{j=1}^{\infty} c_{2k,j} u_j^{(m-1)} + f_{2k} \Big) [(\varphi_{2k}(\tau;x,t),\tau)] d\tau, \quad k = s+1, s+2,\ldots.$$

From the theory of ordinary differential equations it is known that the functions $\varphi_i(\tau;x,t)$ are continuous in the variables (x,t) whenever $\lambda_i \in C(\overline{G}) \cap \mathrm{Lip}_x(\overline{G})$. Therefore, if the functions $u_i(x,t)$ are continuous in both variables for all $i \in \mathbb{N}$ and the conditions of Theorem 1 are satisfied, then all successive approximations (14)–(15) will be continuous functions, as they are compositions of continuous functions ([27]).

We now show that the sequence of these approximations converges uniformly in G. The sequence will be uniformly convergent if and only if the functional series

$$u_i^{(0)}(x,t) + \sum_{m=1}^{\infty} \big(u_i^{(m)}(x,t) - u_i^{(m-1)}(x,t) \big), \quad i \in \mathbb{N}, \tag{16}$$

is uniformly convergent.

We prove the convergence of series (16) by applying the Weierstrass M-test. For $i = 2k-1$, where $k = 1,2,\ldots,s$, the first successive approximation has the form

$$u_{2k-1}^{(1)}(x,t) = h_{2k-1}(t_{2k-1}(x,t)) +$$
$$+ \sum_{j=1}^{s} \alpha_{2k-1,2j-1}(t_{2k-1}(x,t)) \cdot h_{2j-1}(t_{2j-1}(a(t_{2k-1}(x,t)), t_{2k-1}(x,t))) +$$
$$+ \sum_{j=s+1}^{\infty} \alpha_{2k-1,2j}(t_{2k-1}(x,t)) \cdot r_{2j}(t_{2j}(a(t_{2k-1}(x,t)), t_{2k-1}(x,t))) +$$

$$+\int\limits_{t_{2k-1}(x,t)}^{t}\Bigg[\sum_{j=1}^{s}c_{2k-1,2j-1}(\varphi_{2k-1}(\tau;x,t,\tau))\cdot h_{2j-1}(t_{2j-1}(\varphi_{2k-1}(\tau;x,t),\tau))+$$

$$+\sum_{j=s+1}^{\infty}c_{2k-1,2j}(\varphi_{2k-1}(\tau;x,t,\tau))\cdot r_{2j}(t_{2j}(\varphi_{2k-1}(\tau;x,t),\tau))+$$

$$+f_{2k-1}(\varphi_{2k-1}(\tau;x,t,\tau))\Bigg]d\tau.$$

For $i=2k$, where $k=s+1,s+2,\ldots$, the first successive approximation takes a similar form:

$$u_{2k}^{(1)}(x,t)=r_{2k}(t_{2k}(x,t))+$$

$$+\sum_{j=1}^{s}\beta_{2k,2j-1}(t_{2k}(x,t))\cdot h_{2j-1}(t_{2j-1}(b(t_{2k}(x,t)),t_{2k}(x,t)))+$$

$$+\sum_{j=s+1}^{\infty}\beta_{2k,2j}(t_{2k}(x,t))\cdot r_{2j}(t_{2j}(b(t_{2k}(x,t)),t_{2k}(x,t)))+$$

$$+\int\limits_{t_{2k}(x,t)}^{t}\Bigg[\sum_{j=1}^{s}c_{2k,2j-1}(\varphi_{2k}(\tau;x,t,\tau))\cdot h_{2j-1}(t_{2j-1}(\varphi_{2k}(\tau;x,t),\tau))+$$

$$+\sum_{j=s+1}^{\infty}c_{2k,2j}(\varphi_{2k}(\tau;x,t,\tau))\cdot r_{2j}(t_{2j}(\varphi_{2k}(\tau;x,t),\tau))\;+\;f_{2k}(\varphi_{2k}(\tau;x,t),\tau)\Bigg]d\tau.$$

We now estimate the difference $|u_i^{(1)}(x,t)-u_i^{(0)}(x,t)|$ for all $i=1,2,\ldots$ та $(x,t)\in G$. For $i=2k-1$, where $k=1,2,\ldots,s$, we obtain

$$|u_{2k-1}^{(1)}(x,t)-u_{2k-1}^{(0)}(x,t)|\;=$$

$$=\Bigg|\sum_{j=1}^{s}\alpha_{2k-1,2j-1}(t_{2k-1}(x,t))\cdot h_{2j-1}(t_{2j-1}(a(t_{2k-1}(x,t)),t_{2k-1}(x,t)))+$$

$$+\sum_{j=s+1}^{\infty}\alpha_{2k-1,2j}(t_{2k-1}(x,t))\cdot r_{2j}(t_{2j}(a(t_{2k-1}(x,t)),t_{2k-1}(x,t)))+$$

$$+\int\limits_{t_{2k-1}(x,t)}^{t}\Bigg[\sum_{j=1}^{s}c_{2k-1,2j-1}(\varphi_{2k-1}(\tau;x,t,\tau))\cdot h_{2j-1}(t_{2j-1}(\varphi_{2k-1}(\tau;x,t),\tau))+$$

$$+\sum_{j=s+1}^{\infty}c_{2k-1,2j}(\varphi_{2k-1}(\tau;x,t,\tau))\cdot r_{2j}(t_{2j}(\varphi_{2k-1}(\tau;x,t),\tau))+$$

$$+f_{2k-1}(\varphi_{2k-1}(\tau;x,t,\tau))\Bigg]d\tau\Bigg|\leqslant$$

$$\leqslant \sum_{j=1}^{s}\Big|\alpha_{2k-1,2j-1}(t_{2k-1}(x,t))\cdot h_{2j-1}(t_{2j-1}(a(t_{2k-1}(x,t)),t_{2k-1}(x,t)))\Big| +$$

$$\sum_{j=s+1}^{\infty}\Big|\alpha_{2k-1,2j}(t_{2k-1}(x,t))\cdot r_{2j}(t_{2j}(a(t_{2k-1}(x,t)),t_{2k-1}(x,t)))\Big| +$$

$$+\int\limits_{t_{2k-1}(x,t)}^{t}\sum_{j=1}^{s}\Big|c_{2k-1,2j-1}(\varphi_{2k-1}(\tau;x,t,\tau))\cdot h_{2j-1}(t_{2j-1}(\varphi_{2k-1}(\tau;x,t),\tau))\Big|d\tau +$$

$$+\int\limits_{t_{2k-1}(x,t)}^{t}\sum_{j=s+1}^{\infty}\Big|c_{2k-1,2j}(\varphi_{2k-1}(\tau;x,t,\tau))\cdot r_{2j}(t_{2j}(\varphi_{2k-1}(\tau;x,t),\tau))\Big|d\tau +$$

$$\int\limits_{t_{2k-1}(x,t)}^{t}\big|f_{2k-1}(\varphi_{2k-1}(\tau;x,t,\tau))\big|d\tau \leqslant AH+AR+\int\limits_{t_{2k-1}(x,t)}^{t}\Big(CH+CR+F\Big)d\tau =$$

$$= A(H+R)+\big(C(H+R)+F\big)\big(t-t_{2k-1}(x,t)\big) =$$

$$= AP+(CP+F)\big(t-t_{2k-1}(x,t)\big) = AP+M\big(t-t_{2k-1}(x,t)\big).$$

Here and below we use constants, or their combinations, which bound the corresponding functions:

$$H=\max_{t\in\mathbb{R}_+}|h(t)|,\quad R=\max_{t\in\mathbb{R}_+}|r(t)|,\quad F=\max_{(x,t)\in\overline{G}}|f(x,t)|,\quad P=H+R,\quad M=CP+F.$$

For $i=2k$, де $k=s+1,s+2,\ldots$, we obtain a similar estimate

$$|u_{2k}^{(1)}(x,t)-u_{2k}^{(0)}(x,t)| =$$

$$= \Bigg|\sum_{j=1}^{s}\beta_{2k,2j-1}(t_{2k}(x,t))\cdot h_{2j-1}(t_{2j-1}(b(t_{2k}(x,t)),t_{2k}(x,t))) +$$

$$+\sum_{j=s+1}^{\infty}\beta_{2k,2j}(t_{2k}(x,t))\cdot r_{2j}(t_{2j}(b(t_{2k}(x,t)),t_{2k}(x,t))) +$$

$$+\int\limits_{t_{2k}(x,t)}^{t}\Bigg[\sum_{j=1}^{s}c_{2k,2j-1}(\varphi_{2k}(\tau;x,t,\tau))\cdot h_{2j-1}(t_{2j-1}(\varphi_{2k}(\tau;x,t),\tau)) +$$

$$+\sum_{j=s+1}^{\infty}c_{2k,2j}(\varphi_{2k}(\tau;x,t,\tau))\cdot r_{2j}(t_{2j}(\varphi_{2k}(\tau;x,t),\tau)) +$$

$$+ f_{2k}(\varphi_{2k}(\tau;x,t),\tau)\Bigg]d\tau\Bigg| \leqslant$$

$$\leqslant \sum_{j=1}^{s}\Big|\beta_{2k,2j-1}(t_{2k}(x,t))\cdot h_{2j-1}(t_{2j-1}(b(t_{2k}(x,t)),t_{2k}(x,t)))\Big| +$$

$$+\sum_{j=s+1}^{\infty}\Big|\beta_{2k,2j}(t_{2k}(x,t))\cdot r_{2j}(t_{2j}(b(t_{2k}(x,t)),t_{2k}(x,t)))\Big|+$$

$$+\int\limits_{t_{2k}(x,t)}^{t}\sum_{j=1}^{s}\big|c_{2k,2j-1}(\varphi_{2k}(\tau;x,t,\tau))\cdot h_{2j-1}(t_{2j-1}(\varphi_{2k}(\tau;x,t),\tau))\big|d\tau+$$

$$+\int\limits_{t_{2k}(x,t)}^{t}\sum_{j=s+1}^{\infty}\big|c_{2k,2j}(\varphi_{2k}(\tau;x,t,\tau))\cdot r_{2j}(t_{2j}(\varphi_{2k}(\tau;x,t),\tau))\big|d\tau+$$

$$+\int\limits_{t_{2k}(x,t)}^{t}\big|f_{2k}(\varphi_{2k}(\tau;x,t,\tau))\big|d\tau \ \leqslant\ B(H+R)+\int\limits_{t_{2k}(x,t)}^{t}\Big(C(H+R)+F\Big)d\tau\ =$$

$$=B(H+R)+\big(C(H+R)+F\big)\big(t-t_{2k}(x,t)\big)\ =$$

$$=\ BP+(CP+F)\big(t-t_{2k}(x,t)\big)\ =\ BP+M\big(t-t_{2k}(x,t)\big).$$

Thus, for all $(x,t)\in G$ and for any $i=1,2,\ldots,$ we obtain the estimate

$$|u_i^{(1)}(x,t)-u_i^{(0)}(x,t)|\leqslant KP+(CP+F)(t-t_i),$$

where

$$K=\max\{A,B\},$$

and

$$t_i=\begin{cases}t_{2k-1}(x,t), & \text{if } i=2k-1,\ \text{where } k=1,2,\ldots,s;\\ t_{2k}(x,t), & \text{if } i=2k,\ \text{where } k=s+1,s+2,\ldots.\end{cases}$$

Let $m=2$. For $i=2k-1,$ where $k=1,2,\ldots s$, we obtain

$$|u_{2k-1}^{(2)}(x,t)-u_{2k-1}^{(1)}(x,t)|=\Bigg|\sum_{j=1}^{\infty}\alpha_{2k-1,2j}(t_{2k-1}(x,t))\times$$

$$\times\big(u_{2j}^{(1)}(a(t_{2k-1}(x,t)),t_{2k-1}(x,t))-u_{2j}^{(0)}(a(t_{2k-1}(x,t)),t_{2k-1}(x,t))\big)+$$

$$+\int\limits_{t_{2k-1}(x,t)}^{t}\Big(\sum_{j=1}^{\infty}c_{2k-1,j}\big(u_j^{(1)}-u_j^{(0)}\big)+f_{2k-1}\Big)[(\varphi_{2k-1}(\tau;x,t),\tau)]d\tau\Bigg|\leqslant$$

$$\leqslant A^2P+CM\int\limits_{t_{2k-1}(x,t)}^{t}\big(\tau-t_{2k-1}(x,t)\big)d\tau=A^2P+CM\frac{\big(t-t_{2k-1}(x,t)\big)^2}{2!}.$$

For $i=2k$, where $k=s+1,s+2,\ldots$ we obtain a similar estimate

$$|u_{2k}^{(2)}(x,t)-u_{2k}^{(1)}(x,t)|=\Bigg|\sum_{j=1}^{\infty}\beta_{2k,2j-1}(t_{2k}(x,t))\times$$

$$\times\big(u_{2j-1}^{(1)}(b(t_{2k}(x,t)),t_{2k}(x,t))-u_{2j-1}^{(0)}(b(t_{2k}(x,t)),t_{2k}(x,t))\big)+$$

$$+\int\limits_{t_{2k}(x,t)}^{t}\left(\sum_{j=1}^{\infty} c_{2k,j}\cdot\big(u_j^{(1)}-u_j^{(0)}\big)\right)[(\varphi_{2k}(\tau;x,t),\tau)]d\tau\Bigg|\leqslant$$

$$\leqslant B^2P+CM\int\limits_{t_{2k}(x,t)}^{t}\big(\tau-t_{2k}(x,t)\big)d\tau=B^2P+CM\frac{\big(t-t_{2k}(x,t)\big)^2}{2!}.$$

Thus, for all $(x,t)\in G$ and for any $i,m\in\mathbb{N}$, we obtain the estimate

$$|u_i^{(m)}(x,t)-u_i^{(m-1)}(x,t)|\leqslant K^mP+C^{m-1}M\frac{(t-t_i)^m}{m!}.$$

Assume that such an inequality holds for some $m=n$. We prove it for $m=n+1$. For $i=2k-1$, where $k=1,2,\ldots,s$, we obtain

$$|u_{2k-1}^{(n+1)}(x,t)-u_{2k-1}^{(n)}(x,t)|=\Bigg|\sum_{j=1}^{\infty}\alpha_{2k-1,2j}(t_{2k-1}(x,t))\times$$

$$\times\big(u_{2j}^{(n+1)}(a(t_{2k-1}(x,t)),t_{2k-1}(x,t))-u_{2j}^{(n)}(a(t_{2k-1}(x,t)),t_{2k-1}(x,t))\big)+$$

$$+\int\limits_{t_{2k-1}(x,t)}^{t}\Big(\sum_{j=1}^{\infty}c_{2k-1,j}\big(u_j^{(n+1)}-u_j^{(n)}\big)+f_{2k-1}\Big)[(\varphi_{2k-1}(\tau;x,t),\tau)]d\tau\Bigg|\leqslant$$

$$\leqslant A^{n+1}P+C^nM\int\limits_{t_{2k-1}(x,t)}^{t}\frac{\big(\tau-t_{2k-1}(x,t)\big)^n}{n!}d\tau=A^{n+1}P+C^nM\frac{\big(t-t_{2k-1}(x,t)\big)^{n+1}}{(n+1)!}.$$

For $i=2k$, where $k=s+1,s+2,\ldots$, we have

$$|u_{2k}^{(n+1)}(x,t)-u_{2k}^{(n)}(x,t)|=\Bigg|\sum_{j=1}^{\infty}\beta_{2k,2j-1}(t_{2k}(x,t))\times$$

$$\times\big(u_{2j-1}^{(n+1)}(b(t_{2k}(x,t)),t_{2k}(x,t))-u_{2j-1}^{(n)}(b(t_{2k}(x,t)),t_{2k}(x,t))\big)+$$

$$+\int\limits_{t_{2k}(x,t)}^{t}\left(\sum_{j=1}^{\infty}c_{2k,j}\cdot\big(u_j^{(n+1)}-u_j^{(n)}\big)\right)[(\varphi_{2k}(\tau;x,t),\tau)]d\tau\Bigg|\leqslant$$

$$\leqslant B^{n+1}P+C^nM\int\limits_{t_{2k}(x,t)}^{t}\frac{\big(\tau-t_{2k}(x,t)\big)^n}{n!}d\tau=B^{n+1}P+C^nM\frac{\big(t-t_{2k}(x,t)\big)^{n+1}}{(n+1)!}.$$

Thus, for all $(x,t)\in G$ and for any $i,m\in\mathbb{N}$, the following estimate holds

$$|u_i^{(m)}(x,t)-u_i^{(m-1)}(x,t)|\leqslant K^mP+C^{m-1}M\frac{(t-t_i)^m}{m!}\leqslant K^mP+C^{m-1}M\frac{t^m}{m!}.$$

Since the estimate holds for all $(x,t)\in G$ and $i=1,2,\ldots$, we obtain for the norm $\|u\|$

$$||u_i^{(m)}(x,t)-u_i^{(m-1)}(x,t)||\leqslant K^mP+C^{m-1}M\frac{t^m}{m!}.$$

Thus, by the Weierstrass M-test, the series (16) is uniformly convergent on G, which implies that the sequence of approximations under consideration is also uniformly convergent on G, that is $u_i^m(x,t) \rightrightarrows u_i(x,t), \quad \forall i \in \mathbb{N}$.

To show that the limit functions $u_i(x,t)$ provide a continuous solution of system (12)–(13), it is necessary in relations (15) to pass to the limit as $m \to \infty$. Since all functions involved in the system are continuous, such a passage is permissible.

We now prove the uniqueness of the solution. Suppose there exist two different solutions $\hat{u}$ and $\tilde{u}$. Consider their difference $u(x,t) = \hat{u}(x,t) - \tilde{u}(x,t)$. Let us denote by $U(t)$ the function

$$U(t) = \sup_{i,\, x,\, \tau \leqslant t} \{|u_i(x,\tau)|\}.$$

Since $\hat{u}$ and $\tilde{u}$ are solutions of system (12)–(13), for $u(x,t)$ we obtain:

1) for $i = 2k-1$, where $k = 1, 2, \ldots, s$,

$$u_{2k-1}(x,t) \leqslant \sum_{j=1}^{\infty} \alpha_{2k-1,2j}(t_{2k-1}(x,t)) \cdot U\big(t_{2k-1}(a(t),t)\big) + $$

$$+ \int\limits_{t_{2k-1}(x,t)}^{t} \left(\sum_{j=1}^{\infty} c_{2k-1,j}(\varphi_{2k-1}(\tau; x,t,\tau)) \cdot U(\tau) \right) d\tau;$$

2) for $i = 2k$, where $k = s+1, s+2, \ldots$,

$$u_{2k}(x,t) \leqslant \sum_{j=1}^{\infty} \beta_{2k,2j-1}(t_{2k}(x,t)) \cdot U\big(t_{2k}(b(t),t))\big) + \tag{17}$$

$$+ \int\limits_{t_{2k}(x,t)}^{t} \left(\sum_{j=1}^{\infty} c_{2k,j}(\varphi_{2k}(\tau; x,t,\tau)) \cdot U(\tau) \right) d\tau. \tag{18}$$

Taking into account inequalities (17) and the fact that $t_i(x,t) \leqslant t$, we obtain an estimate for the function $U(t)$ valid for all $i \in \mathbb{N}$

$$U(t) \ \leqslant \ \big(\tilde{M}U\big)(t) + \int\limits_0^t \tilde{C}U(\tau)d\tau,$$

where $\big(\tilde{M}U\big)(t) = A(t)B(t)\big(\tilde{P}U\big)(t)$. Here $\tilde{P}$ is the shift operator, which acts according to the formula

$$\big(\tilde{P}U\big)(t) = \begin{cases} U\big(t_{2k-1}(a(t),t\big), & k = 1,2,\ldots,s; \\ U\big(t_{2k}(b(t),t\big), & k = s+1, s+2, \ldots, \end{cases}$$

moreover $\max\limits_{0 \leqslant t \leqslant \epsilon} |(\tilde{P}U)(t)| \leqslant \max\limits_{0 \leqslant t \leqslant \epsilon} |U(t)|$. Hence, $U(t) \leqslant (1-\tilde{M})^{-1} \cdot \int_0^t \tilde{C}U(\tau)d\tau$. Therefore, by the Gronwall–Bellman lemma, we obtain that $U(t) = 0$, and hence the solution of problem (5)–(7) is unique for all $t \in [0,\epsilon]$.

Then, in the corresponding part of the sector $\overline{G} \supset \overline{G}_\epsilon$, we have the initial conditions $u_i(x,\epsilon) = g_i(x)$, $i \in \mathbb{N}$, that is, an initial value problem for which the theorem on existence and uniqueness of a generalized solution was proved in [3, 26].

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ЗАДАЧА ДАРБУ ДЛЯ ЗЛІЧЕННОЇ ГІПЕРБОЛІЧНОЇ 1D СИСТЕМИ РІВНЯНЬ

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Доведено теорему існування та єдиності глобального узагальненого розв'язку крайової задачі для лінійної зліченної системи гіперболічних рівнянь першого порядку в криволінійному секторі.

Ключові слова: гіперболічна система, зліченна система, метод характеристик, інтегральні рівняння Вольтерри, послідовні наближення, задача Дарбу.