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BOUNDARY VALUE PROBLEMS FOR ELLIPTIC EQUATIONS IN UNBOUNDED DOMAINS WITH CONDITIONS AT INFINITY

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The article investigates a boundary value problems for second-order elliptic equations given in unbounded domains. In the class of equations under consideration, in addition to linear ones, there are nonlinear ones with variable nonlinearity exponents.The existence and uniqueness of weak solutions of the studied problems are established under additional conditions on behavior of solutions and the growth of input data at infinity. A priori estimates of weak solutions of the studied problems are obtained. The study uses an analogue of the Saint-Venant principle known from mechanics and the monotonicity method.

Key words: elliptic equation, variable exponent of nonlinearity, Lebesgue space with variable exponent, Sobolev space with variable exponent, unbounded domain, Saint-Venant principle, monotonicity method.

1. **Introduction**

Let n be a natural number, and $\mathbb{R}^n$ be the linear space of ordered collections $x = (x_1, ..., x_n)$ of real numbers with a norm $|x| := (|x_1|^2 + ... + |x_n|^2)^{1/2}$. Suppose that Ω is an unbounded domain in $\mathbb{R}^n$, and $\partial\Omega$ (boundary of the domain Ω) is a piecewise-smooth surface. Let $\nu(x) = (\nu_1(x), ..., \nu_n(x))$ be a outward-pointing normal unit vector on $\partial\Omega$ in point $x \in \partial\Omega$. Suppose $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where Γ_0 is a closure of an open set on $\partial\Omega$ (in particular, $\Gamma_0 = \emptyset$ or $\Gamma_0 = \partial\Omega$), $\Gamma_1 := \partial\Omega \setminus \Gamma_0$. Denote by $Bd(\Omega)$ the set of all bounded subdomains of Ω.

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We consider the problem: *to find the function* $u(x)$, $x \in \overline{\Omega}$, *that satisfies (in some sense) the equation*

$$-\sum_{i=1}^{n} \frac{d}{dx_i}\, a_i(x,u,\nabla u) + a_0(x,u,\nabla u) = f(x), \quad x \in \Omega\,, \tag{1}$$

the boundary conditions

$$u\Big|_{\Gamma_0} = 0, \qquad\qquad \frac{\partial u}{\partial \nu_a}\Big|_{\Gamma_1} = 0, \tag{2}$$

where $a_i\colon \Omega\times\mathbb{R}^{1+n}\to\mathbb{R}$, $i=\overline{0,n}$, $f\colon\Omega\to\mathbb{R}$ *are given real-valued functions,*

$$\frac{\partial u(x)}{\partial \nu_a} := \sum_{i=1}^{n} a_i(x,u,\nabla u)\,\nu_i(x)$$

is an exterior conormal derivative of u *in point* $x\in\Gamma_1$.

Зауваження 1. An simpler example of the equations of type (1) considered here is linear elliptic equation

$$-\sum_{i,j=1}^{n} (\widehat{a}_{ij}(x)u_{x_j})_{x_i} + \sum_{j=1}^{n} \widehat{a}_j(x)u_{x_j} + \widehat{a}_0(x)u = f(x), \quad x\in\Omega, \tag{3}$$

where $\widehat{a}_{ij} = \widehat{a}_{ji} \in L_\infty(\Omega)$, $i,j=\overline{1,n}$, are functions such that for some constant $\omega>0$ and for a.e. $x\in\Omega$ we have

$$\sum_{i,j=1}^{n} \widehat{a}_{ij}(x)\eta_i\eta_j \geqslant \omega\sum_{l=1}^{n}|\eta_l|^2 \quad \forall(\eta_1,...,\eta_n)\in\mathbb{R}^n,$$

and $\widehat{a}_j\in L_\infty(\Omega)$, $j=\overline{0,n}$, $f\colon\Omega\to\mathbb{R}$ is such that $f\in L_2(\Omega')$ for all $\Omega'\in Bd(\Omega)$.

In remark 6, we have given additional conditions for the coefficients of equation (3), which together with those indicated here guarantee the existence and uniqueness of a weak solution of problem (3), (2) in some class of functions, which have corresponding behavior at infinity. □

Зауваження 2. An more complex example of the equations of type (1) considered here is nonlinear elliptic equations with variable exponents of the nonlinearity

$$-\sum_{i,j=1}^{k} (\widehat{a}_{ij}(x)u_{x_j})_{x_i} - \sum_{i=k+1}^{n} (\widehat{a}_i(x)|u_{x_i}|^{p_i(x)-2}u_{x_i})_{x_i} + \widehat{a}_0(x)u = f(x), \quad x\in\Omega, \tag{4}$$

where $k\in\{1,...,n-1\}$, and Ω such that $\Omega\cap\{x=(x_1,...,x_k,x_{k+1},...,x_n)\in\mathbb{R}^n \,\big|\, |x_1|^2+...+|x_k|^2<\tau^2\}$ is bounded for each $\tau>0$; for example, $\Omega=\Omega_1\times\Omega_2$, Ω_1 is an unbounded domain in space $\{(x_1,...,x_k)\,\big|\,x_1,...,x_k\in\mathbb{R}\}$, and Ω_2 is a bounded domain in space $\{(x_{k+1},...,x_n)\,\big|\,x_{k+1},...,x_n\in\mathbb{R}\}$. Also we suppose that 1) $\widehat{a}_{ij}=\widehat{a}_{ji}\in L_\infty(\Omega)$, $i,j=\overline{1,k}$, are functions such that for a.e. $x\in\Omega$ a quadratic form

$$\sum_{i,j=1}^{k} \widehat{a}_{ij}(x)\eta_i\eta_j,\ (\eta_1,...,\eta_k)\in\mathbb{R}^k,$$

is positive, 2) for every $i \in \{0, k+1, ..., n\}$ a function $\widehat{a}_i \colon \Omega \to \mathbb{R}$ is measurable, and

$$0 < \operatorname*{ess\,inf}_{\Omega'} \widehat{a}_i \leqslant \operatorname*{ess\,sup}_{\Omega'} \widehat{a}_i < +\infty$$

for all $\Omega' \in Bd(\Omega)$, 3) for every $i \in \{k+1, ..., n\}$ a function $p_i \colon \Omega \to \mathbb{R}$ is measurable, and

$$1 < \operatorname*{ess\,inf}_{\Omega'} p_i \leqslant \operatorname*{ess\,sup}_{\Omega'} p_i < +\infty$$

for all $\Omega' \in Bd(\Omega)$ (the functions $p_i, i = \overline{k+1, n}$, are called *exponents of the nonlinearity*).

In remark 7, we have given additional conditions for the coefficients of equation (4), which together with those indicated here guarantee the existence and uniqueness of a weak solution of problem (4), (2) in some class of functions, which have corresponding behavior at infinity. □

Boundary value problems for elliptic equations in unbounded domains were studied by many authors. Well known, to guarantee the uniqueness of the solution of these problems for linear equations (3) we need some restrictions on solution's behavior as $|x| \to +\infty$, for example, solution's growth restriction as $|x| \to +\infty$, or belonging of solution to some functional spaces. For example, let $\Omega := \{x \in \mathbb{R}^3 \,\big|\, |x| > R\}$, where $R > 0$ is arbitrary fix number, $\varphi \in C(\partial\Omega)$. Then (see, for example, [1], page 244) the external boundary value problem for Laplace equation

$$\Delta u = 0 \quad \text{in} \quad \Omega, \quad u|_{\partial\Omega} = \varphi, \tag{5}$$

$$u(x) \to 0 \quad \text{as} \quad |x| \to +\infty, \tag{6}$$

has a unique classical solution and it given formula

$$u(x) = \frac{1}{4\pi R} \int_{\partial\Omega} \frac{|x|^2 - R^2}{|x - y|^3} \, \varphi(y) \, dS_y, \quad x \in \Omega. \tag{7}$$

Note that restriction (6) is an essential condition for the solution uniqueness of the problem. Indeed, it is easy to verify that the classical solutions of problem (5) for $\varphi = 0$ are the following functions

$$u(x) = C\,(1 - R/|x|)\,, \quad x \in \overline{\Omega}, \quad C \in \mathbb{R} \ \text{ is arbitrary constant.}$$

Note that restriction (6) can be interpreted as an analog of the boundary condition at infinity.

Similar results for weak solutions of linear elliptic equations from a wide class were obtained in [2]. To confirm these results, it was used an analogue of the principle of Saint-Venant known in mechanics. The similar situation is with nonlinear equations from certain classes (see [3], [4], etc).

However, there are nonlinear elliptic equations for which the corresponding boundary value problems have a unique solution without any conditions at infinity. First result was proven in [5] for equation

$$-\Delta u + |u|^{q-2} u = f \quad \text{in} \quad \mathbb{R}^n,$$

where $q > 2$ is some number, f is locally integrable function. Similar results were obtained for nonlinear elliptic equations in [6], [7], [8], etc.

Nonlinear differential equations with variable exponents of the nonlinearity (for example, equation (4)) appear as mathematical models in various physical processes. In particular, these equations describe electroreological substance flows, image recovering processes, electric current in the conductor with changing temperature field (see [9]). Nonlinear differential equations with variable exponents of the nonlinearity in bounded domains were intensively studied in [10]– [13], etc. The corresponding generalizations of Lebesgue and Sobolev spaces (see [14]) were used in these investigations.

Within this paper, we consider a class of second order elliptic equations in unbounded domains, which require setting conditions for behavior of the solution at infinity for the correct formulation of the boundary value problems. This class contains both linear (see (3)) and nonlinear equations with variable exponents of the nonlinearity (see (4) as an example)). Here we complement and generalize results for linear and nonlinear elliptic (see, for example, [2] and [3]). As we know from the available sources, elliptic equations were not previously investigated in the context of the problem under our consideration. In our research, we use an analog of the well-known in mechanics Saint-Venant principle, which was developed in [2], [3], [15], and others. Moreover, to prove the feasibility of our problem we use the method of exhaustion for unbounded domains, and the monotonicity method (see [16]).

The article is organized as follows. Section 2 introduces the functional spaces used throughout the paper. In Section 3, we formulate the problem under consideration and state the main results. Section 4 is devoted to auxiliary statements that are employed in the subsequent analysis. Finally, Section 5 contains the proofs of the main results.

2. **Main notation**

We introduce some functional spaces. Let $r\colon \Omega \to \mathbb{R}$ be a measurable function, $r(x) \geqslant 1$ for almost every (a.e.) $x \in \Omega$, and

$$\operatorname*{ess\,sup}_{x\in\Omega'} r(x) < \infty$$

for any $\Omega' \in Bd(\Omega)$. For any $\Omega' \in Bd(\Omega)$ we denote by $L_{r(\cdot)}(\Omega')$ the linear space of (classes of equivalent) measurable functions $v\colon \Omega' \to \mathbb{R}$ such that

$$\rho_{\Omega',r}(v) := \int_{\Omega'} |v(x)|^{r(x)}\,dx < \infty.$$

This is the Banach space with a norm

$$\|v\|_{L_{r(\cdot)}(\Omega')} := \inf\{\lambda > 0 \ \mid\ \rho_{\Omega',r}(v/\lambda) \leqslant 1\}.$$

Space $L_{r(\cdot)}(\Omega')$ is called *the Lebesgue space with variable exponent* or *generalized Lebesgue space* (see, for example, [14]). If $r(x) > 1$ for a.e. $x \in \Omega$, put by definition $r'(x) := r(x)/(r(x) - 1)$ for a.e. $x \in \Omega$. As is well known, if $\Omega' \in Bd(\Omega)$, then the dual space $(L_{r(\cdot)}(\Omega'))'$ can be identified with $L_{r'(\cdot)}(\Omega')$ under the condition

$$\operatorname*{ess\,inf}_{x\in\Omega'} r(x) > 1.$$

Note also that in the case $r(x) = r = \text{const} \geqslant 1$ for a.e. $x \in \Omega' \in Bd(\Omega)$, we have $L_{r(\cdot)}(\Omega') = L_r(\Omega')$, and $\|\cdot\|_{L_{r(\cdot)}(\Omega')} = \|\cdot\|_{L_r(\Omega')}$.

Denote by $L_{r(\cdot),\,\mathrm{loc}}(\overline{\Omega})$ the linear space of (classes of equivalent) measurable functions $v\colon \Omega \to \mathbb{R}$ such that their restrictions $v|_{\Omega'}$ on Ω' belong to the space $L_{r(\cdot)}(\Omega')$ for any set $\Omega' \in Bd(\Omega)$. This space with a family of seminorms $\{\|\cdot\|_{L_{r(\cdot)}(\Omega')} \,|\, \Omega' \in Bd(\Omega)\}$ is complete locally convex. Then a sequence $\{v_l\}_{l=1}^{\infty}$ converges to v in $L_{r(\cdot),\,\mathrm{loc}}(\overline{\Omega})$ *strongly* (correspondly, *weakly*), if for any domain $\Omega' \in Bd(\Omega)$ the sequence $\{v_l|_{\Omega'}\}_{l=1}^{\infty}$ converges to $v|_{\Omega'}$ in $L_{r(\cdot)}(\Omega')$ *strongly* (correspondly, *weakly*).

We shall need the following assumption:

(**P**) $p = (p_0, p_1, \ldots, p_n)\colon \Omega \to \mathbb{R}^{1+n}$ is a vector-valued function such that
for all $i \in \{0, 1, \ldots, n\}$ the function $p_i\colon \Omega \to \mathbb{R}$ is measurable,
and $1 < \operatorname*{ess\,inf}_{x\in\Omega'} p_i(x) \le \operatorname*{ess\,sup}_{x\in\Omega'} p_i(x) < +\infty$ for each $\Omega' \in Bd(\Omega)$.

Let $p' = (p_0', p_1', \ldots, p_n')$ be the vector-valued function such that $\frac{1}{p_i(x)} + \frac{1}{p_i'(x)} = 1$ for a.e. $x \in \Omega$, $i = \overline{0,n}$. Obviously, the function p' satisfies condition (**P**) with p_i' instead of p_i, $i = \overline{0,n}$.

For any domain $\Omega' \in Bd(\Omega)$ we define the space

$$W^1_{p(\cdot)}(\Omega') := \{v \in L_{p_0(\cdot)}(\Omega') \mid v_{x_i} \in L_{p_i(\cdot)}(\Omega'),\ i = \overline{1,n}\}.$$

This is the Banach space with a norm

$$\|v\|_{W^1_{p(\cdot)}(\Omega')} := \|v\|_{L_{p_0(\cdot)}(\Omega')} + \sum_{i=1}^{n} \|v_{x_i}\|_{L_{p_i(\cdot)}(\Omega')}.$$

Space $W^1_{p(\cdot)}(\Omega')$ is called the *Sobolev space with variable exponent* or *generalized Sobolev space* (see, for example, [14]).

Denote by $W^1_{p(\cdot),\,\mathrm{loc}}(\overline{\Omega})$ the complete locally convex space of functions $v \in L_{p_0(\cdot),\,\mathrm{loc}}(\overline{\Omega})$ such that $v_{x_i} \in L_{p_i(\cdot),\,\mathrm{loc}}(\overline{\Omega})$, $i = \overline{1,n}$, along with a family of seminorms

$$\big\{\|v\|_{W^1_{p(\cdot)}(\Omega'))} \mid \Omega' \in Bd(\Omega)\big\}.$$

Let $\widetilde{W}^1_{p(\cdot),\,\mathrm{loc}}(\overline{\Omega})$ be the closure of the set $\widetilde{C}^1(\overline{\Omega}) := \{v \in C^1(\overline{\Omega}) \mid v|_{\Gamma_0} = 0\}$ in space $W^1_{p(\cdot),\,\mathrm{loc}}(\overline{\Omega})$. By $\widetilde{W}^1_{p(\cdot),\mathrm{c}}(\Omega)$ we denote a subspace of $\widetilde{W}^1_{p(\cdot),\,\mathrm{loc}}(\overline{\Omega})$ consisting of functions with bounded supports.

3. Statement of the problem and formulation of main results

We will consider weak solutions of the problem (1), (2). To define them, we introduce corresponding data-in classes.

Let $p = (p_0, p_1, \ldots, p_n)$ be a vector-valued function that satisfies condition (**P**). By $\mathbb{A}_p$ we denote all ordered collections $(a_0, a_1, \ldots, a_n)$ of the real functions satisfying the following conditions:

(**A**$_1$)**:** for every $i \in \{0, 1, \ldots, n\}$, function $a_i(x, \rho, \xi)$, $(x, \rho, \xi) \in \Omega \times \mathbb{R}^{1+n}$, is a Carathéodory, i.e., function $a_i(x, \cdot, \cdot)\colon \mathbb{R}^{1+n} \to \mathbb{R}$ is a continuous for a.e. $x \in \Omega$, and function $a_i(\cdot, \rho, \xi)\colon \Omega \to \mathbb{R}$ is a measurable for every $(\rho, \xi) \in \mathbb{R}^{1+n}$; in addition, $a_i(x, 0, 0) = 0$ for a.e. $x \in \Omega, i = \overline{0,n}$;

(**A**$_2$)**:** for every $i \in \{0, 1, \ldots, n\}$, for a.e. $x \in \Omega$, and for every $(\rho, \xi) \in \mathbb{R}^{1+n}$ the following inequality holds

$$|a_i(x, \rho, \xi)| \leqslant h_{i,1}(x)\big(|\rho|^{p_0(x)/p_i'(x)} + \sum_{j=1}^{n} |\xi_j|^{p_j(x)/p_i'(x)}\big) + h_{i,2}(x),$$

where $h_{i,1} \in L_{\infty,\mathrm{loc}}(\overline{\Omega})$, $h_{i,2} \in L_{p_i'(\cdot),\mathrm{loc}}(\overline{\Omega})$.

Now we give a definition of a weak solution of problem (1), (2). We assume that p satisfies condition $(\mathbf{P})$, $(a_0, a_1, ..., a_n) \in \mathbb{A}_p$, $f \in L_{2,\mathrm{loc}}(\overline{\Omega})$.

Означення 1. A weak solution of problem (1) — (2) is called a function $u \in \widetilde{W}^1_{p(\cdot),\mathrm{loc}}(\overline{\Omega})$, which satisfies the integral identity

$$\int\limits_{\Omega} \Big[\sum_{i=1}^{n} a_i(x,u,\nabla u)\psi_{x_i} + a_0(x,u,\nabla u)\psi\Big]\,dxdt = \int\limits_{\Omega} f\psi\,dxdt \quad \forall\,\psi \in \widetilde{W}^1_{p(\cdot),\mathrm{c}}(\Omega). \tag{8}$$

Suppose $0 \in \Omega$. Let $k \in \{1,\ldots,n\}$ be a number such that for any $\tau > 0$ the set $\widetilde{\Omega}_\tau := \Omega \cap \{x \in \mathbb{R}^n \mid |x_1|^2 + \ldots + |x_k|^2 < \tau^2\}$ is bounded. For any $\tau > 0$ we denote by Ω_τ a connected component of the set $\widetilde{\Omega}_\tau$ that contains 0. Obviously,

$$\Omega = \bigcup_{\tau>0} \Omega_\tau.$$

The value k depends on the geometry of the domain Ω (up to the numbering of variables $x_1, ..., x_n$). Obviously, in the general case we can take $k = n$, and, in this case, the class of equations considered below will consist of generalizations of equation (3), or rather, of almost linear equations. But in the case of $k < n$ the class of equations to which the following results apply is wider than in the case of $k = n$, and the smaller the value of k the wider the class of these equations (to confirm this, see (4)).

Let us illustrate possibilities of the value's k considered two examples.

Приклад 1. Assume $\Omega = \Omega_1 \times \Omega_2$, where Ω_1 is an unbounded domain in $\mathbb{R}^l := \{(x_1,\ldots,x_l) \mid x_i \in \mathbb{R},\ i = \overline{1,l}\}$ for some $l \in \{1,\ldots,n-1\}$, Ω_2 is a bounded domain in $\mathbb{R}^{n-l} := \{(x_{l+1},\ldots,x_n) \mid x_i \in \mathbb{R},\ i = \overline{l+1,n}\}$, and $0 \in \Omega$. Then we can take arbitrary $k \in \{l,\ldots,n\}$. If $k = l$, then $\Omega_\tau = \Omega_{1,\tau} \times \Omega_2$ for any $\tau > 0$, where $\Omega_{1,\tau}$ is a connected component of the set $\Omega_1 \cap \{(x_1,\ldots,x_l) \in \mathbb{R}^l \mid |x_1|^2 + \ldots + |x_l|^2 < \tau^2\}$ such that $0 \in \Omega_{1,\tau}$.

Приклад 2. Suppose

$$\Omega := \{(x_1,x_2) \in \mathbb{R}^2 \mid -\infty < x_1 < +\infty, \quad -\phi_1(x_1) < x_2 < \phi_2(x_1)\},$$

where for each $m \in \{1,2\}$ a function ϕ_m is continuous on $\mathbb{R}$, and $\phi_m(s) > 0$ for all $s \in \mathbb{R}$. Then we can take either $k = 1$ or $k = 2$. In case $k = 1$, we have

$$\Omega_\tau = \big\{(x_1,x_2) \in \mathbb{R}^2 \,\big|\, |x_1| < \tau, \quad |x_2| < \phi_2(x_1)\big\}$$

for any $\tau > 0$. If $k = 2$, then

$$\Omega_\tau = \big\{(x_1,x_2) \in \mathbb{R}^2 \,\big|\, |x_1| < \tau, \quad |x_2| < \min\{\phi_2(x_1), \sqrt{\tau^2 - |x_1|^2}\big\}$$

for any $\tau > 0$.

By definition, put

$$\Gamma_{j,\tau} := \Gamma_j \cap \partial\Omega_\tau,\ j = 0,1, \quad \Gamma_{*,\tau} := \Omega \cap \partial\Omega_\tau.$$

We will use a notation

$$\nabla_k v := (v_{x_1},\ldots,v_{x_k}), \quad |\nabla_k v| := (|v_{x_1}|^2 + \ldots + |v_{x_k}|^2)^{1/2}.$$

Everywhere further we will consider that is carried out the following condition:

($\mathbf{P}^*$): *$p = (p_0, p_1, \ldots, p_n)\colon \Omega \to \mathbb{R}^{1+n}$ satisfy condition ($\mathbf{P}$), and $p_0(x) = p_1(x) = \ldots = p_k(x) = 2$ for a.e. $x \in \Omega$.*

Suppose $\mathbb{A}_p^*$ is a subset of $\mathbb{A}_p$, which every element satisfies conditions ($\mathbf{A}_1$), ($\mathbf{A}_2$), and the following condition:

($\mathbf{A}_3$): for a.e. $x \in \Omega$, and for every (ρ_1, ξ^1), $(\rho_2, \xi^2) \in \mathbb{R}^{1+n}$, we have

$$\sum_{i=1}^{k} |a_i(x, \rho_1, \xi^1) - a_i(x, \rho_2, \xi^2)| \leqslant g_1(x)|\xi'^1 - \xi'^2| + g_2(x)|\rho_1 - \rho_2|, \tag{9}$$

$$\sum_{i=1}^{n} (a_i(x, \rho_1, \xi^1) - a_i(x, \rho_2, \xi^2))(\xi_i^1 - \xi_i^2) + (a_0(x, \rho_1, \xi^1) - a_0(x, \rho_2, \xi^2))(\rho_1 - \rho_2) \geqslant$$
$$\geqslant q_1(x)|\xi'^1 - \xi'^2|^2 + q_2(x)|\rho_1 - \rho_2|^2, \tag{10}$$

where $\xi'^j := (\xi_1^j, \ldots, \xi_k^j)$, $|\xi'^j| := (|\xi_1^j|^2 + \ldots + |\xi_k^j|^2)^{1/2}$, $j \in \{1, 2\}$, and g_1, g_2, q_1, $q_2\colon \overline{\Omega} \to \mathbb{R}$ are continuous functions on $\overline{\Omega}$ that satisfy the following conditions:

- $$g_1(x) > 0, \quad g_2(x) \geqslant 0, \quad q_1(x,t) > 0, \quad q_2(x,t) > 0 \quad \text{for all} \quad x \in \overline{\Omega}; \tag{11}$$
- there exist continuous functions d_1, d_2, λ defined on $[1, +\infty)$ such that

$$\text{for all} \ \ \tau \geqslant 1\colon \quad d_1(\tau) \geqslant \max_{\overline{\Gamma_{*,\tau}}} \frac{g_1}{\sqrt{q_1}}, \quad d_2(\tau) \geqslant \max_{\overline{\Gamma_{*,\tau}}} g_2, \tag{12}$$

$$\text{for all} \ \ \tau \geqslant 1\colon \quad 0 < \lambda(\tau) \leqslant \inf_{v} \frac{\int_{\Gamma_{*,\tau}} \left[q_1|\nabla_k v|^2 + q_2|v|^2\right] d\Gamma}{\int_{\Gamma_{*,\tau}} |v|^2 \, d\Gamma}, \tag{13}$$

where the infimum is taken over all functions v that are continuously differentiable in the neighborhood of $\overline{\Gamma_{*,\tau}}$, and $v = 0$ on $\partial\Gamma_{*,\tau} \cap \Gamma_0$ (in particular, $0 < \lambda(\tau) \leqslant \min\limits_{\overline{\Gamma_{*,\tau}}} q_2$),
while

$$\int_1^{+\infty} \frac{d\tau}{A(\tau)} = +\infty, \tag{14}$$

where

$$A(\tau) := \frac{d_1(\tau)}{\sqrt{\lambda(\tau)}} + \frac{d_2(\tau)}{\lambda(\tau)}, \quad \tau \geqslant 1. \tag{15}$$

Зауваження 3. If

$$\sup_{\overline{\Omega}} \frac{g_1}{\sqrt{q_1}} < +\infty, \ \sup_{\overline{\Omega}} g_2 < +\infty, \ \inf_{\overline{Q}} q_2 > 0,$$

then functions d_1, d_2, λ can be chosen as constants. Namely, $d_1(\tau) := d_{1,0}$, $d_2(\tau) := d_{2,0}$, $\lambda(\tau) := \lambda_0$ for all $\tau \geqslant 1$, where $d_{1,0}$, $d_{2,0}$, λ_0 are constants such that

$$d_{1,0} \geqslant \sup_{\overline{\Omega}} \frac{g_1}{\sqrt{q_1}}, \quad d_{2,0} \geqslant \sup_{\overline{\Omega}} g_2, \quad 0 < \lambda_0 \leqslant \inf_{\overline{\Omega}} q_2.$$

Then we can take

$$A(\tau) = A_0 := \frac{d_{1,0}}{\sqrt{\lambda_0}} + \frac{d_{2,0}}{\lambda_0} \quad \text{for all} \quad \tau \geqslant 1.$$

□

Suppose $\mathbb{A}_p^{**}$, in the case of $k < n$, is a subset of $\mathbb{A}_p^*$, which every element satisfies the following condition:

($\mathbf{A}_4$): for a.e. $x \in \Omega$ and for every $(\rho, \xi) \in \mathbb{R}^{1+n}$, we have

$$\sum_{i=0}^{n} a_i(x,\rho,\xi)\xi_i + a_0(x,\rho,\xi)\rho \geqslant q_3(x) \sum_{i=k+1}^{n} |\xi_i|^{p_i(x)} + h(x), \tag{16}$$

where $q_3 \in C(\overline{Q})$, $q_3(x) > 0$ for all $x \in \overline{\Omega}$, $h \in L_{1,\mathrm{loc}}(\overline{\Omega})$.

In the case of $k = n$ we will assume that $\mathbb{A}_p^{**} := \mathbb{A}_p^*$.

It is easy to prove that the initial problem

$$\frac{d\tau}{d\alpha} = A(\tau), \quad \tau(0) = 1 \tag{17}$$

has a unique solution $\tau(\alpha)$, $\alpha \in [0, +\infty)$, and this solution is determined by the equality

$$\int_1^{\tau(\alpha)} \frac{ds}{A(s)} = \alpha, \quad \alpha \geqslant 0. \tag{18}$$

From this and (14) it follows that

$$\tau(\alpha) \to +\infty \quad \text{as} \quad \alpha \to +\infty. \tag{19}$$

Suppose $\tau(\alpha)$, $\alpha \in [0, +\infty)$, is a solution of problem (17) (see (18)), and put

$$\Omega^\alpha := \Omega_{\tau(\alpha)}, \quad \Gamma_j^\alpha := \Gamma_{j,\tau(\alpha)}, \; j = 0, 1, \quad \Gamma_*^\alpha := \Gamma_{*,\tau(\alpha)}.$$

Note that in view of (19) we have $\Omega = \bigcup\limits_{\alpha > 0} \Omega^\alpha$.

Let $\{\Lambda_m\}_{m=1}^\infty$ be a sequence of real numbers such that for all $m \in \mathbb{N}$ we have

$$0 < \Lambda_m \leqslant \inf_v \frac{\int_{\Omega^m} \left[q_1|\nabla_k v|^2 + q_2|v|^2\right] dx}{\int_{\Omega^m} |v|^2\, dx}\,, \tag{20}$$

where the infimum is taken over all functions $v \in C^1(\overline{\Omega^m})$ such that $v = 0$ on $\partial\Omega^m \setminus \Gamma_1^m$ (in particular, $0 < \Lambda_m \leqslant \inf\limits_{\Omega^m} q_2$).

Denote

$$E_k(w) := q_1|\nabla_k w|^2 + q_2|w|^2, \quad \langle w \rangle_\alpha := \Bigg(\int_{\Omega^\alpha} E_k(w)\, dx\Bigg)^{1/2}, \quad \alpha \geqslant 0. \tag{21}$$

Now we formulate our main results.

Теорема 1 (a uniqueness of the solution)**.** *Let* p *satisfies condition* $(\mathbf{P}^*)$, $f \in L_{2,\mathrm{loc}}(\overline{\Omega})$, $(a_0, a_1, \ldots, a_n) \in \mathbb{A}_p^*$. *Then problem (1), (2) has at most one weak solution such that*

$$e^{-R/2}\,\langle u\rangle_R \to 0 \quad \textit{as} \quad R \to +\infty \tag{22}$$

(an analog of the boundary condition at infinity), where $\langle\cdot,\cdot\rangle_R$ *defined in* (21).

Зауваження 4. Assertion (22) is equivalent to the condition

$$e^{-\int_1^r \frac{ds}{A(s)}} \int\limits_{\Omega_r} \big[q_1|\nabla_k u|^2 + q_2|u|^2\big]\,dxdt \to 0 \quad \text{as} \quad r \to +\infty. \tag{23}$$

It follows from (18), if to remark that $\Omega^R = \Omega_r$, if $R = \int_1^r \frac{ds}{A(s)}$. ☐

Теорема 2 (an existence of the solution)**.** *Let* p *satisfies condition* $(\mathbf{P}^*)$, $f \in L_{2,\mathrm{loc}}(\overline{\Omega})$, $(a_0, a_1, \ldots, a_n) \in \mathbb{A}_p^{**}$. *Also suppose for some number* $\varkappa \in (0,1)$ *the following inequality holds*

$$\Lambda_m^{-1} \int\limits_{\Omega^m} |f|^2\,dx \leqslant C_1\,e^{(1-\varkappa)m} \quad \forall\, m \in \mathbb{N}, \tag{24}$$

where $C_1 > 0$ *is a some constant.*

Then there exists a weak solution of problem (1), (2) satisfying condition (22) (it is unique, see Theorem 1). Moreover, for this solution the following estimate is fulfilled:

$$\langle u\rangle_m \;\leqslant\; C_2\,e^{(1-\varkappa)m/2} \quad \forall\, m \in \mathbb{N}, \tag{25}$$

where $C_2 := [(2 + e^{1/2} - e^{-\varkappa/2})/(1 - e^{-\varkappa/2})]\sqrt{C_1}$, $\langle\cdot,\cdot\rangle_m$ *defined in* (21).

Зауваження 5. Estimate (25) is equivalent to the estimate

$$\int\limits_{\Omega_r} \big[q_1|\nabla_k u|^2 + q_2|u|^2\big]\,dxdt \;\leqslant\; C_3\,e^{(1-\varkappa)\int_1^r \frac{ds}{A(s)}} \quad \forall\, r \geqslant 1, \tag{26}$$

where $C_3 > 0$ is a constant depending only on $\varkappa$ and C_1. The statement is substantiated in the same way as (23). ☐

Зауваження 6. For equation (3) the conditions of Theorems 1 and 2 are satisfied if functions $\widehat{a}_{ij}$, $i,j = \overline{1,n}$, $\widehat{a}_i$, $i = \overline{0,n}$, are as in Remark 1, and for a.e. $x \in \Omega$ following hold

$$g_1(x) \geqslant \sum_{i=1}^n \Big(\sum_{j=1}^n |\widehat{a}_{ij}(x)|^2\Big)^{1/2}, \quad g_2(x) = 0,$$

$$q_1(x) = \omega/2, \quad q_2(x) \leqslant \Big(\widehat{a}_0(x,t) - \frac{1}{2\omega}\sum_{i=1}^n |\widehat{a}_i(x)|^2\Big), \tag{27}$$

where g_1, g_2, q_1, q_2 are as in $(\mathbf{A}_3)$, and f satisfy (24). ☐

Зауваження 7. For equation (4) the conditions of Theorems 1 and 2 are satisfied if functions $\widehat{a}_{ij}$, $i,j=\overline{1,k}$, $\widehat{a}_i$, $i=\overline{k+1,n}$, $\widehat{a}_0$ are as in Remark 2, and for a.e. $x\in\Omega$ following inequalities hold

$$\sqrt{k}\sum_{i=1}^{k}\max_{j\in\{1,\dots,k\}}\big|\widehat{a}_{ij}(x)\big|\leqslant g_1(x),\quad g_2(x)=0,$$

$$\sum_{i,j=1}^{k}\widehat{a}_{ij}(x)\eta_i\eta_j\geqslant q_1(x)\sum_{i=1}^{k}|\eta_i|^2\quad\forall\,(\eta_1,...,\eta_k)\in\mathbb{R}^k, \tag{28}$$

$$\widehat{a}_0(x,t)\geqslant q_2(x),\qquad \min_{i\in\{k+1,\dots,n\}}\widehat{a}_i(x)\geqslant q_3(x), \tag{29}$$

where g_1, q_1, q_2, q_3 are as in $(\mathbf{A}_3)$, $(\mathbf{A}_4)$, and f satisfy (24). □

4. **Auxiliary statements**

Here we give some auxiliary results which will be used in Section 5. We denote

$$a_i(v)(x):=a_i(x,v(x),\nabla v(x)),\quad x\in\Omega,\quad i=\overline{0,n}, \tag{30}$$

$$\partial_0 v=v,\ \ \partial_i v=\partial_i v,\quad i=\overline{1,n}. \tag{31}$$

Лема 1 (an analog of Saint-Venant principle)**.** *Assume p satisfies condition* $(\mathbf{P}^*)$, $(a_0,a_1,\dots,a_n)\in\mathbb{A}_p^*$, *and* $f\in L_{2,\mathrm{loc}}(\overline{Q})$. *Suppose $R>0$ is an arbitrary number, and* $u_1,u_2\in\widetilde{W}^1_{p(\cdot),\mathrm{loc}}(\overline{\Omega})$ *such that for each $l\in\{1,2\}$ we have*

$$\int\limits_{\Omega^R}\sum_{i=0}^{n}a_i(u_l)\partial_i\psi\,dx=\int\limits_{\Omega^R}f\psi\,dx\quad\forall\,\psi\in\widetilde{W}^1_{p(\cdot),c}(\Omega),\ supp\,\psi\subset\overline{\Omega^R}. \tag{32}$$

Then for every R_1,R_2, $0<R_1<R_2\leqslant R$, the following inequality holds

$$\langle u_1-u_2\rangle_{R_1}\leqslant e^{(R_1-R_2)/2}\,\langle u_1-u_2\rangle_{R_2}. \tag{33}$$

Доведення. For an arbitrary $x\in\mathbb{R}^n$ we set $x=(x',x'')$, where $x'=(x_1,...,x_k)\in\mathbb{R}^k$, $x''=(x_{k+1},...,x_n)\in\mathbb{R}^{n-k}$. Let $|x'|=(|x_1|^2+\ldots+|x_k|^2)^{1/2}$. For any $\delta\in(0,1)$, $\tau\in[1,+\infty)$, $x'\in\mathbb{R}^k$ we denote

$$\psi_\delta(x',\tau):=\begin{cases}1, & \text{if } |x'|\leqslant\tau-\delta;\\ (\tau-|x'|)/\delta, & \text{if } \tau-\delta<|x'|<\tau;\\ 0, & \text{if } |x'|\geqslant\tau.\end{cases}$$

Obviously, for every $i\in\{1,\dots,k\}$ we have $\partial_i\psi_\delta(x',\tau):=0$ if $|x'|<\tau-\delta$ or $|x'|>\tau$, and

$$\partial_i\psi_\delta(x',\tau)=-\frac{x_i}{\delta|x'|},\quad\text{if }\ \tau-\delta<|x'|<\tau. \tag{34}$$

By definition, put $w:=u_1-u_2$. Let $\delta\in(0,1)$, $\tau\in(1,\tau(R))$ be arbitrary fixed. We subtract the integral identity (32) for $l=2$ from this identity for $l=1$. Putting $\psi(x):=w(x)\psi_\delta(x',\tau)$, $x=(x',x'')\in\Omega$, we obtain

$$\int\limits_{\Omega_\tau}\sum_{i=0}^{n}(a_i(u_1)-a_i(u_2))\partial_i w\psi_\delta\,dx=-\int\limits_{\Omega_\tau}\sum_{i=1}^{k}(a_i(u_1)-a_i(u_2))w\partial_i\psi_\delta\,dx. \tag{35}$$

Let $\nabla_k w := (\partial_1 w, \dots, \partial_k w)$, $|\nabla_k w| := (\sum_{i=1}^k |\partial_i w|^2)^{1/2}$. In view of (9) we have

$$\sum_{i=1}^{k} |a_i(u_1) - a_i(u_2)| \leqslant g_1 |\nabla_k w| + g_2 |w| \quad \text{a.e. on } Q. \tag{36}$$

From (35), taking into account (10), (34), and (36), we deduce

$$\int\limits_{\Omega_\tau} \big[q_1 |\nabla_k w|^2 + q_2 |w|^2\big] \psi_\delta \, dx \leqslant \frac{1}{\delta} \int\limits_{\Omega_\tau \setminus \Omega_{\tau-\delta}} \big[g_1 |\nabla_k w| + g_2 |w|\big] |w| \, dx. \tag{37}$$

Note that for an arbitrary function $P \in L_{1,\mathrm{loc}}(\overline{\Omega})$ we have

$$\int\limits_{\Omega_\tau \setminus \Omega_{\tau-\delta}} P(x) \, dx dt = \int\limits_{\tau-\delta}^{\tau} \Big(\int\limits_{\Gamma_{*,\sigma}} P(x) \, d\Gamma \Big) \, d\sigma, \quad \tau > 0.$$

Using the latter assertion, we pass to the limit in (37) as $\delta \to 0+$. So, we get

$$\int\limits_{\Omega_\tau} \big[q_1 |\nabla_k w|^2 + q_2 |w|^2\big] \, dx \leqslant \int\limits_{\Gamma_{*,\tau}} \big[g_1 |\nabla_k w| + g_2 |w|\big] |w| \, d\Gamma \quad \text{for a.e. } \tau \in (0, \tau(R)). \tag{38}$$

From Cauchy-Bunyakovsky-Schvartz inequality it follows that for a.e. $\tau \in (0, \tau(R))$

$$\int\limits_{\Gamma_{*,\tau}} \big[g_1 |\nabla_k w| + g_2 |w|\big] |w| \, d\Gamma \, dt \leqslant$$

$$\leqslant \Big(\int\limits_{\Gamma_{*,\tau}} |g_1|^2 |\nabla_k w|^2 \, d\Gamma \Big)^{1/2} \Big(\int\limits_{\Gamma_{*,\tau}} |w|^2 \, d\Gamma \Big)^{1/2} + \int\limits_{\Gamma_{*,\tau}} g_2 |w|^2 \, d\Gamma.$$

By virtue of (11), (12) and (13), for a.e. $\tau \in (0, \tau(R))$ and for a.e. $t \in (0, T)$ we obtain

$$\int\limits_{\Gamma_{*,\tau}} |g_1|^2 |\nabla_k w|^2 \, d\Gamma \leqslant \int\limits_{\Gamma_{*,\tau}} [|g_1|^2 / q_1] q_1 |\nabla_k w|^2 \, d\Gamma \leqslant (d_1(\tau))^2 \int\limits_{\Gamma_{*,\tau}} \big[q_1 |\nabla_k w|^2 + q_2 |w|^2\big] \, d\Gamma, \tag{39}$$

$$\int\limits_{\Gamma_{*,\tau}} |w|^2 \, d\Gamma \leqslant \int\limits_{\Gamma_{*,\tau}} \big[q_1 |\nabla_k w|^2 + q_2 |w|^2\big] \, d\Gamma \Big/ \Big[\int\limits_{\Gamma_{*,\tau}} \big[q_1 |\nabla_k w|^2 + q_2 |w|^2\big] \, d\Gamma \Big/ \int\limits_{\Gamma_{*,\tau}} |w|^2 \, d\Gamma \Big] \leqslant$$

$$\leqslant \lambda^{-1}(\tau) \int\limits_{\Gamma_{*,\tau}} \big[q_1 |\nabla_k w|^2 + q_2 |w|^2\big] \, d\Gamma, \tag{40}$$

$$\int\limits_{\Gamma_{*,\tau}} g_2 |w|^2 \, d\Gamma \leqslant d_2(\tau) \int\limits_{\Gamma_{*,\tau}} |w|^2 \, d\Gamma \leqslant d_2(\tau) \lambda^{-1}(\tau) \int\limits_{\Gamma_{*,\tau}} \big[q_1 |\nabla_k w|^2 + q_2 |w|^2\big] \, d\Gamma. \tag{41}$$

From (38), taking into account (4)–(41), we infer

$$\int\limits_{\Omega_\tau} \big[q_1 |\nabla_k w|^2 + q_2 |w|^2\big] \, dx \leqslant \big[d_1(\tau) \lambda^{-1/2}(\tau) + d_2(\tau) \lambda^{-1}(\tau)\big] \int\limits_{\Sigma_{*,\tau}} \big[q_1 |\nabla_k w|^2 + q_2 |w|^2\big] \, d\Gamma. \tag{42}$$

In view of (15), (21), and (42) we establish for a.e. $\tau \in (0,\tau(R))$

$$\int\limits_{\Omega_\tau} E_k(w)\,dx \leqslant A(\tau) \int\limits_{\Gamma_{*,\tau}} E_k(w)\,d\Gamma. \tag{43}$$

Denote

$$F(\tau) := \int\limits_{\Omega_\tau} E_k(w)\,dx \equiv \int\limits_0^\tau \Big(\int\limits_{\Gamma_{*,\sigma}} E_k(w)\,d\Gamma\Big)\,d\sigma, \quad \tau \in [1,\tau(R)]. \tag{44}$$

Then for a.e. $\tau \in (1,\tau(R))$

$$\int\limits_{\Gamma_{*,\tau}} E_k(w)\,d\Gamma = \frac{d}{d\tau}\int\limits_0^\tau \Big(\int\limits_{\Gamma_{*,\sigma}} E_k(w)\,d\Gamma\Big)\,d\sigma = \frac{dF(\tau)}{d\tau}. \tag{45}$$

From (43), using (44), and (45), we obtain

$$F(\tau) \leqslant A(\tau)\frac{dF(\tau)}{d\tau} \quad \text{for a.e.} \ \ \tau \in [1,\tau(R)]. \tag{46}$$

Suppose $\tau = \tau(\alpha)$, $\alpha \in [0,+\infty)$, is a solution of problem (17), and R_1, R_2 are arbitrary real numbers such that $0 < R_1 < R_2 \leqslant R$. In view of (17) and (46) we get

$$F(\tau(\alpha)) \leqslant \frac{dF(\tau(\alpha))}{d\tau}\frac{d\tau(\alpha)}{d\alpha}, \quad \alpha \in [R_1,R_2].$$

It follows that

$$0 \leqslant \frac{dF(\tau(\alpha))}{d\alpha} - F(\tau(\alpha)), \quad \alpha \in [R_1,R_2]. \tag{47}$$

Multiplying (47) by $e^{-\alpha}$, we deduce $0 \leqslant \frac{d}{d\alpha}\Big(e^{-\alpha}F(\tau(\alpha))\Big)$, $\alpha \in [R_1,R_2]$. Integrating the latter inequality in α from R_1 to R_2, we infer

$$F(\tau(R_1)) \leqslant e^{R_1-R_2}F(\tau(R_2)). \tag{48}$$

From (48), taking into account $\langle w\rangle_\alpha = \sqrt{F(\tau(\alpha))}$, we imply (33). □

5. Proofs of the main results

Proof of Theorem 1. Let us show that problem (1), (2) has no more than one weak solution satisfying condition (22). Assume the opposite. Let u_1 and u_2 be different weak solutions of problem (1), (2), which satisfy condition (22). It is clear that for arbitrary $R > 0$ a functional $\langle\cdot,\cdot\rangle_R$ is a seminorm in space $\widetilde{W}^1_{p(\cdot),\mathrm{loc}}(\overline{\Omega})$. From this fact and (22) we deduce

$$e^{-R/2}\,\langle u_1-u_2\rangle_R \leqslant e^{-R/2}(\langle u_1\rangle_R + \langle u_2\rangle_R) = e^{-R/2}\langle u_1\rangle_R + e^{-R/2}\langle u_2\rangle_R = \beta(R),$$

where $\beta(R) \to 0$ as $R \to +\infty$. Using this assertion and Lemma 1 (see (33)) for arbitrary R_1, R_2 such that $R_1 < R_2$, we obtain the estimate

$$\langle u_1-u_2\rangle_{R_1} \leqslant e^{(R_1-R_2)/2}\langle u_1-u_2\rangle_{R_2} = e^{R_1/2}\beta(R_2). \tag{49}$$

We fix R_1, and tend R_2 to $+\infty$. From (49) it follows that $\langle u_1 - u_2\rangle_{R_1} = 0$. Thus $u_1 = u_2$ almost everywhere on Ω^{R_1}. As R_1 is arbitrary, we get $u_1 = u_2$ almost everywhere on Ω. This contradiction proves Theorem 1. $\square$

Proof of Theorem 2. The proof is in four steps.
Step 1 (the solution's approximations). Let $\alpha > 0$ be an arbitrary number. By $\widehat{W}^1_{p(\cdot)}(\Omega^\alpha)$ define the closure of space $\{v \in C^1\big(\overline{\Omega^\alpha}\big) \mid v\big|_{\partial\Omega^\alpha\setminus\Gamma_1^\alpha} = 0\}$ in $W^1_{p(\cdot)}(\Omega^\alpha)$.

For every $l \in \mathbb{N}$ we consider the problem: *to find the function* $u_l \in \widehat{W}^1_{p(\cdot)}(\Omega^l)$ *that satisfies the integral identity*

$$\int\limits_{\Omega^l} \sum_{i=0}^{n} a_i(u_l)\partial_i\psi\, dx = \int\limits_{\Omega^l} f\psi\, dx \quad \forall\, \psi \in \widehat{W}^1_{p(\cdot)}(\Omega^l). \tag{50}$$

To prove the existence of the function $u_l \in \widehat{W}^1_{p(\cdot)}(\Omega^l)$ we use Galerkin method (see, for example, [16]). In view of $(\mathbf{A}_3)$ it is easy to show that the function u_l is a unique. For every $l \in \mathbb{N}$ the function u_l is extended by zero to Ω, and the extension denote by u_l again. Obviously, that $u_l \in \widetilde{W}^1_{p(\cdot),\mathrm{loc}}(\overline{\Omega})$, and identity

$$\int\limits_{\Omega} \sum_{i=0}^{n} a_i(u_l)\partial_i\psi\, dx = \int\limits_{\Omega} f\psi\, dx \quad \forall\, \psi \in \widetilde{W}^1_{p(\cdot),\mathrm{c}}(\Omega),\ \operatorname{supp}\psi \subset \overline{\Omega^l}, \tag{51}$$

holds.

Now we show that there exists a subsequence of the sequence $\{u_l\}_{l=1}^{\infty}$ converging (in some sense) to the weak solution of problem (1), (2), which satisfy condition (22). We use an approach from [2], [3], [15], [16].

Step 2 (the convergence of the sequence of solution's approximations). First we estimate $\langle u_l\rangle_l$ for an arbitrary fixed $l \in \mathbb{N}$. From (50), putting $\psi = u_l$, we obtain

$$\int\limits_{\Omega^l} \sum_{i=0}^{n} a_i(u_l)\, \partial_i u_l\, dx = \int\limits_{\Omega^l} f\, u_l\; dx. \tag{52}$$

From this assertion, taking into account $(\mathbf{A}_1)$ (or rather, the condition $a_i(0) = 0$, $i = \overline{0,n}$), $(\mathbf{A}_3)$ (see (10)), and Cauchy inequality

$$ab \leqslant \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2, \quad a, b \in \mathbb{R},\ \varepsilon > 0, \tag{53}$$

we infer

$$\int\limits_{\Omega^l} \big[q_1|\nabla_k u_l|^2 + q_2|u_l|^2\big]\, dx \leqslant \frac{\varepsilon_1}{2}\int\limits_{\Omega^l} |u_l|^2\, dx + \frac{1}{2\varepsilon_1}\int\limits_{\Omega^l} |f|^2\, dx, \tag{54}$$

where $\varepsilon_1 > 0$ is an arbitrary constant.

We have

$$\int\limits_{\Omega^l} |u_l|^2\, dx = \int\limits_{\Omega^l} \big[q_1|\nabla_k u_l|^2 + q_2|u_l|^2\big]\, dx \Big/ \Big[\int\limits_{\Omega^l} \big[q_1|\nabla_k u_l|^2 + q_2|u_l|^2\big]\, dx \Big/ \int\limits_{\Omega^l} |u_l|^2\, dx\Big] \leqslant$$

$$\leqslant \frac{1}{\Lambda_l}\int\limits_{\Omega^l}\big[q_1|\nabla_k u_l|^2+q_2|u_l|^2\big]\,dx, \tag{55}$$

where Λ_l is defined in (20).

From (54) and (55), putting $\varepsilon_1=\Lambda_l$, we get

$$\int\limits_{Q^l}E_k(u_l)\,dx\leqslant \Lambda_l^{-1}\int\limits_{Q^l}|f|^2\,dx.$$

The latter inequality and (24) imply the estimate

$$\langle u_l\rangle_l\leqslant \sqrt{C_1}\,e^{(1-\varkappa)l/2},\quad l\in\mathbb{N}. \tag{56}$$

Let $m\in\mathbb{N}$ be an arbitrary fixed number, and let $l,r\in\mathbb{N}$ be arbitrary numbers, while $l\geqslant m$. We have

$$\langle\, u_{l+r}-u_l\rangle_m\leqslant\sum_{i=0}^{r-1}\langle\, u_{l+i+1}-u_{l+i}\rangle_m. \tag{57}$$

For each $i\in\{0,\dots,r-1\}$ and the functions u_{l+i+1}, u_{l+i}, using Lemma 1 with $R=l+i$, we obtain

$$\langle\, u_{l+i+1}-u_{l+i}\rangle_m\leqslant e^{-1/2}\langle\, u_{l+i+1}-u_{l+i}\rangle_{m+1}\leqslant\ldots\leqslant e^{-(l+i-m)/2}\langle\, u_{l+i+1}-u_{l+i}\rangle_{l+i}. \tag{58}$$

In view of (56), we have

$$\langle\, u_{l+i+1}-u_{l+i}\rangle_{l+i}\leqslant\langle\, u_{l+i+1}\rangle_{l+i+1}+\langle\, u_{l+i}\rangle_{l+i}\leqslant\sqrt{C_1}\big[e^{(1-\varkappa)(l+i+1)/2}+e^{(1-\varkappa)(l+i)/2}\big]\leqslant$$
$$\leqslant\sqrt{C_1}\big[e^{1/2}+1\big]e^{(1-\varkappa)(l+i)/2}=C_4\,e^{(1-\varkappa)(l+i)/2}, \tag{59}$$

where $C_4:=\sqrt{C_1}\big(e^{1/2}+1\big)$.

Using (57)–(59), we find

$$\langle\, u_{l+r}-u_l\rangle_m\leqslant C_4\sum_{i=0}^{r-1}e^{-(l+i-m)/2}\,e^{(1-\varkappa)(l+i)/2}\leqslant$$
$$\leqslant C_4e^{(m-\varkappa l)/2}\sum_{i=0}^{\infty}(e^{-\varkappa/2})^i\leqslant C_5e^{(m-\varkappa l)/2}, \tag{60}$$

where

$$C_5:=C_4/(1-e^{-\varkappa/2})=\sqrt{C_1}(e^{1/2}+1)/(1-e^{-\varkappa/2}). \tag{61}$$

From (60) it follows that $\langle\, u_{l+r}-u_l\rangle_m\to 0$ as $l\to+\infty$ uniformly by $r\in\mathbb{N}$, that is, $\{\partial_i u_l\}$, $i=\overline{0,k}$, are Cauchy sequences in space $L_2(\Omega^m)$, where $m\in\mathbb{N}$ is an arbitrary fixed. Hence, there exists a function $u\in L_{2,\,\mathrm{loc}}(\overline{\Omega})$ such that $\partial_i u\in L_{2,\mathrm{loc}}(\overline{\Omega})$, $i=\overline{1,k}$, and

$$\partial_i u_l\underset{l\to\infty}{\longrightarrow}\partial_i u\quad\text{strongly in}\quad L_{2,\,\mathrm{loc}}(\overline{\Omega}),\quad i=\overline{0,k}. \tag{62}$$

Taking into account $(\mathbf{A}_3)$ (see (9)), from (62) we get

$$a_i(u_l)\underset{l\to\infty}{\longrightarrow}a_i(u)\quad\text{strongly in}\quad L_{2,\,\mathrm{loc}}(\overline{\Omega}),\quad i=\overline{1,k}. \tag{63}$$

Assume $m \in \mathbb{N}$ is an arbitrary fixed number, and $l \in \mathbb{N}$ is an arbitrary number such that $l \geqslant m$. Putting $\psi(x) := u_l(x)\psi_{1/2}(x', \tau(m))$, $x = (x', x'') \in \overline{\Omega}$, from (50) we obtain

$$\int\limits_{\Omega^m} \sum_{i=0}^{n} a_i(u_l)\partial_i u_l \psi_{1/2}\, dx = - \int\limits_{\Omega^m} \sum_{i=1}^{k} a_i(u_l) u_l\, \partial_i \psi_{1/2}\, dx + \int\limits_{\Omega^m} f u_l \psi_{1/2}\, dx. \qquad (64)$$

Estimating the terms of (64) with conditions $(\mathbf{A}_1)$, $(\mathbf{A}_3)$ (see (9)), $(\mathbf{A}_4)$, (34) and Cauchy-Bunyakovsky-Schvartz inequality, we get

$$\int\limits_{\Omega_{\tau(m)-1/2}} q_3 \sum_{i=k+1}^{n} |\partial_i u_l|^{p_i(x)}\, dx \leqslant 2 \int\limits_{\Omega^m} \big(g_1 |\nabla_k u_l| |u_l| + g_2 |u_l|^2\big)\, dx + \int\limits_{\Omega^m} \big(|f||u_l| + |h|\big)\, dx \leqslant$$

$$\leqslant C_6 \Big(\int\limits_{\Omega^m} \sum_{i=0}^{k} |\partial_i u_l|^2\, dx + \int\limits_{\Omega^m} \big(|f|^2 + |h|\big)\, dx \Big), \qquad (65)$$

where constant $C_6 > 0$ is independent of l, but it may be depended on m.

From (65) we obtain

$$\int\limits_{\Omega_{\tau(m)-1/2}} \sum_{i=k+1}^{n} |\partial_i u_l|^{p_i(x)}\, dx \leqslant C_7, \quad m, l \in \mathbb{N},\ l \geqslant m, \qquad (66)$$

where constant $C_7 > 0$ is independent of l, but it may be depended on m.

By virtue of $(\mathbf{A}_2)$, (62), (66), and discrete Hölder inequality we deduce that for every $i \in \{0, k+1, \ldots, n\}$ and arbitrary $m, l \in \mathbb{N}$, $l \geqslant m$,

$$\int\limits_{\Omega_{\tau(m)-1/2}} |a_i(u_l)|^{p_i'(x)}\, dx \leqslant \int\limits_{\Omega_{\tau(m)-1/2}} \Big| h_{i,1} \sum_{j=0}^{n} |\partial_j u_l|^{p_j(x)/p_i'(x)} + h_{i,2} \Big|^{p_i'(x)}\, dx \leqslant$$

$$\leqslant C_8 \int\limits_{\Omega_{\tau(m)-1/2}} \sum_{j=0}^{n} |\partial_j u_l|^{p_j(x)}\, dx + C_9 \leqslant C_{10}, \qquad (67)$$

where positive constants C_8, C_9, C_{10} are independent of l, but they may be depended on m.

In view of (66), (67), and the reflexivity of spaces $L_{p_i(\cdot)}(\Omega_\tau)$, $L_{p_i'(\cdot)}(\Omega_\tau)$, $i = \overline{k+1, n}$, $\tau > 0$, it follows that there exists a subsequence of the sequence $\{u_l\}_{l=1}^{\infty}$ (without loss of generality we use the notation $\{u_l\}_{l=1}^{\infty}$ for this subsequence), and functions $\chi_0 \in L_{2,\mathrm{loc}}(\overline{Q})$, $\chi_i \in L_{p_i'(\cdot),\mathrm{loc}}(\overline{\Omega})$, $i = \overline{k+1, n}$, such that

$$\partial_i u_l \underset{l\to\infty}{\longrightarrow} \partial_i u \quad \text{weakly in } L_{p_i(\cdot),\mathrm{loc}}(\overline{\Omega}), \quad i = \overline{k+1, n}, \qquad (68)$$

$$a_0\big(u_l\big) \underset{l\to\infty}{\longrightarrow} \chi_0 \quad \text{weakly in } L_{2,\mathrm{loc}}(\overline{\Omega}), \qquad (69)$$

$$a_i\big(u_l\big) \underset{l\to\infty}{\longrightarrow} \chi_i \quad \text{weakly in } L_{p_i'(\cdot),\mathrm{loc}}(\overline{\Omega}), \quad i = \overline{k+1, n}. \qquad (70)$$

Put

$$\chi_i := a_i(u), \quad i = \overline{1, k}. \qquad (71)$$

Remark that for every $l \in \mathbb{N}$ we have the identity (51). In (51) we fix an arbitrary $\psi \in \widetilde{W}^1_{p(\cdot),\mathrm{c}}(\Omega)$, and pass to the limit as $l \to \infty$, taking into account (63), (69)–(71). So, we get

$$\int\limits_\Omega \sum_{i=0}^n \chi_i \partial_i \psi \, dx = \int\limits_\Omega f\psi \, dx. \tag{72}$$

To conclude that u is a weak solution of problem (1), (2). It remains to show that the following identity holds

$$\int\limits_\Omega \sum_{i=0}^n \chi_i \partial_i \psi \, dx = \int\limits_\Omega \sum_{i=0}^n a_i(u) \partial_i \psi \, dx \quad \forall\, \psi \in \widetilde{W}^1_{p(\cdot),\mathrm{c}}(\Omega). \tag{73}$$

Indeed, if (73) is true, then from this and (72) we obtain the integral identity (8). In view of (62), (68) we have that $u \in \widetilde{W}^1_{p(\cdot),\,\mathrm{loc}}(\overline{\Omega})$. Hence, the function u is a weak solution of problem (1), (2).

Step 3 (the correctness of identity (73)). To verify the correctness of identity (73) we use the monotonicity method [16].

Let $v \in L_{2,\mathrm{loc}}(\overline{\Omega})$ be an arbitrary function such that $\partial_i v \in L_{p_i(\cdot),\,\mathrm{loc}}(\overline{\Omega})$, $i = \overline{1,n}$, let $\zeta(x')$, $x' = (x_1, \ldots, x_k) \in \mathbb{R}^k$, be a nonnegative continuously differentiable function with bounded support. By virtue of condition $(\mathbf{A}_3)$ (see (10), (11)), for every $l \in \mathbb{N}$ we have

$$\int\limits_\Omega \sum_{i=0}^n (a_i(u_l) - a_i(v))(\partial_i u_l - \partial_i v)\zeta \, dx \geqslant 0. \tag{74}$$

We rewrite inequality (74) as

$$\int\limits_\Omega \sum_{i=0}^n a_i(u_l)\partial_i u_l \zeta \, dx - \int\limits_\Omega \sum_{i=0}^n \big(a_i(u_l)\partial_i v + a_i(v)(\partial_i u_l - \partial_i v)\big)\zeta \, dx \geqslant 0 \quad \forall\, l \in \mathbb{N}. \tag{75}$$

Assume $m \in \mathbb{N}$ such that $\operatorname{supp}\zeta \subset \{x' \mid |x'| \leqslant \tau(m)\}$. From identity (51) we obtain

$$\int\limits_\Omega \sum_{i=0}^n a_i(u_l)\partial_i u_l \zeta \, dx = -\int\limits_\Omega \sum_{i=1}^k a_i(u_l) u_l \partial_i \zeta \, dx + \int\limits_\Omega f u_l \zeta \, dx, \quad l \geqslant m. \tag{76}$$

From (75) and (76) we get

$$-\int\limits_\Omega \sum_{i=1}^k a_i(u_l) u_l \partial_i \zeta + \int\limits_\Omega f u_l \zeta \, dx - \int\limits_\Omega \left[\sum_{i=0}^n \big(a_i(u_l)\partial_i v + a_i(v)(\partial_i u_l - \partial_i v)\big)\right]\zeta \, dx \geqslant 0. \tag{77}$$

In (77) we pass to the limit as $l \to \infty$, and by virtue of (62), (63), (69)–(71) we infer

$$-\int\limits_\Omega \sum_{i=1}^k \chi_i u \partial_i \zeta + \int\limits_\Omega f u \zeta \, dx - \int\limits_\Omega \left[\sum_{i=0}^n \big(\chi_i \partial_i v + a_i(v)(\partial_i u - \partial_i v)\big)\right]\zeta \, dx \geqslant 0. \tag{78}$$

From (72) it follows next equality

$$\int\limits_\Omega \sum_{i=0}^{n} \chi_i \partial_i u \, \zeta \, dx = -\int\limits_\Omega \sum_{i=1}^{k} \chi_i u \partial_i \zeta \, dx + \int\limits_\Omega f u \zeta \, dx. \tag{79}$$

Assertions (78) and (79) imply

$$\int\limits_\Omega \sum_{i=0}^{n} \chi_i \, \partial_i u \zeta \, dx - \int\limits_\Omega \sum_{i=0}^{n} \big(\chi_i \partial_i v + a_i(v)(\partial_i u - \partial_i v)\big)\zeta \, dx \geqslant 0,$$

that is,

$$\int\limits_\Omega \sum_{i=0}^{n} (\chi_i - a_i(v))(\partial_i u - \partial_i v)\zeta \, dx \geqslant 0. \tag{80}$$

In (80) we put $v = u - \lambda\psi$, where λ is an arbitrary number, and $\psi \in \widetilde{W}^1_{p(\cdot),\mathrm{c}}(\Omega)$ is arbitrary function. So, taking into account the arbitrariness of λ, we obtain the equality

$$\int\limits_\Omega \sum_{i=0}^{n} \big(\chi_i - a_i(u - \lambda\psi\varphi)\big)\partial_i \psi \zeta \, dx = 0.$$

Here we tend λ to 0, using conditions $(\mathbf{A}_1)$, $(\mathbf{A}_2)$, and Lebesgue dominated convergence theorem. Thus, taking into account the arbitrariness of ζ, we deduce

$$\int\limits_\Omega \sum_{i=0}^{n} \big(\chi_i - a_i(u)\big)\partial_i \psi \, dx = 0 \quad \forall \psi \in \widetilde{W}^1_{p(\cdot),\mathrm{c}}(\Omega). \tag{81}$$

From (81) it follows (73).

Step 4 (the solution's estimate). Estimate (25) is obtained from (56), (60) and (61) by this way:

$$\langle u \rangle_m \leqslant \langle u - u_m \rangle_m + \langle u_m \rangle_m = \lim_{l \to \infty} \langle u_l - u_m \rangle_m + \langle u_m \rangle_m \leqslant C_2 e^{(1-\varkappa)m/2},$$

where $C_2 := \sqrt{C_1} + C_5 = (2 + e^{1/2} - e^{-\varkappa/2})/(1 - e^{-\varkappa/2})\sqrt{C_1}$.

Now it is easy to see that the function u satisfies (22). Indeed, let $R > 0$ be an arbitrary number, and m be a natural number such that $m - 1 < R \leqslant m$. Using (25), we get

$$\langle u \rangle_R \leqslant \langle u \rangle_m \leqslant C_2 e^{(1-\varkappa)m/2} = C_2 e^{(1-\varkappa)(m-R)/2} e^{(1-\varkappa)R/2} \leqslant$$

$$\leqslant C_2 e^{(1-\varkappa)/2} e^{-\varkappa R/2} e^{R/2} = \beta(R) e^{R/2}, \quad R \geqslant 1,$$

where $\beta(R) := C_2 e^{(1-\varkappa)/2} e^{-\varkappa R/2}$. Since $\beta(R) \to 0$ as $R \to +\infty$, then we have (22).

So, we shown that u is a weak solution of problem (1), (2) that satisfies (22) and (25). Theorem 2 is proved. □

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КРАЙОВІ ЗАДАЧІ ДЛЯ ЕЛІПТИЧНИХ РІВНЯНЬ В НЕОБМЕЖЕНИХ ОБЛАСТЯХ З УМОВАМИ НА НЕСКІНЧЕННОСТІ

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В статті досліджена крайова задача для еліптичних рівнянь другого порядку, заданих в необмежених областях. В класі досліджуваних рівнянь, крім лінійних, є нелінійні зі змінними показниками нелінійності. Встановлено існування та єдиність слабких розв'язків досліджуваних задач при додаткових умовах на їх поведінку і зростання вхідних даних на нескінченності. Отримано апріорні оцінки слабких розв'язків досліджуваних задач. В дослідженні використовується аналог відомого з механіки принципу Сен-Венана і метод монотонності.

Ключові слова: еліптичне рівняння, змінний показник нелінійності, простір Лебега зі змінним показником сумовності, простір Соболєва зі змінним показником сумовності, необмежена область, принцип Сен-Венана, метод монотонності.