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ON CERTAIN CLASSES OF SERIES IN SYSTEMS OF FUNCTIONS

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Let $f(z) = z^p - \sum_{k=1}^{\infty} f_k z^{k+p}$ with $f_k > 0$ be an entire transcendental function and (λ_n) be a sequence of positive numbers increasing to $+\infty$. Suppose that the series $A(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z)$ with $a_n > 0$ regularly converges in $\mathbb{D} = \{z : |z| < 1\}$. For $p \in \mathbb{N}$, $\alpha \ge 0$ and $0 \le \beta < p$ by $\mathfrak{F}_p(\alpha, \beta)$ denote the class of an analytic in $\mathbb{D}$ functions $g(z) = z^p - \sum\limits_{n=1}^{\infty} g_n z^{n+p}$ with $g_n > 0$ such that $\text{Re}\left\{(1-\alpha)\dfrac{g(z)}{z^p} + \alpha\dfrac{g'(z)}{pz^{p-1}}\right\} > \dfrac{\beta}{p}$ for all $z \in \mathbb{D}$, and say that $g \in \mathfrak{G}_p(\alpha, \beta)$ if $\text{Re}\left\{(p + \alpha(1-p))\dfrac{g'(z)}{pz^{p-1}} + \alpha\dfrac{g''(z)}{pz^{p-2}}\right\} > \beta$ for all $z \in \mathbb{D}$. Conditions under which the function A belongs either to $\mathfrak{F}_p(\alpha, \beta)$ or to $\mathfrak{G}_p(\alpha, \beta)$ are investigated.

Key words: regularly converging series, analytic functions in the unit disk.

1. **Introduction**

Let $p \in \mathbb{N}$, $\alpha \ge 0$ and $0 \le \beta < p$. We say that an analytic in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ function

$$g(z) = z^p - \sum_{n=1}^{\infty} g_n z^{n+p}, \quad g_n > 0, \tag{1}$$

belongs to the class $\mathfrak{F}_p(\alpha, \beta)$ if and only if

$$\text{Re}\left\{(1-\alpha)\frac{g(z)}{z^p} + \alpha\frac{g'(z)}{pz^{p-1}}\right\} > \frac{\beta}{p} \quad (z \in \mathbb{D}), \tag{2}$$

2020 Mathematics Subject Classification: 30B50, 30C45 and belongs to the class $\mathfrak{G}_p(\alpha,\beta)$ if and only if

$$\mathrm{Re}\left\{(p+\alpha(1-p))\frac{g'(z)}{pz^{p-1}}+\alpha\frac{g''(z)}{pz^{p-2}}\right\}>\beta \quad (z\in\mathbb{D}). \tag{3}$$

It is clear that $g\in\mathfrak{G}_p(\alpha,\beta)$ if and only if $zg'(z)/p\in\mathfrak{F}_p(\alpha,\beta)$.

The class $\mathfrak{F}_p(\alpha,\alpha)$ was introduced and studied earlier by S.K. Lee, S. Owa and H.M. Srivastava [1] and was further investigated by M.K. Aouf and H.E. Darwish [2]. The class $\mathfrak{G}_p(\alpha,\beta)$ was studied recently by M.K. Aouf [3]. In particular, the class $\mathfrak{G}_1(\alpha,\beta)$ was considered earlier by O. Altintas [4], [5]. The research of classes $\mathfrak{F}_p(\alpha,\beta)$ and $\mathfrak{G}_p(\alpha,\beta)$ was continued in the works [6], [7]. In [8] analogues of the classes $\mathfrak{F}_p(\alpha,\beta)$ and $\mathfrak{G}_p(\alpha,\beta)$ have been studied for Dirichlet series absolutely convergent in the left half-plane.

Let $f(z)=\sum_{k=0}^{\infty}f_kz^k$ be an entire transcendental function, (λ_n) be a sequence of positive numbers increasing to $+\infty$ and the series

$$A(z)=\sum_{n=1}^{\infty}a_nf(\lambda_nz). \tag{4}$$

regularly converges in $\{z:\,|z|<R[A]\}$, i.e. for all $r\in[0,R[A])$

$$\sum_{n=1}^{\infty}|a_n|M_f(r\lambda_n)<+\infty,\quad M_f(r)=\max\{|f(z)|:\,|z|=r\}.$$

It is clear that many functional series arising in various sections of the analysis can be written as series by a system of functions $\{f(\lambda_nz)\}$. In particular, in articles [9, 10, 11, 12] B.V. Vinnitskii investigated under the most general conditions on a function f itself and on the sequence (λ_n), both the basicity of this system of functions and the properties of series on this system. In [18, 19] there were obtained the conditions under which, for series of the form (2), as well as integrals of the form $\int_0^{+\infty}a(t)f(tx)\nu(dt)$ that are a generalization of such series, the Borel-type asymptotic relation holds outside some set of finite Lebesgue measure, where f is a positive functions on $(0,+\infty)$ such that the function $\ln f(x)$ is convex on $(0,+\infty)$.

This article continues the study of the properties series of form (2), which was started by the author in articles [13, 14, 15, 16, 17].

In the end, modern e-search systems will allow the reader to easily find both other articles about the series on this general system of functions, and on the specific systems of functions, such as the Mittag-Leffler functions, the Bessel functions, and many others.

The starlikeness and the convexity of the function (4) in $\mathbb{D}$ are studied in [16]. In the proposed note we will assume that $R[A]\ge1$ and find the conditions under which the function A belongs to the class $\mathfrak{F}_p(\alpha,\beta)$ and, consequently, to the class $\mathfrak{G}_p(\alpha,\beta)$.

2. **Preliminary result.**

Now suppose that all $a_n>0$ and

$$f(z)=z^p-\sum_{k=1}^{\infty}f_kz^{k+p},\quad f_k>0. \tag{5}$$

Then

$$A(z)=\sum_{n=1}^{\infty}a_n\left(\lambda_n^pz^p-\sum_{k=1}^{\infty}f_k\lambda_n^{k+p}z^{k+p}\right)=\sum_{n=1}^{\infty}a_n\lambda_n^pz^p-\sum_{k=1}^{\infty}f_k\left(\sum_{n=1}^{\infty}a_n\lambda_n^{k+p}\right)z^{k+p}=$$

$$= \left(\sum_{n=1}^{\infty} a_n\lambda_n^p\right)\left(z^p - \sum_{k=1}^{\infty}\frac{f_k\sum\limits_{n=1}^{\infty} a_n\lambda_n^{k+p}}{\sum\limits_{n=1}^{\infty} a_n\lambda_n^p} z^{k+p}\right) = bg(z),$$

where $b = \sum\limits_{n=1}^{\infty} a_n\lambda_n^p$ and the function g is represented by a power series (1) with coefficients

$$g_k = \frac{f_k}{b}\sum_{n=1}^{\infty} a_n\lambda_n^{k+p}. \tag{6}$$

Therefore, in view of (2) and (3) it follows that $A(z)/b \in \mathfrak{F}_p(\alpha,\beta)$ if and only if $g \in \mathfrak{F}_p(\alpha,\beta)$, and $A(z)/b \in \mathfrak{G}_p(\alpha,\beta)$ if and only if $g \in \mathfrak{G}_p(\alpha,\beta)$.

The following lemma is known [1].

Lemma 1. *Function* (1) *belongs to* $\mathfrak{F}_p(\alpha,\beta)$ *if and only if*

$$\sum_{k=1}^{\infty}(p+\alpha k)g_k < p-\beta,$$

and belongs to $\mathfrak{G}_p(\alpha,\beta)$ *if and only if*

$$\sum_{k=1}^{\infty}(p+\alpha k)(p+k)g_k < p(p-\beta).$$

Therefore, we need to estimate the coefficients given by formula (6). Let $\tau \in (0,+\infty)$ be the index of convergence of the sequence (λ_n), i.e.

$$Q := \sum_{n=1}^{\infty}\frac{1}{\lambda_n^{\omega}} < +\infty \tag{7}$$

for every $\omega \in (\tau,+\infty)$. Then

$$\sum_{n=1}^{\infty}|a_n|\lambda_n^{k+p} = \sum_{n=1}^{\infty}\frac{|a_n|\lambda_n^{k+p+\omega}}{\lambda_n^{\omega}} \le Q\max\{|a_n|\lambda_n^{k+p+\omega} : n \ge 1\}.$$

Denote by Ω a class of positive unbounded on $(-\infty,+\infty)$ functions Φ such that the derivative Φ' is positive, continuously differentiable and increasing to $+\infty$ on $(-\infty,+\infty)$. Let φ be the function inverse to Φ' and $\Psi(x) = x-\Phi(x)/\Phi'(x)$ be the function associated with Φ in the sense of Newton. Suppose that the Dirichlet series $D(\sigma) = \sum\limits_{n=1}^{\infty}|a_n|\exp\{\mu_n\sigma\}$ with $0 \le \mu_n \uparrow +\infty$ converges for all $\sigma < +\infty$, and let $\mu(\sigma) = \max\{|a_n|\exp\{\mu_n\sigma\} : n \ge 2\}$ be the maximal term. Then [20] in order that $\ln\,\mu(\sigma) \le \Phi(\sigma) \in \Omega$ for all σ it is necessary and sufficient that $\ln\,|a_n| \le -\mu_n\Psi(\varphi(\mu_n))$ for all n. Therefore, if we put $\mu_n = \ln\,\lambda_n$ and $\sigma = k+p+\omega$ then for hence we obtain $\max\{|a_n|\lambda_n^{k+p+\omega} : n \ge 2\} \le e^{\Phi(k+p+\omega)}$ provided $\lambda_n \ge 1$ and $\ln\,|a_n| \le -\ln\,\lambda_n\Psi(\varphi(\ln\,\lambda_n))$ for all $n \ge 2$.

Thus, the following statement is true.

Lemma 2. *Let* $\Phi \in \Omega$, $\lambda_n \ge 1$ *for* $n \ge 1$, $\ln\,|a_n| \le -\ln\,\lambda_n\Psi(\varphi(\ln\,\lambda_n))$ *for* $n \ge 1$ *and* (7) *holds. Then for the coefficients* g_k, *given by equality* (6), *there is a correct estimate*

$$|g_k| \le \frac{Qf_k}{b}e^{\Phi(k+p+\omega)}, \quad k \ge 1. \tag{8}$$

3. Main theorem and corollaries.

The following theorem is true,

Theorem 1. *Let $\Phi \in \Omega$, $\lambda_n \geq 1$, $\ln|a_n| \leq -\ln\lambda_n \Psi(\varphi(\ln\lambda_n))$ for $n \geq 1$ and (7) holds. Suppose that the function f is given by power series (5). If*

$$\sum_{k=1}^{\infty}(p+\alpha k)f_k e^{\Phi(k+p+\omega)} < \frac{(p-\beta)b}{Q}, \quad b = \sum_{n=1}^{\infty} a_n\lambda_n^p, \tag{9}$$

then $A/b \in \mathfrak{F}_p(\alpha,\beta)$, and if

$$\sum_{k=1}^{\infty}(p+\alpha k)(p+k)f_k e^{\Phi(k+p+\omega)} < \frac{p(p-\beta)b}{Q}\sum_{n=1}^{\infty} a_n\lambda_n \tag{10}$$

then $A/b \in \mathfrak{G}_p(\alpha,\beta)$.

Indeed, from (8) and (9) we obtain

$$\sum_{k=1}^{\infty}(p+\alpha k)g_k \leq \sum_{k=1}^{\infty}(p+\alpha k)\frac{Qf_k}{b}e^{\Phi(k+p+\omega)} \leq p-\beta.$$

Therefore, $g \in \mathfrak{F}_p(\alpha,\beta)$ and, thus, $A/b \in \mathfrak{F}_p(\alpha,\beta)$. Similarly, from (8) and (10) it follows that $g \in \mathfrak{G}_p(\alpha,\beta)$ and $A/b \in \mathfrak{G}_p(\alpha,\beta)$.

Let us consider several consequences of Theorem 1. At first let $\Phi(x) = e^x$ for $x > 0$. Then $\Psi(x) = x - 1$, $\varphi(x) = \ln x$ and $x\Psi(\varphi(x)) = x\ln(x/e)$. Therefore, we get the following statement.

Corollary 1. *Let $\lambda_n > 1$ and $\ln|a_n| \leq -\ln\lambda_n \ln\left(\frac{\ln\lambda_n}{e}\right)$ for $n \geq 1$, (7) holds and the function f is given by power series (5). If*

$$\sum_{k=1}^{\infty}(p+\alpha k)f_k \exp\{e^{k+p+\omega}\} < \frac{(p-\beta)b}{Q}$$

then $A/b \in \mathfrak{F}_p(\alpha,\beta)$, and if

$$\sum_{k=1}^{\infty}(p+\alpha k)(p+k)f_k \exp\{e^{k+p+\omega}\} < \frac{p(p-\beta)b}{Q}$$

then $A/b \in \mathfrak{G}_p(\alpha,\beta)$.

Now let $\Phi(x) = x^\varrho$ for $x > 1$, where $\varrho > 1$. Then $\Psi(x) = \frac{\varrho-1}{\varrho}x$, $\varphi(x) = (x/\varrho)^{1/(\varrho-1)}$ and $x\Psi(\varphi(x)) = (\varrho-1)(x/\varrho)^{\varrho/(\varrho-1)}$. Therefore, we get the following statement.

Corollary 2. *Let $\lambda_n > 1$ and $\ln|a_n| \leq -(\varrho-1)((\ln\lambda_n)/\varrho)^{\varrho/(\varrho-1)}$ for $n \geq 1$, (7) holds and the function f is given by power series (5). If*

$$\sum_{k=1}^{\infty}(p+\alpha k)f_k \exp\{(k+p+\omega)^\varrho\} < \frac{(p-\beta)b}{Q}$$

then $A/b \in \mathfrak{F}_p(\alpha,\beta)$*, and if*

$$\sum_{k=1}^{\infty}(p+\alpha k)(p+k)f_k\exp\{(k+p+\omega)^{\varrho}\} < \frac{p(p-\beta)b}{Q}$$

then $A/b \in \mathfrak{G}_p(\alpha,\beta)$.

Finally, consider the most interesting case when $\Phi(x) = x\ln x$ for $x > e$. Then $\varphi(x) = e^{x-1}$ and $x\Psi(\varphi(x)) = x\varphi(x) - \Phi(\varphi(x)) = e^{x-1}$. Therefore, we get the following statement.

Corollary 3. *Let* $\lambda_n \ge 1$ *and* $\ln|a_n| \le -e\lambda_n$ *for* $n \ge 1$*,* (7) *holds and the function* f *is given by power series* (5)*. If*

$$\sum_{k=1}^{\infty}(p+\alpha k)f_k(k+p+\omega)^{k+p+\omega} < \frac{(p-\beta)b}{Q}$$

then $A/b \in \mathfrak{F}_p(\alpha,\beta)$*, and if*

$$\sum_{k=1}^{\infty}(p+\alpha k)(p+k)f_k(k+p+\omega)^{k+p+\omega} < \frac{p(p-\beta)b}{Q}$$

then $A/b \in \mathfrak{G}_p(\alpha,\beta)$.

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ПРО ДЕЯКІ КЛАСИ РЯДІВ ЗА СИСТЕМОЮ ФУНКЦІЙ

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Нехай функція $f(z) = z^p - \sum_{k=1}^{\infty} f_k z^{k+p}$ з $f_k > 0$ є цілою трансцендентною, а (λ_n) - зростаюча до $+\infty$ послідовність додатних чисел. Припустимо, що ряд $A(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z)$ з $a_n > 0$ регулярно збігається в $\mathbb{D} = \{z : |z| < 1\}$. Для $p \in \mathbb{N}$, $\alpha \geq 0$ і $0 \leq \beta < p$ через $\mathfrak{F}_p(\alpha, \beta)$ позначимо такий клас аналітичних в $\mathbb{D}$ функцій $g(z) = z^p - \sum\limits_{n=1}^{\infty} g_n z^{n+p}$ з $g_n > 0$, що $\mathrm{Re}\left\{(1-\alpha)\dfrac{g(z)}{z^p} + \alpha\dfrac{g'(z)}{pz^{p-1}}\right\} > \dfrac{\beta}{p}$ для всіх $z \in \mathbb{D}$, і будемо говорити, що $g \in \mathfrak{G}_p(\alpha, \beta)$, якщо $\mathrm{Re}\left\{(p + \alpha(1-p))\dfrac{g'(z)}{pz^{p-1}} + \alpha\dfrac{g''(z)}{pz^{p-2}}\right\} > \beta$ для всіх $z \in \mathbb{D}$. Досліджено, за яких умов функція A належить до $\mathfrak{F}_p(\alpha, \beta)$, або до $\mathfrak{G}_p(\alpha, \beta)$.

Ключові слова: регулярно збіжний ряд, аналітична в одиничному крузі функція.