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# ANALOGUES OF WARING-GIRARD FORMULAS FOR THE BLOCK-SYMMETRIC POLYNOMIALS ON THE SPACES $\ell_1(\mathbb{C}^2)$ AND $\ell_p(\mathbb{C}^2)$

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Classical Waring-Girard formulas gives a representation of elementary and complete symmetric polynomials though the power symmetric polynomials. In this paper, we propose an analog of the Waring-Girard formulas for block-symmetric polynomials on the spaces $\ell_1(\mathbb{C}^2)$ and $\ell_p(\mathbb{C}^2)$ and show the application of the obtained formula in combinatorics.

*Key words:* symmetric polynomials, block-symmetric polynomials, Waring-Girard formulas.

## 1. Introduction

Symmetric functions on finite-dimensional vector spaces are standard objects in combinatorics and classical invariant theory (see e.g. [1]). For infinite-dimensional spaces, investigations of symmetric polynomials started by Nemirovski and Semenov in [2]. In particular, in [2] the authors constructed algebraic bases of algebras of symmetric real-valued polynomials on real Banach spaces $\ell_p$ and $L_p[0,1]$ for $1 \leqslant p < \infty$. In [3], these results were generalized to separable sequence real Banach spaces with symmetric bases and to separable rearrangement invariant real Banach spaces respectively.

Block-symmetric or MacMahon polynomials are natural generalizations of symmetric polynomials and can be considered as symmetric polynomials on vector-sequences linear spaces. Combinatorial properties of such polynomials are described in [4]. An algebraic basis of the algebra of all symmetric continuous complex-valued polynomials on the Cartesian power of the complex Banach space $\ell_p$ for some fixed $1 \leq p < \infty$ was

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*2020 Mathematics Subject Classification*: 46J15, 46E10, 46E50 constructed in [5]. Algebras of symmetric continuous polynomials on Cartesian products $\ell_{p_1} \times \cdots \times \ell_{p_s}$ for different $p_1, \ldots, p_s$ were considered in [6]. Some generalizations of the Newton formulas for algebraic bases of block-symmetric polynomials were obtained in [7, 8].

### 1.1. **Symmetric polynomials**

Let $X$ be a complex Banach spaces and $S_{\mathbb{N}}$ is the semigroup of all permutations on the set of all natural numbers $\mathbb{N}$. A mapping $f$ on $X$ is said to be *symmetric* if it is $S_{\mathbb{N}}$-symmetric (or just symmetric), that is,

$$f(x_1, x_2, \ldots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots)$$

for each $\sigma \in S_{\mathbb{N}}$. We denote by $\mathcal{P}_s(X)$ the algebra of all continuous symmetric polynomials on $X$.

It is known that polynomials

$$F_k(x) = \sum_{n=1}^{\infty} x_n^k, \quad k \in \mathbb{N},$$

form an algebraic basis in the algebra $\mathcal{P}_s(\ell_1)$ [3]. That is, for any polynomials $P \in \mathcal{P}_s(\ell_1)$ there is a unique polynomial of several complex variables $Q(t_1, \ldots, t_m)$ such that $P(x) = Q(F_1(x), \ldots, F_m(x))$. Polynomials $F_k$ are called *power* symmetric polynomials. The algebraic basis is not unique, and we will use also the following bases in $\mathcal{P}_s(\ell_1)$:

$$G_n(x) = \sum_{i_1 < \ldots < i_n} x_{i_1} \cdots x_{i_n},$$

which is called the *basis of elementary symmetric polynomials* and

$$H_n(x) = \sum_{i_1 \leqslant \ldots \leqslant i_n} x_{i_1} \cdots x_{i_n},$$

which is called the *basis of complete symmetric polynomials*. These bases are connected by known Newton formulas :

$$nG_n = \sum_{k=1}^{n} (-1)^{k-1} G_{n-k} F_k, \quad n \in \mathbb{N}, \tag{1}$$

$$nH_n = \sum_{k=1}^{n} H_{n-k} F_k, \quad n \in \mathbb{N}. \tag{2}$$

We denote by $\mathbb{Z}_+$ the set of all nonnegative integers. It is well-known that the elementary and complete symmetric polynomials can be expressed in terms of the power symmetric polynomials (see e.g. [9], p.6-7) by the Waring-Girard formulas:

$$G_n = \sum_{\lambda_1 + 2\lambda_2 + \ldots + n\lambda_n = n} \frac{(-1)^{n + (\lambda_1 + \lambda_2 + \ldots + \lambda_n)}}{1^{\lambda_1} \cdot 2^{\lambda_2} \cdot \ldots \cdot n^{\lambda_n} \lambda_1! \cdot \lambda_2! \cdot \ldots \cdot \lambda_n!} (F_1)^{\lambda_1} (F_2)^{\lambda_2} \ldots (F_n)^{\lambda_n} \tag{3}$$

and

$$H_n = \sum_{\lambda_1 + 2\lambda_2 + \ldots + n\lambda_n = n} \frac{1}{1^{\lambda_1} \cdot 2^{\lambda_2} \cdot \ldots \cdot n^{\lambda_n} \lambda_1! \cdot \lambda_2! \cdot \ldots \cdot \lambda_n!} (F_1)^{\lambda_1} (F_2)^{\lambda_2} \ldots (F_n)^{\lambda_n}, \tag{4}$$

where $\lambda_j \in \mathbb{Z}_+, j = 1, \ldots, n$.

In [10] was defined the Waring-Girard formulas in the case of space $\ell_p, p>1$ :

$$G_n^{(p)} = \sum_{p\lambda_p+\ldots+n\lambda_n=n} \frac{(-1)^{n+(\lambda_p+\ldots+\lambda_n)}}{p^{\lambda_p}\cdot\ldots\cdot n^{\lambda_n}\lambda_p!\cdot\ldots\cdot\lambda_n!}(F_p)^{\lambda_p}\ldots(F_n)^{\lambda_n}.$$

and

$$H_n^{(p)} = \sum_{p\lambda_p+\ldots+n\lambda_n=n} \frac{1}{p^{\lambda_p}\cdot\ldots\cdot n^{\lambda_n}\lambda_p!\cdot\ldots\cdot\lambda_n!}(F_p)^{\lambda_p}\ldots(F_n)^{\lambda_n}.$$

1.2. **Block-Symmetric Polynomials**

We denote by $\ell_p(\mathbb{C}^s)=\ell_p(\mathbb{N},\mathbb{C}^s)$, $1\le p<\infty$ the linear space of all sequences

$$x=(x_1,x_2,\ldots,x_j,\ldots), \tag{5}$$

such that $x_j=(x_j^{(1)},\ldots,x_j^{(s)})\in\mathbb{C}^s$ for $j\in\mathbb{N}$, and the series $\sum\limits_{j=1}^{\infty}\sum\limits_{r=1}^{s}\left|x_j^{(r)}\right|^p$ converges. We will use also representation

$$x=\left(\begin{pmatrix} x_1^{(1)}\\ x_1^{(2)}\\ \vdots\\ x_1^{(s)}\end{pmatrix},\begin{pmatrix} x_2^{(1)}\\ x_2^{(2)}\\ \vdots\\ x_2^{(s)}\end{pmatrix},\ldots,\begin{pmatrix} x_j^{(1)}\\ x_j^{(2)}\\ \vdots\\ x_j^{(s)}\end{pmatrix},\ldots\right)$$

for $x$. Vectors $x_j$ in (5) are called vector coordinates of $x$. The linear space $\ell_p(\mathbb{C}^s)$ endowed with the norm

$$\|x\|=\left(\sum_{j=1}^{\infty}\sum_{r=1}^{s}\left|x_j^{(r)}\right|^p\right)^{1/p}$$

is a Banach space. A polynomial $P$ on the space $\ell_p(\mathbb{C}^s)$ is called *block-symmetric (or vector-symmetric)* if:

$$P(x_1,x_2,\ldots,x_m,\ldots)=P(x_{\sigma(1)},x_{\sigma(2)},\ldots,x_{\sigma(m)},\ldots)$$

for every permutation $\sigma\in S_{\mathbb{N}}$ and $x_m\in\mathbb{C}^s$. Let us denote by $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$ the algebra of all block-symmetric polynomials on $\ell_p(\mathbb{C}^s)$.

More information about algebra $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$ can be found in [5, 6, 11] and in references therein. Note that in combinatorics, block-symmetric polynomials on finite-dimension spaces are called *MacMahon symmetric polynomials* (see [4]).

Throughout this paper, we consider multi-indexes $\mathbf{k}=(k_1,k_2,\ldots,k_s)$ with non-negative integer entries $k_1,k_2,\ldots,k_s$ and we will use the standard notations $|\mathbf{k}|=k_1+k_2+\cdots+k_s$, and $\mathbf{k}!=k_1!k_2!\cdots k_s!$.

According to [5], polynomials

$$H^{\mathbf{k}}(x)=H^{k_1,k_2,\ldots,k_s}(x)=\sum_{j=1}^{\infty}\prod_{\substack{r=1\\ |k|\ge\lceil p\rceil}}^{s}(x_j^{(r)})^{k_r}$$

form an algebraic basis in $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$, $1\le p<\infty$, where $x=(x_1,\ldots,x_m,\ldots)$ are in $\ell_p(\mathbb{C}^s)$, and $x_j=(x_j^{(1)},\ldots,x_j^{(s)})\in\mathbb{C}^s$. According to [4], these polynomials are called

The Power Sum MacMahon Symmetric Functions. For example, $H^{n,0,\ldots,0}(x) = \big(x_1^{(1)}\big)^n + \big(x_2^{(1)}\big)^n + \cdots$ .

In the case of the space $\ell_1(\mathbb{C}^s)$ there is another important algebraic basis (see [1, 4, 12]):

$$R^{\mathbf{k}}(x) = R^{k_1,k_2,\ldots,k_s}(x) = \sum_{\substack{i_1^j<\cdots<i_{k_j}^j\\ 1\leqslant j\leqslant s}}^{\infty} \prod_{j=1}^{s} x_{i_1^j}^{(j)} \ldots x_{i_{k_j}^j}^{(j)}.$$

In [4] these polynomials are called The Elementary MacMahon Symmetric Functions. For example,

$$R^{1,2,0\ldots,0}(x) = \sum_{i,\,j_1<j_2} x_i^{(1)} x_{j_1}^{(2)} x_{j_2}^{(2)}.$$

Let $\mathcal{H}(x)(t)$ and $\mathcal{R}(x)(t)$ be formal series

$$\mathcal{H}(x)(t) = \sum_{|k|=1}^{\infty} \prod_{i=1}^{s} t_i^{k_i} H^{\mathbf{k}}(x),$$

$$\mathcal{R}(x)(t) = \sum_{|k|=0}^{\infty} \prod_{i=1}^{s} t_i^{k_i} R^{\mathbf{k}}(x), \quad R^{\mathbf{0}} = 1,$$

which also are called generating function (see [4]). From [4] and [12] we know that

$$\mathcal{R}(x)(t) = \sum_{|k|=0}^{\infty} \prod_{i=1}^{s} t_i^{k_i} R^{\mathbf{k}}(x) = \prod_{i=1}^{\infty} (1 + x_i^{(1)} t_1 + \ldots + x_i^{(s)} t_s).$$

From [4] it follows that there is one more algebraic basis of homogeneous polynomials $E^{\mathbf{k}}(x)$, which can be defined from the generating function

$$\mathcal{E}(x)(t) = \sum_{|k|=0}^{\infty} \prod_{i=1}^{s} t_i^{k_i} E^{\mathbf{k}}(x) = \prod_{i=1}^{\infty} \frac{1}{1 - x_i^{(1)} t_1 - \cdots - x_i^{(s)} t_s}, \quad E^{\mathbf{0}} = 1.$$

These polynomials are called The Complete Homogeneous MacMahon Symmetric Functions.

On the other hand, each of polynomials $F_m(t_1 x^{(1)} + t_2 x^{(2)} + \cdots + t_s x^{(s)})$ and $G_m(t_1 x^{(1)} + t_2 x^{(2)} + \cdots + t_s x^{(s)})$ have a representation as a linear combination of block-symmetric polynomials $H^{\mathbf{k}}(x)$ and $R^{\mathbf{k}}(x)$, respectively. Indeed, from direct calculations,

$$G_n(t_1 x^{(1)} + t_2 x^{(2)} + \cdots + t_s x^{(s)}) = \sum_{|\mathbf{k}|=n} t_1^{k_1} t_2^{k_2} \cdots t_s^{k_s} R^{\mathbf{k}}(x) \tag{6}$$

and

$$F_n(t_1 x^{(1)} + t_2 x^{(2)} + \cdots + t_s x^{(s)}) = \sum_{|\mathbf{k}|=n} \frac{|\mathbf{k}|!}{\mathbf{k}!} t_1^{k_1} t_2^{k_2} \cdots t_s^{k_s} H^{\mathbf{k}}(x), \tag{7}$$

where $x = (x^{(1)}, \ldots, x^{(s)})$.

Let $\omega$ be the isomorphism of $\mathcal{P}_{vs}(\ell_1(\mathbb{C}^s))$ to itself defined so that $\omega(H^{\mathbf{k}}) = (-1)^{|k|+1}H^{\mathbf{k}}$ for every multi-index $\mathbf{k}$. In other words, if $P \in \mathcal{P}_{vs}(\ell_1(\mathbb{C}^s))$ is of the form

$$P(x) = Q(H^{\mathbf{k}}, H^{\mathbf{m}}, \ldots, H^{\mathbf{r}})$$

for some polynomial $Q$ of several variables, then

$$\omega(P)(x) = Q\big(\omega(H^{\mathbf{k}}), \omega(H^{\mathbf{m}}), \ldots, \omega(H^{\mathbf{r}})\big).$$

The following proposition was proved in [13].

**Proposition 1.** *For every multi-index* $\mathbf{k}$,

$$\omega(R^{\mathbf{k}}) = E^{\mathbf{k}} \quad \textit{and} \quad \omega(E^{\mathbf{k}}) = R^{\mathbf{k}}.$$

For given multi-indexes $\mathbf{k}$ and $\mathbf{q}$ we denote by $\mathbf{k} - \mathbf{q} = (k_1 - q_1, k_2 - q_2, \ldots, k_s - q_s)$. In addition, we write $\mathbf{k} \geq \mathbf{q}$ whenever $k_1 \geq q_1$, $k_2 \geq q_2$, ..., $k_s \geq q_s$. In [7, 8], it is proved the following generalization of Newton's formula (1).

$$nR^{\mathbf{k}} = \sum_{j=1}^{|\mathbf{k}|} (-1)^{j-1} \sum_{\substack{|\mathbf{q}|=j \\ \mathbf{k} \geqslant \mathbf{q}}} \frac{|\mathbf{q}|!}{\mathbf{q}!} H^{\mathbf{q}} R^{\mathbf{k}-\mathbf{q}}.$$

In [13], it is proved the following generalization of Newton's formula (2).

$$nE^{\mathbf{k}} = \sum_{j=1}^{|\mathbf{k}|} \sum_{\substack{|\mathbf{q}|=j \\ \mathbf{k} \geqslant \mathbf{q}}} \frac{|\mathbf{q}|!}{\mathbf{q}!} H^{\mathbf{q}} E^{\mathbf{k}-\mathbf{q}}.$$

## 2. Main results

Let $\ell_1(\mathbb{C}^2)$ be the linear space of all sequences

$$x = (x_1, x_2, \ldots, x_j, \ldots),$$

such that $x_j = (x_j^{(1)}, x_j^{(2)}) \in \mathbb{C}^2$ for $j \in \mathbb{N}$, and the series $\sum\limits_{j=1}^{\infty} \sum\limits_{r=1}^{2} \left| x_j^{(r)} \right|$ converges.

**Theorem 1.** *For every* $\lambda_i, \lambda_i^j, p \in \mathbb{Z}_+$, $i \in \{1, \ldots, n\}, j \in \{0, \ldots, i\}$ *we have*

$$\begin{aligned} R^{n-q,q} = &\sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} \frac{(-1)^{n+(\lambda_1+\ldots+\lambda_n)}}{1^{\lambda_1}\ldots n^{\lambda_n}\lambda_1!\ldots\lambda_n!} \times \\ &\times \sum_{\substack{\lambda_i^0+\ldots+\lambda_i^i=\lambda_i \\ 1\leqslant i\leqslant n \\ \lambda_1^0+2\lambda_2^0+\lambda_2^1+\ldots+n\lambda_n^0+\ldots+\lambda_n^{n-1}=n-q}} \prod_{i=1}^{n} \lambda_i! \prod_{j=0}^{i} \frac{\left(C_i^j\right)^{\lambda_i^j}}{\lambda_i^j!} \left(H^{i-j,j}\right)^{\lambda_i^j} \end{aligned} \tag{8}$$

*and*

$$E^{n-q,q} = \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} \frac{1}{1^{\lambda_1}\ldots n^{\lambda_n}\lambda_1!\ldots\lambda_n!}\times$$

$$\times \sum_{\substack{\lambda_i^0+\ldots+\lambda_i^i=\lambda_i \\ 1\leqslant i\leqslant n \\ \lambda_1^0+2\lambda_2^0+\lambda_2^1+\ldots+n\lambda_n^0+\ldots+\lambda_n^{n-1}=n-q}} \prod_{i=1}^n \lambda_i! \prod_{j=0}^i \frac{\left(C_i^j\right)^{\lambda_i^j}}{\lambda_i^j!}\left(H^{i-j,j}\right)^{\lambda_i^j} \tag{9}$$

*Proof.* From formulas (6) and (7) we have that

$$F_n(tx^{(1)}+x^{(2)}) = \sum_{q=0}^n t^{n-q}C_n^q H^{n-q,q}(x_1,x_2)$$

and

$$G_n(tx^{(1)}+x^{(2)}) = \sum_{q=0}^n t^{n-q}R^{n-q,q}(x_1,x_2). \tag{10}$$

From direct calculations we obtain

$$\Big(F_k(tx^{(1)}+x^{(2)})\Big)^{\lambda_k} = \left(\sum_{q=0}^k t^{k-q}C_n^q H^{k-q,q}(x_1,x_2)\right)^{\lambda_k} =$$

$$= \sum_{\lambda_k^0+\ldots+\lambda_k^k=\lambda_k} t^{k\lambda_k^0+(k-1)\lambda_k^1+\ldots+\lambda_k^{k-1}}\frac{\lambda_k!}{\lambda_k^0!\ldots\lambda_k^k!}\times \tag{11}$$

$$\times \left(C_k^0\right)^{\lambda_k^0}\ldots\left(C_k^s\right)^{\lambda_k^s}\ldots\left(C_k^k\right)^{\lambda_k^k}\left(H^{k,0}\right)^{\lambda_k^0}\ldots\left(H^{k-s,s}\right)^{\lambda_k^s}\ldots\left(H^{0,k}\right)^{\lambda_k^k}.$$

If we substitute (10) and (11) to formula (3), we obtain

$$\sum_{q=0}^n t^{n-q}R^{n-q,q}(x_1,x_2) =$$

$$= \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} \frac{(-1)^{n+(\lambda_1+\lambda_2+\ldots+\lambda_n)}}{1^{\lambda_1}\cdot 2^{\lambda_2}\cdot\ldots\cdot n^{\lambda_n}\lambda_1!\cdot\lambda_2!\cdot\ldots\cdot\lambda_n!}\times$$

$$\times \Big(F_1(tx^{(1)}+x^{(2)})\Big)^{\lambda_1}\times\ldots\times\Big(F_n(tx^{(1)}+x^{(2)})\Big)^{\lambda_n} =$$

$$= \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} \frac{(-1)^{n+(\lambda_1+\lambda_2+\ldots+\lambda_n)}}{1^{\lambda_1}\cdot 2^{\lambda_2}\cdot\ldots\cdot n^{\lambda_n}\lambda_1!\cdot\lambda_2!\cdot\ldots\cdot\lambda_n!}\times \tag{12}$$

$$\times \left(\sum_{\lambda_1^0+\lambda_1^1=\lambda_1} t^{\lambda_1^0}\frac{\lambda_1!}{\lambda_1^0!\lambda_1^1!}\left(C_1^0\right)^{\lambda_1^0}\left(C_1^1\right)^{\lambda_1^1}\left(H^{1,0}\right)^{\lambda_1^0}\left(H^{0,1}\right)^{\lambda_1^1}\right)\times\ldots\times$$

$$\times \left(\sum_{\lambda_n^0+\ldots+\lambda_n^n=\lambda_n} t^{n\lambda_n^0+(n-1)\lambda_n^1+\ldots+\lambda_n^{n-1}}\frac{\lambda_n!}{\lambda_n^0!\ldots\lambda_n^n!}\times\right.$$

$$\times \left(C_n^0\right)^{\lambda_n^0} \dots \left(C_n^s\right)^{\lambda_n^s} \dots \left(C_n^n\right)^{\lambda_n^n} \left(H^{n,0}\right)^{\lambda_n^0} \dots \left(H^{n-s,s}\right)^{\lambda_n^s} \dots \left(H^{0,n}\right)^{\lambda_n^n} \Bigg) =$$

$$= \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} \frac{(-1)^{n+(\lambda_1+\lambda_2+\ldots+\lambda_n)}}{1^{\lambda_1}\cdot 2^{\lambda_2}\cdot \ldots \cdot n^{\lambda_n}\lambda_1!\cdot\lambda_2!\cdot\ldots\cdot\lambda_n!} \times$$

$$\times \sum_{\substack{\lambda_1^0+\lambda_1^1=\lambda_1\\ \ldots \\ \lambda_n^0+\ldots+\lambda_n^n=\lambda_n}} t^{\lambda_1^0+\ldots+n\lambda_n^0+(n-1)\lambda_n^1+\ldots+\lambda_n^{n-1}} \frac{\lambda_1!}{\lambda_1^0!\lambda_1^1!} \cdots \frac{\lambda_n!}{\lambda_n^0!\ldots\lambda_n^n!} \times$$

$$\times \left(C_1^0\right)^{\lambda_1^0} \left(C_1^1\right)^{\lambda_1^1} \dots \left(C_n^0\right)^{\lambda_n^0} \dots \left(C_n^s\right)^{\lambda_n^s} \dots \left(C_n^n\right)^{\lambda_n^n} \times$$

$$\times \left(H^{1,0}\right)^{\lambda_1^0} \left(H^{0,1}\right)^{\lambda_1^1} \dots \left(H^{n,0}\right)^{\lambda_n^0} \dots \left(H^{n-s,s}\right)^{\lambda_n^s} \dots \left(H^{0,n}\right)^{\lambda_n^n}.$$

Equating multipliers at all powers of $t$, we obtain the required formula (8).
Applying the isomorphism $\omega$ to equation (8), we have

$$E^{n-q,q} = \omega(R^{n-q,q}) =$$

$$= \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} \frac{(-1)^{n+(\lambda_1+\ldots+\lambda_n)}}{1^{\lambda_1}\ldots n^{\lambda_n}\lambda_1!\ldots\lambda_n!} \times$$

$$\times \sum_{\substack{\lambda_i^0+\ldots+\lambda_i^i=\lambda_i\\ 0\leqslant i\leqslant n\\ \lambda_1^0+2\lambda_2^0+\lambda_2^1+\ldots+n\lambda_n^0+\ldots+\lambda_n^{n-1}=n-q}} \prod_{i=1}^{n} \lambda_i! \prod_{j=0}^{i} \frac{\left(C_i^j\right)^{\lambda_i^j}}{\lambda_i^j!} \left(\omega(H^{i-j,j})\right)^{\lambda_i^j} =$$

$$= \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} \frac{(-1)^{n+(\lambda_1+\ldots+\lambda_n)}}{1^{\lambda_1}\ldots n^{\lambda_n}\lambda_1!\ldots\lambda_n!} \times$$

$$\times \sum_{\substack{\lambda_i^0+\ldots+\lambda_i^i=\lambda_i\\ 0\leqslant i\leqslant n\\ \lambda_1^0+2\lambda_2^0+\lambda_2^1+\ldots+n\lambda_n^0+\ldots+\lambda_n^{n-1}=n-q}} \prod_{i=1}^{n} \lambda_i! \prod_{j=0}^{i} \frac{\left(C_i^j\right)^{\lambda_i^j}}{\lambda_i^j!} (-1)^{i\lambda_i^j} \left(H^{i-j,j}\right)^{\lambda_i^j} =$$

$$= \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} \frac{(-1)^{n+(\lambda_1+\ldots+\lambda_n)}}{1^{\lambda_1}\ldots n^{\lambda_n}\lambda_1!\ldots\lambda_n!} \times$$

$$\times \sum_{\substack{\lambda_i^0+\ldots+\lambda_i^i=\lambda_i\\ 0\leqslant i\leqslant n\\ \lambda_1^0+2\lambda_2^0+\lambda_2^1+\ldots+n\lambda_n^0+\ldots+\lambda_n^{n-1}=n-q}} (-1)^{2\lambda_1+\ldots+(n+1)\lambda_n} \prod_{i=1}^{n} \lambda_i! \prod_{j=0}^{i} \frac{\left(C_i^j\right)^{\lambda_i^j}}{\lambda_i^j!} \left(H^{i-j,j}\right)^{\lambda_i^j} =$$

$$= \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} \frac{(-1)^{n+(\lambda_1+2\lambda_2+\ldots+n\lambda_n)+2(\lambda_1+\ldots+\lambda_n)}}{1^{\lambda_1}\ldots n^{\lambda_n}\lambda_1!\ldots\lambda_n!} \times$$

$$\times \sum_{\substack{\lambda_i^0+\ldots+\lambda_i^i=\lambda_i\\ 0\leqslant i\leqslant n\\ \lambda_1^0+2\lambda_2^0+\lambda_2^1+\ldots+n\lambda_n^0+\ldots+\lambda_n^{n-1}=n-q}} \prod_{i=1}^n \lambda_i! \prod_{j=0}^i \frac{\left(C_i^j\right)^{\lambda_i^j}}{\lambda_i^j!} \left(H^{i-j,j}\right)^{\lambda_i^j} =$$

$$= \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} \frac{1}{1^{\lambda_1}\ldots n^{\lambda_n}\lambda_1!\ldots\lambda_n!}\times$$

$$\times \sum_{\substack{\lambda_i^0+\ldots+\lambda_i^i=\lambda_i\\ 0\leqslant i\leqslant n\\ \lambda_1^0+2\lambda_2^0+\lambda_2^1+\ldots+n\lambda_n^0+\ldots+\lambda_n^{n-1}=n-q}} \prod_{i=1}^n \lambda_i! \prod_{j=0}^i \frac{\left(C_i^j\right)^{\lambda_i^j}}{\lambda_i^j!} \left(H^{i-j,j}\right)^{\lambda_i^j}.$$

□

Formulas (3) and (4) are useful in combinatorics. For example, the well-known combinatorial identities

$$\sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} \frac{(-1)^{n+(\lambda_1+\lambda_2+\ldots+\lambda_n)}}{1^{\lambda_1}\cdot 2^{\lambda_2}\cdot\ldots\cdot n^{\lambda_n}\lambda_1!\cdot\lambda_2!\cdot\ldots\cdot\lambda_n!} = 0$$

and

$$\sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} \frac{1}{1^{\lambda_1}\cdot 2^{\lambda_2}\cdot\ldots\cdot n^{\lambda_n}\lambda_1!\cdot\lambda_2!\cdot\ldots\cdot\lambda_n!} = 1$$

can be obtained if we compute $G_n(e_1)$ using (3) and (4), where $e_1 = (1, 0, 0, \ldots)$. By the similar way, we can get some new relations using (8). Let

$$e_1 = \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots \right).$$

Then $R^{n-q,q}(e_1) = 0, H^{n-q,q}(e_1) = 1$ for all $n > 1, 0 \leqslant q \leqslant n$. Now we obtain new combinatorial identities:

$$\sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} \frac{(-1)^{n+(\lambda_1+\ldots+\lambda_n)}}{1^{\lambda_1}\ldots n^{\lambda_n}\lambda_1!\ldots\lambda_n!}\times$$

$$\times \sum_{\substack{\lambda_i^0+\ldots+\lambda_i^i=\lambda_i\\ 0\leqslant i\leqslant n\\ \lambda_1^0+2\lambda_2^0+\lambda_2^1+\ldots+n\lambda_n^0+\ldots+\lambda_n^{n-1}=n-q}} \prod_{i=1}^n \lambda_i! \prod_{j=0}^i \frac{\left(C_i^j\right)^{\lambda_i^j}}{\lambda_i^j!} = 0.$$

In the case of spaces $\ell_p(\mathbb{C}^2)$, where $p$ is positive integer numbers we can obtain analog of the Waring-Girard formulas. If $p$ is not integer, then we can take $\lceil p \rceil$ instead of $p$. If we put $H^{i-j,j} = 0$, for all $i \leqslant p-1, 0 \leqslant j \leqslant i$ to the formulas (8) and (9) we obtain the following analog of the Waring-Girard formulas in the case of $\ell_p(\mathbb{C}^2)$ for $p \geq 1$:

$$R_{(p)}^{n-q,q} = \sum_{p\lambda_p+\ldots+n\lambda_n=n} \frac{(-1)^{n+(\lambda_p+\ldots+\lambda_n)}}{p^{\lambda_p}\ldots n^{\lambda_n}\lambda_p!\ldots\lambda_n!} \times$$

$$\times \sum_{\substack{\lambda_i^0+\ldots+\lambda_i^i=\lambda_i \\ p\leqslant i\leqslant n \\ p\lambda_p^0+\ldots+\lambda_p^{p-1}+\ldots+n\lambda_n^0+\ldots+\lambda_n^{n-1}=n-q}} \prod_{i=p}^{n} \lambda_i! \prod_{j=0}^{i} \frac{\left(C_i^j\right)^{\lambda_i^j}}{\lambda_i^j!} \left(H^{i-j,j}\right)^{\lambda_i^j}$$

and

$$E_{(p)}^{n-q,q} = \sum_{p\lambda_p+\ldots+n\lambda_n=n} \frac{1}{p^{\lambda_p}\ldots n^{\lambda_n}\lambda_p!\ldots\lambda_n!} \times$$

$$\times \sum_{\substack{\lambda_i^0+\ldots+\lambda_i^i=\lambda_i \\ p\leqslant i\leqslant n \\ p\lambda_p^0+\ldots+\lambda_p^{p-1}+\ldots+n\lambda_n^0+\ldots+\lambda_n^{n-1}=n-q}} \prod_{i=p}^{n} \lambda_i! \prod_{j=0}^{i} \frac{\left(C_i^j\right)^{\lambda_i^j}}{\lambda_i^j!} \left(H^{i-j,j}\right)^{\lambda_i^j}.$$

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# АНАЛОГИ ФОРМУЛ ВАРІНГА-ГІРАРДА ДЛЯ БЛОЧНО-СИМЕТРИЧНИХ ПОЛІНОМІВ НА ПРОСТОРАХ $\ell_1(\mathbb{C}^2)$ ТА $\ell_p(\mathbb{C}^2)$

**Вікторія КРАВЦІВ, Марія МАРЦІНКІВ, Андрій ЯСЕЛЬСЬКИЙ**

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Класичні формули Варінга-Гірарда дають зображення елементарних і повних симетричних поліномів через степеневі симетричні поліноми. Отримано аналоги формул Варінга-Гірарда для блочно-симетричних поліномів на просторах $\ell_1(\mathbb{C}^2)$ і $\ell_p(\mathbb{C}^2)$ та запропоновано застосування отриманих формул до комбінаторики.

*Ключові слова:* симетричні поліноми, блочно-симетричні поліноми, формули Варінга-Гірарда.