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ON NON-TOPOLOGIZABLE SEMIGROUPS

Dedicated to 75th Anniversary of Professor Mykola Komarnytyskyy

Oleg GUTIK

*Ivan Franko National University of Lviv,
Universytetska Str., 1, 79000, Lviv, UKRAINE
e-mails: oleg.gutik@lnu.edu.ua*

We find anti-isomorphic submonoids $\mathcal{C}_+(a, b)$ and $\mathcal{C}_-(a, b)$ of the bicyclic monoid $\mathcal{C}(a, b)$ with the following properties: every Hausdorff left-continuous (right-continuous) topology on $\mathcal{C}_+(a, b)$ ($\mathcal{C}_-(a, b)$) is discrete and there exists a compact Hausdorff topological monoid S which contains $\mathcal{C}_+(a, b)$ ($\mathcal{C}_-(a, b)$) as a submonoid. Also, we construct a non-discrete right-continuous (left-continuous) topology τ_P^+ (τ_P^-) on the semigroup $\mathcal{C}_+(a, b)$ ($\mathcal{C}_-(a, b)$) which is not left-continuous (right-continuous).

Key words: semitopological semigroup, topological semigroup, left semitopological semigroup, right semitopological semigroup, discrete topology, bicyclic monoid, compact, locally compact, Baire space.

1. INTRODUCTION, MOTIVATION AND MAIN DEFINITIONS

In this paper we shall follow the terminology of [6, 8, 14, 25].

By ω we denote the set of all non-negative integers. Throughout these notes we always assume that all topological spaces involved are Hausdorff — unless explicitly stated otherwise.

Definition 1. Let X , Y and Z be topological spaces. A map $f: X \times Y \rightarrow Z$, $(x, y) \mapsto f(x, y)$, is called

- (i) *right [left] continuous* if it is continuous in the right [left] variable; i.e., for every fixed $x_0 \in X$ [$y_0 \in Y$] the map $Y \rightarrow Z$, $y \mapsto f(x_0, y)$ [$X \rightarrow Z$, $x \mapsto f(x, y_0)$] is continuous;
- (ii) *separately continuous* if it is both left and right continuous;

(iii) *jointly continuous* if it is continuous as a map between the product space $X \times Y$ and the space Z .

Definition 2 ([6, 25]). Let S be a non-void topological space which is provided with an associative multiplication (a semigroup operation) $\mu: S \times S \rightarrow S$, $(x, y) \mapsto \mu(x, y) = xy$. Then the pair (S, μ) is called

- (i) a *right topological semigroup* if the map μ is right continuous, i.e., all interior left shifts $\lambda_s: S \rightarrow S$, $x \mapsto sx$, are continuous maps, $s \in S$;
- (ii) a *left topological semigroup* if the map μ is left continuous, i.e., all interior right shifts $\rho_s: S \rightarrow S$, $x \mapsto xs$, are continuous maps, $s \in S$;
- (iii) a *semitopological semigroup* if the map μ is separately continuous;
- (iv) a *topological semigroup* if the map μ is jointly continuous.

We usually omit the reference to μ and write simply S instead of (S, μ) . It goes without saying that every topological semigroup is also semitopological and every semitopological semigroup is both a right and left topological semigroup.

A topology τ on a semigroup S is called:

- a *semigroup topology* if (S, τ) is a topological semigroup;
- a *shift-continuous topology* if (S, τ) is a semitopological semigroup;
- an *left-continuous topology* if (S, τ) is a left topological semigroup;
- an *right-continuous topology* if (S, τ) is a right topological semigroup.

The *bicyclic monoid* $\mathcal{C}(a, b)$ is the semigroup with the identity 1 generated by two elements a and b subjected only to the condition $ab = 1$. The semigroup operation on $\mathcal{C}(a, b)$ is determined as follows:

$$b^k a^l \cdot b^m a^n = \begin{cases} b^{k-l+m} a^n, & \text{if } l < m; \\ b^k a^n, & \text{if } l = m; \\ b^k a^{l-m+n}, & \text{if } l > m. \end{cases}$$

It is well known that the bicyclic monoid $\mathcal{C}(a, b)$ is a bisimple (and hence simple) combinatorial E -unitary inverse semigroup and every non-trivial congruence on $\mathcal{C}(a, b)$ is a group congruence [8].

For a semigroups S and T a map $\alpha: S \rightarrow T$ is said to be an *anti-homomorphism* if $\alpha(s \cdot t) = \alpha(t) \cdot \alpha(s)$. A bijective anti-homomorphism of semigroups is called an *anti-isomorphism*.

It is well known that topological algebra studies the influence of topological properties of its objects on their algebraic properties and the influence of algebraic properties of its objects on their topological properties. There are two main problems in topological algebra: the problem of non-discrete topologization and the problem of embedding into objects with some topological-algebraic properties.

In mathematical literature the question about non-discrete (Hausdorff) topologization was posed by Markov in [22]. Pontryagin gave well known conditions on a base at the unity of a group for its non-discrete topologization (see Theorem 4.5 of [18]). Various authors have refined Markov's question: *can a given infinite group G endowed with a non-discrete group topology be embedded into a compact topological group?* Again, for an

arbitrary Abelian group G the answer is affirmative, but there is a non-Abelian topological group that cannot be embedded into any compact topological group (see Section 9 of [9]).

Also, Ol'shanskiy [24] constructed an infinite countable group G such that every Hausdorff group topology on G is discrete. Taimanov presented in [26] a commutative semigroup \mathfrak{T} which admits only discrete Hausdorff semigroup topology. Also in [27] he gave sufficient conditions on a commutative semigroup to have a non-discrete semigroup topology. In [16] it is proved that each shift-continuous T_1 -topology on the Taimanov semigroup \mathfrak{T} is discrete.

The bicyclic monoid admits only the discrete shift-continuous Hausdorff topology [13, 5]. In [7] non-discrete shift-continuous (semigroup, semigroup inverse) T_1 -topologies on the bicyclic monoid $\mathcal{C}(a, b)$ are constructed. Also in [7] are presented the sufficient conditions when a shift-continuous T_1 -topology on $\mathcal{C}(a, b)$ is discrete. If a Hausdorff (semi)topological semigroup T contains the bicyclic monoid $\mathcal{C}(a, b)$ as a dense proper semigroup then $T \setminus \mathcal{C}(a, b)$ is a closed ideal of T [13, 15]. Moreover, the closure of $\mathcal{C}(a, b)$ in a locally compact topological inverse semigroup can be obtained (up to isomorphism) from $\mathcal{C}(a, b)$ by adjoining the additive group of integers in a suitable way [13].

Stable and Γ -compact topological semigroups do not contain the bicyclic monoid [1, 19, 20]. The problem of embedding the bicyclic monoid into compact-like topological semigroups was studied in [2, 3, 4, 17].

Subsemigroups of the bicyclic monoid are studied in [10, 11, 21].

We define the following subsets of the bicyclic monoid

$$\mathcal{C}_+(a, b) = \{b^i a^j \in \mathcal{C}(a, b) : i \leq j\} \quad \text{and} \quad \mathcal{C}_-(a, b) = \{b^i a^j \in \mathcal{C}(a, b) : i \geq j\}.$$

Proposition 1. $\mathcal{C}_+(a, b)$ and $\mathcal{C}_-(a, b)$ are submonoids of $\mathcal{C}(a, b)$.

Proof. For arbitrary $b^{i_1} a^{j_1}, b^{i_2} a^{j_2} \in \mathcal{C}_+(a, b)$ by the semigroup operation of the bicyclic monoid $\mathcal{C}(a, b)$ we have that

$$b^{i_1} a^{j_1} \cdot b^{i_2} a^{j_2} = \begin{cases} b^{i_1 - j_1 + i_2} a^{j_2}, & \text{if } j_1 \leq i_2; \\ b^{i_1} a^{j_1 - i_2 + j_2}, & \text{if } j_1 > i_2. \end{cases}$$

If $j_1 \leq i_2$ we have that $i_1 - j_1 + i_2 \geq i_2 \geq 0$. Since $j_1 \leq j_1$ and $j_2 \leq j_2$ we get that $i_1 - j_1 + i_2 \leq i_2$, and hence $i_1 - j_1 + i_2 \leq j_2$.

If $j_1 > i_2$ we get that $j_1 - i_2 + j_2 \geq j_1$ because $i_2 \leq j_2$. Hence $j_1 - i_2 + j_2 \geq i_1$.

It is obvious that $1 = b^0 a^0$ is the identity element of $\mathcal{C}_+(a, b)$. This and above arguments imply that $\mathcal{C}_+(a, b)$ is a submonoid of $\mathcal{C}(a, b)$.

The proof the statement that $\mathcal{C}_-(a, b)$ is a submonoid of $\mathcal{C}(a, b)$ is similar. \square

In this paper we prove that every Hausdorff left-continuous (right-continuous) topology on the monoid $\mathcal{C}_+(a, b)$ ($\mathcal{C}_-(a, b)$) is discrete and show that there exists a compact Hausdorff topological monoid S which contains $\mathcal{C}_+(a, b)$ ($\mathcal{C}_-(a, b)$) as a submonoid. Also, we construct a non-discrete right-continuous (left-continuous) topology τ_p^+ (τ_p^-) on the semigroup $\mathcal{C}_+(a, b)$ ($\mathcal{C}_-(a, b)$) which is not left-continuous (right-continuous).

2. ALGEBRAIC PROPERTIES AND TOPOLOGIZATIONS OF MONOIDS $\mathcal{C}_+(a, b)$ AND $\mathcal{C}_-(a, b)$

Proposition 2. *The monoids $\mathcal{C}_+(a, b)$ and $\mathcal{C}_-(a, b)$ are anti-isomorphic.*

Proof. We define a map $\alpha: \mathcal{C}_+(a, b) \rightarrow \mathcal{C}_-(a, b)$ by the formula $\alpha(b^i a^j) = b^j a^i$. Then for any $b^{i_1} a^{j_1}, b^{i_2} a^{j_2} \in \mathcal{C}_+(a, b)$ we have that

$$\begin{aligned} \alpha(b^{i_1} a^{j_1} \cdot b^{i_2} a^{j_2}) &= \begin{cases} \alpha(b^{i_1-j_1+i_2} a^{j_2}), & \text{if } j_1 \leq i_2; \\ \alpha(b^{i_1} a^{j_1-i_2+j_2}), & \text{if } j_1 > i_2. \end{cases} = \\ &= \begin{cases} b^{j_2} a^{i_1-j_1+i_2}, & \text{if } j_1 < i_2; \\ b^{j_2} a^{i_1}, & \text{if } j_1 = i_2; \\ b^{j_1-i_2+j_2} a^{i_1}, & \text{if } j_1 > i_2 \end{cases} \end{aligned}$$

and

$$\begin{aligned} \alpha(b^{i_1} a^{j_1}) \cdot \alpha(b^{i_2} a^{j_2}) &= b^{j_1} a^{i_1} \cdot b^{j_2} a^{i_2} = \\ &= \begin{cases} b^{j_2} a^{i_1-j_1+i_2}, & \text{if } j_1 < i_2; \\ b^{j_2} a^{i_1}, & \text{if } j_1 = i_2; \\ b^{j_1-i_2+j_2} a^{i_1}, & \text{if } j_1 > i_2. \end{cases} \end{aligned}$$

This implies that the map α is an anti-homomorphism. It is obvious that the so defined map $\alpha: \mathcal{C}_+(a, b) \rightarrow \mathcal{C}_-(a, b)$ is bijective, and hence it is an anti-isomorphism of monoids $\mathcal{C}_+(a, b)$ and $\mathcal{C}_-(a, b)$, because $\alpha(b^0 a^0) = b^0 a^0 = 1$. \square

If S is a semigroup, then we shall denote the Green relations on S by \mathcal{R} , \mathcal{L} , \mathcal{J} , \mathcal{D} and \mathcal{H} (see [8, Section 2.1]):

$$\begin{aligned} a\mathcal{R}b &\text{ if and only if } aS^1 = bS^1; \\ a\mathcal{L}b &\text{ if and only if } S^1 a = S^1 b; \\ a\mathcal{J}b &\text{ if and only if } S^1 a S^1 = S^1 b S^1; \\ \mathcal{D} &= \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}; \\ \mathcal{H} &= \mathcal{L} \cap \mathcal{R}. \end{aligned}$$

The following proposition describes Green's relations on monoids $\mathcal{C}_+(a, b)$ and $\mathcal{C}_-(a, b)$.

Proposition 3. *Green's relations \mathcal{R} , \mathcal{L} , \mathcal{J} , \mathcal{D} and \mathcal{H} on monoids $\mathcal{C}_+(a, b)$ and $\mathcal{C}_-(a, b)$ coincide with the equality relation.*

Proof. Suppose that $b^i a^j \mathcal{L} b^k a^l$ in $\mathcal{C}_+(a, b)$ for some $i, j, k, l \in \omega$, $i \leq j$, and $k \leq l$. Then there exist $b^x a^y, b^u a^v \in \mathcal{C}_+(a, b)$ such that $b^i a^j = b^u a^v \cdot b^k a^l$ and $b^k a^l = b^x a^y \cdot b^i a^j$. Then the equalities

$$b^k a^l = b^x a^y \cdot b^i a^j = \begin{cases} b^{x-y+i} a^j, & \text{if } y < i; \\ b^x a^j, & \text{if } y = i; \\ b^x a^{y-i+j}, & \text{if } y > i \end{cases} \quad (1)$$

imply that $l \geq j$, and by the equalities

$$b^i a^j = b^u a^v \cdot b^k a^l = \begin{cases} b^{u-v+k} a^l, & \text{if } v < k; \\ b^u a^l, & \text{if } v = k; \\ b^u a^{k-v+l}, & \text{if } v > k \end{cases} \quad (2)$$

we get that $l \leq j$, and hence $l = j$. Also, the equalities (1) imply that $k - l = (x - y) + (i - j)$. Since $x \leq y$, we have that $k - l \leq i - j$. Similar, the equalities (2) imply that $i - j = (u - v) + (k - l)$, and hence $i - j \leq k - l$. Thus, $i - j = k - l$. Since $l = j$, we obtain that $k = i$. Hence \mathcal{L} is the equality relation on the monoid $\mathcal{C}_+(a, b)$.

Suppose that $b^i a^j \mathcal{R} b^k a^l$ in $\mathcal{C}_+(a, b)$ for some $i, j, k, l \in \omega$, $i \leq j$, and $k \leq l$. Then there exist $b^x a^y, b^u a^v \in \mathcal{C}_+(a, b)$ such that $b^i a^j = b^k a^l \cdot b^u a^v$ and $b^k a^l = b^i a^j \cdot b^x a^y$. By the equalities

$$b^k a^l = b^i a^j \cdot b^x a^y = \begin{cases} b^{i-j+x} a^y, & \text{if } j < x; \\ b^i a^y, & \text{if } j = x; \\ b^i a^{j-x+y}, & \text{if } j > x \end{cases} \quad (3)$$

we get that $k \geq i$, and by the equalities

$$b^i a^j = b^k a^l \cdot b^u a^v = \begin{cases} b^{k-l+u} a^v, & \text{if } l < u; \\ b^k a^v, & \text{if } l = u; \\ b^k a^{l-u+v}, & \text{if } l > u \end{cases} \quad (4)$$

we get that $k \leq i$, and hence $k = i$. The equalities (3) imply that $k - l = (i - j) + (x - y)$. Since $x \leq y$, we have that $k - l \geq i - j$. Similar, the equalities (4) imply that $i - j = (k - l) + (u - v)$, and hence $i - j \geq k - l$. Thus, $i - j = k - l$. Since $l = j$, we obtain that $k = i$. Hence \mathcal{R} is the equality relation on the monoid $\mathcal{C}_+(a, b)$.

Since $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$, the previous part of the proof imply that $\mathcal{H} = \mathcal{D} = \mathcal{L} = \mathcal{R}$ in $\mathcal{C}_+(a, b)$.

Suppose that $b^i a^j \not\mathcal{J} b^k a^l$ in $\mathcal{C}_+(a, b)$ for some $i, j, k, l \in \omega$, $i \leq j$, and $k \leq l$. Then there exist $b^{k_1} a^{l_1}, b^{k_2} a^{l_2}, b^{i_1} a^{j_1}, b^{i_2} a^{j_2} \in \mathcal{C}_+(a, b)$ such that

$$b^i a^j = b^{k_1} a^{l_1} \cdot b^k a^l \cdot b^{k_2} a^{l_2} \quad (5)$$

and

$$b^k a^l = b^{i_1} a^{j_1} \cdot b^i a^j \cdot b^{i_2} a^{j_2}. \quad (6)$$

The semigroup operation of $\mathcal{C}_+(a, b)$ implies that

$$i - j = (k_1 - l_1) + (k - l) + (k_2 - l_2), \quad (7)$$

and since $k_1 \leq l_1$ and $k_2 \leq l_2$, we get that $i - j \leq k - l$. Similar we get that

$$k - l = (i_1 - j_1) + (i - j) + (i_2 - j_2), \quad (8)$$

and since $i_1 \leq j_1$ and $i_2 \leq j_2$, we have that $k - l \leq i - j$. Hence we obtain that $i - j = k - l$. The last equality and equalities (7) and (8) imply that

$$i_1 = j_1, \quad i_2 = j_2, \quad k_1 = l_1, \quad \text{and} \quad k_2 = l_2,$$

because $i_1 \leq j_1$, $i_2 \leq j_2$, $k_1 \leq l_1$, and $k_2 \leq l_2$. Then the semigroup operation of $\mathcal{C}_+(a, b)$ implies that

$$i_1 = j_1 \leq k, \quad i_2 = j_2 \leq l, \quad k_1 = l_1 \leq i, \quad \text{and} \quad k_2 = l_2 \leq j.$$

Then we have that

$$b^i a^j = b^{k_1} a^{k_1} \cdot b^k a^l \cdot b^{k_2} a^{k_2} = b^{k_1} a^{k_1} \cdot b^{i_1} a^{i_1} \cdot b^i a^j \cdot b^{i_2} a^{i_2} \cdot b^{k_2} a^{k_2}$$

and

$$b^k a^l = b^{i_1} a^{i_1} \cdot b^i a^j \cdot b^{i_2} a^{i_2} = b^{i_1} a^{i_1} \cdot b^{k_1} a^{k_1} \cdot b^k a^l \cdot b^{k_2} a^{k_2} \cdot b^{i_2} a^{i_2}.$$

Since idempotents commute in $\mathcal{C}_+(a, b)$, the last two equalities and the semigroup operation of $\mathcal{C}_+(a, b)$ imply that $\max\{k_1, i_1\} \leq i, k$ and $\max\{k_2, i_2\} \leq j, l$. Hence, again using the semigroup operation of $\mathcal{C}_+(a, b)$ we get that $b^i a^j = b^k a^l$. Hence \mathcal{J} is the equality relation on the monoid $\mathcal{C}_+(a, b)$.

Applying Proposition 2 we obtain that the statement of the proposition holds for the monoid $\mathcal{C}_-(a, b)$. \square

Since

$$b^x a^y \cdot b^i a^i = \begin{cases} b^{x-y+i} a^i, & \text{if } y < i; \\ b^x a^i, & \text{if } y = i; \\ b^x a^y, & \text{if } y > i, \end{cases}$$

we have that

$$\mathcal{C}_+(a, b) \cdot b^i a^i = \{b^s a^t \in \mathcal{C}_+(a, b) : t \geq i\} \quad (9)$$

for any $i \in \omega$. This implies that if τ is a Hausdorff left-continuous topology on the semigroup $\mathcal{C}_+(a, b)$ then for any idempotent $b^i a^i \in \mathcal{C}_+(a, b)$ the right shift $\rho_{b^i a^i} : \mathcal{C}_+(a, b) \rightarrow \mathcal{C}_+(a, b)$ is a retraction, $\mathcal{C}_+(a, b) \cdot b^i a^i$ is a retract of $(\mathcal{C}_+(a, b), \tau)$, and hence $\mathcal{C}_+(a, b) \cdot b^i a^i$ is a closed subset of the topological space $(\mathcal{C}_+(a, b), \tau)$ (see [14, Ex. 1.5.C]). The above arguments imply that every element $b^k a^l$ of the monoid $\mathcal{C}_+(a, b)$ has a finite open neighbourhood in the space $(\mathcal{C}_+(a, b), \tau)$. Since the topology τ is Hausdorff, $b^l a^l$ is an isolated point $(\mathcal{C}_+(a, b), \tau)$. Hence we proved the following theorem.

Theorem 1. *Every Hausdorff left-continuous topology on the monoid $\mathcal{C}_+(a, b)$ is discrete.*

Proposition 2 and Theorem 1 imply the following theorem.

Theorem 2. *Every Hausdorff right-continuous topology on the monoid $\mathcal{C}_-(a, b)$ is discrete.*

In the paper [7] the following two examples are constructed.

Example 1 ([7, Example 2]). The topology τ_2 on the bicyclic monoid $\mathcal{C}(a, b)$ is defined in the following way. For any $b^i a^j \in \mathcal{C}(a, b)$ and any non-negative integer n put

$$O_n(b^i a^j) = \{b^i a^j\} \cup \{b^{i+l} a^{j+l} : l > n\}.$$

Let $\mathcal{B}_2(b^i a^j) = \{O_n(b^i a^j) : n \in \omega\}$ be the system of open neighbourhoods at the point $b^i a^j \in \mathcal{C}(a, b)$. It is obvious that the family $\mathcal{B}_2 = \bigcup_{i,j \in \omega} \mathcal{B}_2(b^i a^j)$ satisfies the properties (BP1)–(BP3) of [14], and hence it generates a topology on $\mathcal{C}(a, b)$.

Example 2 ([7, Example 3]). The topology τ_c on the bicyclic semigroup $\mathcal{C}(a, b)$ is defined in the following way. For any non-negative integer n put

$$C_n = \{b^i a^j \in \mathcal{C}(a, b) : i, j \leq n\}.$$

Let

$$\mathcal{B}_c(b^i a^j) = \{W_n(b^i a^j) = \{b^i a^j\} \cup \mathcal{C}(a, b) \setminus C_n : n \in \omega\}$$

be the system of open neighbourhoods at the point $b^i a^j \in \mathcal{C}(a, b)$. It is obvious that the family $\mathcal{B}_c = \bigcup_{i,j \in \omega} \mathcal{B}_c(b^i a^j)$ satisfies the properties (BP1)–(BP3) of [14], and hence it generates the topology τ_c on $\mathcal{C}(a, b)$.

By Proposition 2 of [7], τ_2 is a locally compact semigroup T_1 -topology on the bicyclic semigroup $\mathcal{C}(a, b)$. Simple verifications show that τ_2 induces on the monoid $\mathcal{C}_+(a, b)$ a locally compact semigroup T_1 -topology. Also, by Proposition 3 of [7], τ_c is a shift-continuous compact T_1 -topology on $\mathcal{C}(a, b)$. It is obvious that τ_c induces on the monoid $\mathcal{C}_+(a, b)$ a shift-continuous compact T_1 -topology.

Lemma I.1 of [13] implies

Lemma 1. *For each $v, w \in \mathcal{C}_+(a, b)$ both sets $\{u \in \mathcal{C}_+(a, b) : vu = w\}$ and $\{u \in \mathcal{C}_+(a, b) : uv = w\}$ are finite.*

Proposition 4 describes the closure of the monoid $\mathcal{C}_+(a, b)$ in a Hausdorff semi-topological monoid.

Proposition 4. *If the monoid $\mathcal{C}_+(a, b)$ is a dense subsemigroup of a Hausdorff semi-topological monoid S and $I = S \setminus \mathcal{C}_+(a, b) \neq \emptyset$ then I is a closed two-sided ideal of the semigroup S .*

Proof. Since every discrete space is locally compact, Theorem 3.3.9 of [14] implies that $\mathcal{C}_+(a, b)$ is an open subset of S .

Fix an arbitrary element $y \in I$. If $xy = z \notin I$ for some $x \in \mathcal{C}_+(a, b)$ then there exists an open neighbourhood $U(y)$ of the point y in the space S such that $\{x\} \cdot U(y) = \{z\} \subset \mathcal{C}_+(a, b)$. The neighbourhood $U(y)$ contains infinitely many elements of the semigroup $\mathcal{C}_+(a, b)$. This contradicts Lemma 1. The obtained contradiction implies that $xy \in I$ for all $x \in \mathcal{C}_+(a, b)$ and $y \in I$. The proof of the statement that $yx \in I$ for all $x \in \mathcal{C}_+(a, b)$ and $y \in I$ is similar.

Suppose to the contrary that $xy = w \notin I$ for some $x, y \in I$. Then $w \in \mathcal{C}_+(a, b)$ and the separate continuity of the semigroup operation in S implies that there exist open neighbourhoods $U(x)$ and $U(y)$ of the points x and y in S , respectively, such that $\{x\} \cdot U(y) = \{w\}$ and $U(x) \cdot \{y\} = \{w\}$. Since both neighbourhoods $U(x)$ and $U(y)$ contain infinitely many elements of the semigroup $\mathcal{C}_+(a, b)$, both equalities $\{x\} \cdot U(y) = \{w\}$ and $U(x) \cdot \{y\} = \{w\}$ contradict mentioned above Lemma 1. The obtained contradiction implies that $xy \in I$. \square

The proof of Proposition 5 is similar to Proposition 4.

Proposition 5. *If the monoid $\mathcal{C}_-(a, b)$ is a dense subsemigroup of a Hausdorff semi-topological monoid S and $I = S \setminus \mathcal{C}_-(a, b) \neq \emptyset$ then I is a closed two-sided ideal of the semigroup S .*

It is well known that neither the bicyclic monoid nor the Taimanov semigroup \mathfrak{T} do not embed into compact Hausdorff topological semigroups [16, 20]. Later we show that there exists a compact Hausdorff topological monoid S which contains the monoid $\mathcal{C}_+(a, b)$ as a dense submonoid.

Example 3. Put $S = \mathcal{C}_+(a, b) \sqcup \{0\}$ is the monoid $\mathcal{C}_+(a, b)$ with the adjoined zero 0. We define the topology τ_S on the semigroup S in the following way. All points of $\mathcal{C}_+(a, b)$ are isolated in (S, τ_S) and put the family $\mathcal{B}_S = \{U_p(0) : p \in \omega\}$, where

$$U_p(0) = \{0\} \cup \{b^i a^j \in \mathcal{C}_+(a, b) : j \geq p\},$$

is the system of open neighbourhoods at zero 0. It is obvious that the family \mathcal{B}_S satisfies the properties (BP1)–(BP3) of [14], and hence it generates the topology τ_S on the monoid S . It is obvious that (S, τ_S) is a Hausdorff topological space, and since the set $U_p(0) \setminus \{0\}$ has the finite complement in $\mathcal{C}_+(a, b)$, the space (S, τ_S) is compact.

The following equality

$$b^x a^y \cdot b^i a^i = \begin{cases} b^{x-y+i} a^i, & \text{if } y < i; \\ b^x a^i, & \text{if } y = i; \\ b^x a^y, & \text{if } y > i, \end{cases} \quad (10)$$

implies that $U_p(0) \cdot U_p(0) \subseteq U_p(0)$ and $b^k a^l \cdot U_p(0) \subseteq U_p(0)$ for any $b^k a^l \in \mathcal{C}_+(a, b)$. Also by (10) we have that $U_p(0) \cdot b^i a^i \subseteq U_p(0)$ for $p \geq i$. Hence (S, τ_S) is a topological semigroup.

3. SOME EXAMPLES

In this section we construct non-discrete Hausdorff right-continuous topology on the monoid $\mathcal{C}_+(a, b)$ which is not left-continuous.

By $(\omega, +)$ we denote the additive semigroup of non-negative integers.

The semigroup operation of $\mathcal{C}_+(a, b)$ implies that

$$R_k = \{b^k a^{k+s} \in \mathcal{C}_+(a, b) : s \in \omega\}$$

is a subsemigroup of $\mathcal{C}_+(a, b)$ for any $k \in \omega$. For any $k \in \omega$ we define the map $\iota_k : (\omega, +) \rightarrow R_k$ by the formula $\iota_k(s) = b^k a^{k+s}$. Since $\iota_k(s_1 + s_2) = b^k a^{k+s_1+s_2}$ and

$$\iota_k(s_1) \cdot \iota_k(s_2) = b^k a^{k+s_1} \cdot b^k a^{k+s_2} = b^k a^{s_1} \cdot a^{k+s_2} = b^k a^{k+s_1+s_2},$$

we obtain that the map ι_k is a monoid homomorphism. It is obvious that the map ι_k is bijective, and hence it is an isomorphism.

Let p be a prime positive integer. Then the family of subgroups $\{p^n \mathbb{Z} : n \in \mathbb{N}\}$ of the additive group of integers \mathbb{Z} form a fundamental system of neighborhoods of the zero 0 for a linear precompact topology τ_p on \mathbb{Z} , which is usually called the *p-adic topology* (see [12, p. 45]). It is well known that for any prime integer p the *p-adic topology* τ_p on the additive group of integers \mathbb{Z} is a group topology, i.e., the group operation and the inversion are continuous in (\mathbb{Z}, τ_p) .

Example 4. Fix an arbitrary prime positive integer p . Put τ_p^i is the induced topology on $(\omega, +)$ from (\mathbb{Z}, τ_p) . Since τ_p is a group topology on the additive group of integers \mathbb{Z} and $(\omega, +)$ is a subsemigroup of \mathbb{Z} , we have that τ_p^i is a semigroup topology on $(\omega, +)$. Since the T_0 -space of a topological group is completely regular (Tychonoff), Theorem 2.1.6 of [14] implies that $(\omega, +, \tau_p^i)$ is a completely regular space. Also, by Hausdorffness of (ω, τ_p^i) we have that the family $\mathcal{B}_p^i(s) = \{U_n(s) : n \in \omega\}$, where $U_n(s) = \{s + p^n j : j \in \omega\}$, determines the system of open neighbourhoods at the point s in the space (ω, τ_p^i) .

We define the topology τ_p^+ on the semigroup $\mathcal{C}_+(a, b)$ in the following way. For any $k, s \in \omega$ we denote $W_n(b^k a^{k+s}) = \iota_k(U_n(s))$ and put the family

$$\mathcal{B}_p^+(b^k a^{k+s}) = \{W_n(b^k a^{k+s}) : n \in \omega\}$$

is the system of open neighbourhoods at the point $b^k a^{k+s}$. It is obvious that the family $\mathcal{B}_p^+ = \{\mathcal{B}_p^+(b^k a^{k+s}) : k, s \in \omega\}$ satisfies the properties (BP1)–(BP3) of [14], and hence it generates the topology τ_p^+ on the monoid $\mathcal{C}_+(a, b)$.

Simple calculations show that

$$W_n(b^k a^{k+s}) = \{b^k a^{k+s+p^n j} : j \in \omega\}$$

for any $k, s \in \omega$. This implies that the topological space $(\mathcal{C}_+(a, b), \tau_p^+)$ is homeomorphic to a countable topological sum of spaces (ω, τ_p^+) . By the Birkhoff-Kakutani Theorem (see [23, Section 1.22]) the topological group (\mathbb{Z}, τ_p) is metrizable, and hence by Theorem 4.2.1 of [14] the space $(\mathcal{C}_+(a, b), \tau_p^+)$ is metrizable as well. Also by Corollary 4.1.13 of [14] the space $(\mathcal{C}_+(a, b), \tau_p^+)$ is perfectly normal, i.e., $(\mathcal{C}_+(a, b), \tau_p^+)$ is a normal space and every closed subset of $(\mathcal{C}_+(a, b), \tau_p^+)$ is a G_δ -set.

Proposition 6. *$(\mathcal{C}_+(a, b), \tau_p^+)$ is a right topological semigroup. Moreover $(\mathcal{C}_+(a, b), \tau_p^+)$ is not a left topological semigroup.*

Proof. Fix arbitrary $b^{k_1} a^{k_1+s_1}, b^{k_2} a^{k_2+s_2} \in \mathcal{C}_+(a, b)$, $k_1, k_2, s_1, s_2 \in \omega$.

We consider the possible cases.

If $k_1 + s_1 < k_2$, then

$$b^{k_1} a^{k_1+s_1} \cdot b^{k_2} a^{k_2+s_2} = b^{k_1-k_1-s_1+k_2} a^{k_2+s_2} = b^{k_2-s_1} a^{k_2+s_2},$$

and for any $n \in \omega$ we get that

$$\begin{aligned} b^{k_1} a^{k_1+s_1} \cdot W_n(b^{k_2} a^{k_2+s_2}) &= \{b^{k_1} a^{k_1+s_1} \cdot b^{k_2} a^{k_2+s_2+p^n j} : j \in \omega\} = \\ &= \{b^{k_2-s_1} a^{k_2+s_2+p^n j} : j \in \omega\} = \\ &= W_n(b^{k_2-s_1} a^{k_2+s_2}). \end{aligned}$$

If $k_1 + s_1 = k_2$, then

$$b^{k_1} a^{k_1+s_1} \cdot b^{k_2} a^{k_2+s_2} = b^{k_1} a^{k_2+s_2},$$

and for any $n \in \omega$ we obtain that

$$\begin{aligned} b^{k_1} a^{k_1+s_1} \cdot W_n(b^{k_2} a^{k_2+s_2}) &= \{b^{k_1} a^{k_1+s_1} \cdot b^{k_2} a^{k_2+s_2+p^n j} : j \in \omega\} = \\ &= \{b^{k_1} a^{k_2+s_2+p^n j} : j \in \omega\} = \\ &= W_n(b^{k_1} a^{k_2+s_2}). \end{aligned}$$

If $k_1 + s_1 > k_2$, then

$$b^{k_1} a^{k_1+s_1} \cdot b^{k_2} a^{k_2+s_2} = b^{k_1} a^{k_1+s_1-k_2+k_2+s_2} = b^{k_1} a^{k_1+s_1+s_2},$$

and for any $n \in \omega$ we get that

$$\begin{aligned} b^{k_1} a^{k_1+s_1} \cdot W_n(b^{k_2} a^{k_2+s_2}) &= \{b^{k_1} a^{k_1+s_1} \cdot b^{k_2} a^{k_2+s_2+p^n j} : j \in \omega\} = \\ &= \{b^{k_1} a^{k_1+s_1+s_2+p^n j} : j \in \omega\} = \\ &= W_n(b^{k_1} a^{k_1+s_1+s_2}). \end{aligned}$$

The above arguments imply that $(\mathcal{C}_+(a, b), \tau_p^+)$ is a right topological semigroup.

Next we show that the second statement holds. It is obvious that $1 \cdot ba = ba$. For any open neighbourhood $W_n(1) = \{a^{p^n j} : j \in \omega\}$ of the unit element 1 in $(\mathcal{C}_+(a, b), \tau_p^+)$ we have that

$$W_n(1) \cdot ba = \{a^{p^n j} \cdot ba : j \in \omega\}.$$

Then for any positive integer j we get that

$$a^{p^n j} \cdot ba = a^{p^n j} \notin W_m(ba)$$

for any $m \in \omega$. This completes the second part of the proposition. \square

Remark 1. Simple verifications show that for arbitrary $b^{k_1} a^{k_1+s_1}, b^{k_2} a^{k_2+s_2} \in \mathcal{C}_+(a, b)$, $k_1, k_2, s_1, s_2 \in \omega$ with $k_1 + s_1 \leq k_2$, the semigroup operation is not right-continuous in $(\mathcal{C}_+(a, b), \tau_p^+)$. The proof of this statement is similar to the second statement of Proposition 6.

Propositions 2 and 6 imply the following.

Proposition 7. *The semigroup $\mathcal{C}_-(a, b)$ admits a non-discrete left-continuous topology τ_p^- which is not right-continuous.*

Remark 2. Alex Ravsky in Topological Algebra Seminar at Lviv University posed the following question: *Is any Hausdorff left-continuous (right-continuous) topology on the semigroup $\mathcal{C}_+(a, b)$ ($\mathcal{C}_-(a, b)$) discrete?* Propositions 6 and 7 give negative answer on this question.

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REFERENCES

1. L. W. Anderson, R. P. Hunter, and R. J. Koch, *Some results on stability in semigroups*, Trans. Amer. Math. Soc. **117** (1965), 521–529. DOI: 10.2307/1994222
2. T. Banakh, S. Dimitrova, and O. Gutik, *The Rees-Suschkewitsch Theorem for simple topological semigroups*, Mat. Stud. **31** (2009), no. 2, 211–218.
3. T. Banakh, S. Dimitrova, and O. Gutik, *Embedding the bicyclic semigroup into countably compact topological semigroups*, Topology Appl. **157** (2010), no. 18, 2803–2814. DOI: 10.1016/j.topol.2010.08.020
4. S. Bardyla and A. Ravsky, *Closed subsets of compact-like topological spaces*, Appl. Gen. Topol. **21** (2020), no. 2, 201–214. DOI: 10.4995/agt.2020.12258.
5. M. O. Bertman and T. T. West, *Conditionally compact bicyclic semitopological semigroups*, Proc. Roy. Irish Acad. **A76** (1976), no. 21–23, 219–226.
6. J. H. Carruth, J. A. Hildebrandt, and R. J. Koch, *The theory of topological semigroups*, Vol. I, Marcel Dekker, Inc., New York and Basel, 1983.
7. A. Chornenka and O. Gutik, *On topologization of the bicyclic monoid*, Visnyk L'viv Univ., Ser. Mech.-Math. **95** (2023), 45–56. DOI: 10.30970/vmm.2023.95.046-056
8. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol. I, Amer. Math. Soc. Surveys **7**, Providence, R.I., 1961.

9. W. W. Comfort, *Topological groups*, Handbook of set-theoretic topology, Kunen K., Vaughan J. (eds.) Elsevier, 1984, pp. 1143–1263.
10. L. Descalço and N. Ruškuc, *Subsemigroups of the bicyclic monoid*, Int. J. Algebra Comput. **15** (2005), no. 1, 37–57. DOI: 10.1142/S0218196705002098
11. L. Descalço and N. Ruškuc, *Properties of the subsemigroups of the bicyclic monoid*, Czech. Math. J. **58** (2008), no. 2, 311–330. DOI: 10.1007/s10587-008-0018-7
12. D. N. Dikranjan, L. N. Stoyanov, and I. R. Prodanov, *Topological groups. Characters, dualities, and minimal group topologies*, Pure and Appl. Math., **130**. Marcel Dekker, Inc., New York, 1990, 287p.
13. C. Eberhart and J. Selden, *On the closure of the bicyclic semigroup*, Trans. Amer. Math. Soc. **144** (1969), 115–126. DOI: 10.1090/S0002-9947-1969-0252547-6
14. R. Engelking, *General topology*, 2nd ed., Heldermann, Berlin, 1989.
15. O. Gutik, *On the dichotomy of a locally compact semitopological bicyclic monoid with adjoined zero*, Visnyk L'viv Univ., Ser. Mech.-Math. **80** (2015), 33–41.
16. O. Gutik, *Topological property of the Taimanov semigroup*, Math. Bull. T. Shevchenko Sci. Soc. **13** (2016), 29–34.
17. O. Gutik and D. Repovš, *On countably compact 0-simple topological inverse semigroups*, Semigroup Forum **75** (2007), no. 2, 464–469. DOI: 10.1007/s00233-007-0706-x
18. E. Hewitt and K. A. Roos, *Abstract harmonic analysis*, Vol. **1**, Springer, Berlin, 1963.
19. J. A. Hildebrandt and R. J. Koch, *Swelling actions of Γ -compact semigroups*, Semigroup Forum **33** (1986), 65–85. DOI: 10.1007/BF02573183
20. R. J. Koch and A. D. Wallace, *Stability in semigroups*, Duke Math. J. **24** (1957), no. 2, 193–195. DOI: 10.1215/S0012-7094-57-02425-0
21. K. H. Hovsepyan, *Inverse subsemigroups of the bicyclic semigroup*, Math. Notes **108** (2020), no. 4, 550–556. DOI: 10.1134/S0001434620090278
22. A. A. Markov, *On free topological groups*, Izvestia Akad. Nauk SSSR **9** (1945), 3–64 (in Russian); **English version in:** Transl. Amer. Math. Soc. **8** (1962), no. 1, 195–272.
23. D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience Publ., New York-London, 1955. 282p.
24. A. Yu. Ol'shanskiy, *Remark on countable non-topologized groups*, Vestnik Moscow Univ. Ser. Mech. Math. **39** (1980), 1034 (in Russian).
25. W. Ruppert, *Compact semitopological semigroups: an intrinsic theory*, Lect. Notes Math., **1079**, Springer, Berlin, 1984. DOI: 10.1007/BFb0073675
26. A. D. Taimanov, *An example of a semigroup which admits only the discrete topology*, Algebra i Logika **12** (1973), no. 1, 114–116 (in Russian); **English version in:** Algebra and Logic **12** (1973), no. 1, 64–65. DOI: 10.1007/BF02218642
27. A. D. Taimanov, *On the topologization of commutative semigroups*, Mat. Zametki **17** (1975), no. 5, 745–748 (in Russian); **English version in:** Math. Notes **17** (1975), no. 5, 443–444. DOI: 10.1007/BF01155800

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ПРО НЕТОПОЛОГІЗОВНІ НАПІВГРУПИ

Олег ГУТІК

Львівський національний університет імені Івана Франка,
вул. Університетська 1, 79000, м. Львів
e-mails: oleg.gutik@lnu.edu.ua

We find anti-isomorphic submonoids $\mathcal{C}_+(a, b)$ and $\mathcal{C}_-(a, b)$ of the bicyclic monoid $\mathcal{C}(a, b)$ with the following properties: every Hausdorff left-continuous (right-continuous) topology on $\mathcal{C}_+(a, b)$ ($\mathcal{C}_-(a, b)$) is discrete and there exists a compact Hausdorff topological monoid S which contains $\mathcal{C}_+(a, b)$ ($\mathcal{C}_-(a, b)$) as a submonoid. Also, we construct a non-discrete right-continuous (left-continuous) topology τ_p^+ (τ_p^-) on the semigroup $\mathcal{C}_+(a, b)$ ($\mathcal{C}_-(a, b)$) which is not left-continuous (right-continuous).

Ключові слова: semitopological semigroup, topological semigroup, left semitopological semigroup, right semitopological semigroup, discrete topology, bicyclic monoid, compact, locally compact, Baire space.