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ON NONLINEAR INTEGRO-DIFFERENTIAL OSKOLKOV-STOKES SYSTEM WITH VARIABLE EXPONENT OF NONLINEARITY AND SMALL PARAMETER

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Some nonlinear parabolic integro-differential system with the variable exponent of the nonlinearity is considered. The initial-boundary value problem for this system is investigated and the existence and uniqueness theorems for the problem are proved. Some convergence results if parameter tends to zero are also obtained.

Key words: nonlinear parabolic system, integro-differential system, generalized Lebesgue space, variable exponents of nonlinearity.

1. Introduction

Let $n \in \mathbb{N}$ and $T > 0$ be fixed numbers, $\Omega \subset \mathbb{R}^n$ be a bounded domain with the boundary $\partial\Omega$, $Q_{0,\tau} := \Omega \times (0,\tau)$, $\Omega_\tau := \{(x,t) \mid x \in \Omega, \ \ t = \tau\}$, $\tau \in [0,T]$. We seek weak solutions $\{u^\varepsilon, \pi^\varepsilon\}$, where $u^\varepsilon = (u_1^\varepsilon, \dots, u_n^\varepsilon) : Q_{0,T} \to \mathbb{R}^n$ and $\pi^\varepsilon : Q_{0,T} \to \mathbb{R}$, of the problem

$$u_t^\varepsilon - \Delta u^\varepsilon + g(x,t)\,|u^\varepsilon|^{q(x)-2}u^\varepsilon + \Phi\Big(\int\limits_\Omega \mathfrak{Z}(x,t,y)u^\varepsilon(y,t)\,dy\Big) + \nabla\pi^\varepsilon = F(x,t) \quad \text{in} \quad Q_{0,T}, \tag{1}$$

$$\varepsilon(\pi_t^\varepsilon - \Delta\,\pi^\varepsilon) + \operatorname{div} u^\varepsilon = f(x,t) \quad \text{in} \quad Q_{0,T}, \tag{2}$$

$$u^\varepsilon|_{\partial\Omega\times(0,T)} = 0, \tag{3}$$

$$\pi^\varepsilon|_{\partial\Omega\times(0,T)} = 0, \tag{4}$$

$$u^\varepsilon|_{t=0} = u_0(x) \quad \text{in} \quad \Omega, \tag{5}$$

$$\pi^\varepsilon|_{t=0} = \pi_0(x) \quad \text{in} \quad \Omega. \tag{6}$$

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Here $\varepsilon > 0$ is a number, $\Delta := \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2} + \ldots + \dfrac{\partial^2}{\partial x_n^2}$ is the Laplacian, $\nabla := \left(\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}, \ldots, \dfrac{\partial}{\partial x_n}\right)$ is a gradient, $\operatorname{div} u^\varepsilon := \dfrac{\partial u_1^\varepsilon}{\partial x_1} + \dfrac{\partial u_2^\varepsilon}{\partial x_2} + \ldots + \dfrac{\partial u_n^\varepsilon}{\partial x_n}$ is a divergence, $\mathfrak{Z}$ is a n-order square matrix-function and $g, q, \Phi, F, f, u_0, \pi_0$ are some functions, in particular, q is a variable exponent of the nonlinearity of system (1). Also we investigate the behaviour of the weak solutions $\{u^\varepsilon, \pi^\varepsilon\}$ to problem (1)–(6) if $\varepsilon \to +0$. The Navier-Stokes systems with the constant exponents of the nonlinearities are investigated in [1], [2], [3], etc. Various equations with the variable exponents of the nonlinearity are considered in [4], [5], [6], [7], [8], [9] (see also the references given there). The integro-differential equations are investigate in [10], [11], [12], [13], [14]. In his paper [15], Oskolkov proposed to approximate the Navier-Stokes system by certain parabolic systems with small parameter. This approach is very useful in finite-difference methods for the solutions of the initial boundary value problems for the Navier-Stokes systems (see [15] for more details). We transfer these results on the Stokes system with the variable exponent of the nonlinearity and call the obtained systems *the Oskolkov-Stokes systems*. The results obtained here are generalizations and complements of the results from [16]-[17], where some Stokes systems (that is (1)-(2) with $\varepsilon = 0$) with the variable exponent of the nonlinearity are considered.

2. Notation and statement of main result

Let $(\cdot,\cdot)_{\mathcal{H}}$ be a scalar product of some Hilbert space $\mathcal{H}$, $||\cdot||_X \equiv ||\cdot; X||$ be a norm of some Banach space X, the Cartesian product of space X by itself n times we denote by X^n, $||(z_1, ..., z_n); X^n|| := \sum\limits_{l=1}^{n} ||z_l||_X$, X^* be a dual space for X, $\langle\cdot,\cdot\rangle_X$ be a scalar product between X^* and X, and $|\cdot| \equiv |\cdot|_{\mathbb{R}^n} := ||\cdot||_{\mathbb{R}^n}$. The notation $X \overset{-}{\circlearrowright} Y$ means the continuous and densely embedding X into another space Y.

Suppose that $m, N \in \mathbb{N}$, $\mathfrak{p} \in [1, \infty]$, $\dfrac{1}{\mathfrak{p}} + \dfrac{1}{\mathfrak{p}'} = 1$, $\mathcal{O} \subset \mathbb{R}^N$ is a bounded measurable set (for example, $\mathcal{O} = \Omega$ or $\mathcal{O} = Q_{0,T}$), $\mathcal{M}(\mathcal{O})$ is a set of all measurable functions $v : \mathcal{O} \to \mathbb{R}$, $C^m(\mathcal{O})$, $C(\mathcal{O})$, and $D(\mathcal{O})$ are the spaces of the smooth functions (see [18, p. 9, 19]), $L^{\mathfrak{p}}(\mathcal{O})$ is the standard Lebesgue space (see [18, p. 22, 24]), $W^{m,\mathfrak{p}}(\mathcal{O})$ and $W_0^{m,\mathfrak{p}}(\mathcal{O})$ are the Sobolev spaces (see [18, p. 45]), $W^{-m,\mathfrak{p}'}(\mathcal{O}) := [W_0^{m,\mathfrak{p}}(\mathcal{O})]^*$, $H^m(\mathcal{O}) := W^{m,2}(\mathcal{O})$, $H_0^m(\mathcal{O}) := W_0^{m,2}(\mathcal{O})$, $H^{-m}(\mathcal{O}) := W^{-m,2}(\mathcal{O})$, $C([0,T]; X)$ is the spaces of the X-valued continuous functions defined on $[0,T]$ (see [19, p. 147]), $L^{\mathfrak{p}}(0,T;X)$ is the Lebesgue-Bochner space (see [19, p. 155]), $W^{m,\mathfrak{p}}(0,T;X)$ is the Sobolev-Bochner space (see [20, p. 286]). Note that for every function $u \in L^1(Q_{0,T}) = L^1(0,T;L^1(\Omega))$ we have $u(\cdot,t) \in L^1(\Omega)$, $t \in (0,T)$. For the sake of convenience we will write $u(t)$ instead of $u(\cdot,t)$ and $L^1(0,T)$ instead of $L^1((0,T))$, etc.

Let $D^*(\mathcal{O})$ and $D^*(0,T;X)$ be spaces of distributions (see [19, p. 166-170]). We identify every $f \in L^{\mathfrak{p}}(0,T;X)$ with the distribution (we call it f again) defined by the following rule:

$$\langle f, \varphi\rangle_{D(0,T)} := \int\limits_0^T f(t)\varphi(t)\, dt, \ \varphi \in D(0,T).$$

This provides a topological embedding $L^{\mathfrak{p}}(0,T;X) \subset D^*(0,T;X)$ and allows to define the derivatives of f to be f_t. Now, we define $W^{0,\mathfrak{p}}(0,T;X) := L^{\mathfrak{p}}(0,T;X)$ and (see [3, Sect. 3.6-3.8] and [16, Sect. 2.4] for more details)

$$W^{-1,\mathfrak{p}}(0,T;X) := \{f \in D^*(0,T;X) \mid f = f_0 + (f_1)_t, \ \ f_0, f_1 \in L^{\mathfrak{p}}(0,T;X)\}.$$

Let $\mathcal{B}_+(\mathcal{O}) := \Big\{\mathfrak{q} \in L^\infty(\mathcal{O}) \mid \operatorname*{ess\,inf}_{y\in\mathcal{O}} \mathfrak{q}(y) > 0\Big\}$. For every $\mathfrak{q} \in \mathcal{B}_+(\mathcal{O})$, we put

$$\mathfrak{q}_0 := \operatorname*{ess\,inf}_{y\in\mathcal{O}} \mathfrak{q}(y), \quad \mathfrak{q}^0 := \operatorname*{ess\,sup}_{y\in\mathcal{O}} \mathfrak{q}(y), \qquad S_{\mathfrak{q}}(r) := \max\Big\{r^{\mathfrak{q}_0},\, r^{\mathfrak{q}^0}\Big\}, \quad r \geq 0, \tag{7}$$

$$\mathfrak{q}'(y) := \frac{\mathfrak{q}(y)}{\mathfrak{q}(y)-1} \text{ for a.e. } y \in \mathcal{O} \ \Big(\frac{1}{\mathfrak{q}(y)} + \frac{1}{\mathfrak{q}'(y)} = 1 \ \text{ and } \ \mathfrak{q}' \in \mathcal{B}_+(\mathcal{O}), \ \text{ if } \ \mathfrak{q}_0 > 1\Big), \tag{8}$$

$$\rho_{\mathfrak{q}}(v;\mathcal{O}) := \int_{\mathcal{O}} |v(y)|^{\mathfrak{q}(y)}\, dy, \quad v \in \mathcal{M}(\mathcal{O}). \tag{9}$$

Assume that $\mathfrak{q} \in \mathcal{B}_+(\mathcal{O})$ and $\mathfrak{q}_0 > 1$. The set

$$L^{\mathfrak{q}(y)}(\mathcal{O}) := \{v \in \mathcal{M}(\mathcal{O}) \mid \rho_{\mathfrak{q}}(v;\mathcal{O}) < +\infty\}$$

with the Luxemburg norm

$$\|v; L^{\mathfrak{q}(y)}(\mathcal{O})\| := \inf\{\lambda > 0 \mid \rho_{\mathfrak{q}}(v/\lambda;\mathcal{O}) \leq 1\}$$

is called a *generalized Lebesgue space*. It is well known that $L^{\mathfrak{q}(y)}(\mathcal{O})$ is the Banach space which is reflexive and separable. The properties of these spaces were studied in [4], [5], [6].

Assume that the following conditions are fulfilled.

(G): $g \in \mathcal{B}_+(Q_{0,T})$, $q \in \mathcal{B}_+(\Omega)$, $q_0 > 1$;

(E): $\mathfrak{Z}$ is a n-order square matrix with the elements from $L^\infty(Q_{0,T} \times \Omega)$, $\Phi = (\phi_1, \dots, \phi_n)$, $\Phi(0) = 0$, $\phi_1, \dots, \phi_n \in C(\mathbb{R})$, and there exist a constant $L > 0$ such that for every $\xi_1, \xi_2 \in \mathbb{R}^n$ we have: $|\Phi(\xi_1) - \Phi(\xi_2)| \leq L|\xi_1 - \xi_2|$;

(U): $F \in [L^2(Q_{0,T})]^n$, $u_0 \in H$;

(W): $f \in L^2(Q_{0,T})$, $\pi_0 \in L^2(\Omega)$.

By definition, put $Y_1 := [H_0^1(\Omega) \cap L^{q(x)}(\Omega)]^n$, $Y_2 := H_0^1(\Omega)$,

$$U_1(Q_{0,T}) := L^2(0,T;[H_0^1(\Omega)]^n) \cap [L^{q(x)}(Q_{0,T})]^n, \quad U_2(Q_{0,T}) := L^2(0,T;Y_2);$$

$$(u,v)_\Omega := \begin{cases} \displaystyle\int_\Omega (u(x),v(x))_{\mathbb{R}^n}\, dx, & \text{if} \quad u = (u_1,\dots,u_n), v = (v_1,\dots,v_n) : \Omega \to \mathbb{R}^n, \\ \displaystyle\int_\Omega u(x)v(x)\, dx, & \text{if} \quad u, v : \Omega \to \mathbb{R}; \end{cases} \tag{10}$$

$$\chi_{0,\tau}(t) = \begin{cases} 1, & t \in (0,\tau), \\ 0, & t \notin (0,\tau), \end{cases} \qquad \tau \in (0,T]; \tag{11}$$

$$(\mathcal{N}(t)z)(x) := g(x,t)|z(x)|^{q(x)-2}z(x), \quad z = z(x), \quad x \in \Omega, \ \ t \in (0,T); \tag{12}$$

$$(\mathtt{N}u)(x,t) := \big(\mathcal{N}(t)u(t)\big)(x) = g(x,t)|u(x,t)|^{q(x)-2}u(x,t), \ u = u(x,t), \ (x,t) \in Q_{0,T}; \tag{13}$$

$$(E(t)z)(x) := \int\limits_{\Omega} \mathfrak{Z}(x,t,y)z(y)\,dy, \quad z = z(x), \quad x \in \Omega, \quad t \in (0,T); \tag{14}$$

$$(\mathtt{E}u)(x,t) := \big(E(t)u(t)\big)(x) = \int\limits_{\Omega} \mathfrak{Z}(x,t,y)u(y,t)\,dy, \quad u = u(x,t), \quad (x,t) \in Q_{0,T}. \tag{15}$$

Remark 1. For smooth functions $\pi : \Omega \to \mathbb{R}^1$, $z : \Omega \to \mathbb{R}^n$ such that $z|_{\partial\Omega} = 0$ we have

$$\int\limits_{\Omega} (\nabla\pi, z)_{\mathbb{R}^n}\,dx = \int\limits_{\Omega} \sum_{l=1}^{n} \pi_{x_l} z_l\,dx = -\int\limits_{\Omega} \pi \sum_{l=1}^{n} (z_l)_{x_l}\,dx = -\int\limits_{\Omega} \pi\,\operatorname{div} z\,dx.$$

Using Remark 1, we give the following definition.

Definition 1. The functions $\{u^\varepsilon, \pi^\varepsilon\}$ are called a *weak solution* of problem (1)–(6) if

$$u^\varepsilon \in U_1(Q_{0,T}) \cap C([0,T];[L^2(\Omega)]^n), \quad u^\varepsilon_t \in [U_1(Q_{0,T})]^*,$$

$$\pi^\varepsilon \in U_2(Q_{0,T}) \cap C([0,T];L^2(\Omega)), \quad \pi^\varepsilon_t \in [U_2(Q_{0,T})]^*;$$

for all $w \in U_1(Q_{0,T})$, $\eta \in U_2(Q_{0,T})$, and $\tau \in (0,T]$ we have that

$$\langle u^\varepsilon_t, \chi_{0,\tau} w\rangle_{U_1(Q_{0,T})} + \int\limits_{Q_{0,\tau}} \Big[\sum_{i=1}^{n} (u^\varepsilon_{x_i}, w_{x_i})_{\mathbb{R}^n} + (\mathtt{N}u^\varepsilon, w)_{\mathbb{R}^n} + (\Phi(\mathtt{E}u^\varepsilon), w)_{\mathbb{R}^n} - \\ - \pi^\varepsilon \operatorname{div} w\Big]\,dxdt = \int\limits_{Q_{0,\tau}} (F, w)_{\mathbb{R}^n}\,dxdt, \tag{16}$$

$$\varepsilon\langle \pi^\varepsilon_t, \chi_{0,\tau}\eta\rangle_{U_2(Q_{0,T})} + \int\limits_{Q_{0,\tau}} \Big[\varepsilon \sum_{i=1}^{n} \pi^\varepsilon_{x_i}\eta_{x_i} + \eta \operatorname{div} u^\varepsilon\Big]\,dxdt = \int\limits_{Q_{0,\tau}} (f, \eta)_{\mathbb{R}^n}\,dxdt \tag{17}$$

hold; initial conditions (5)–(6) hold.

Theorem 1 (**uniqueness**). *Let $\varepsilon > 0$ and conditions* **(G)**–**(E)** *hold. Then, problem* (1)–(6) *can't have more that one weak solution* $\{u^\varepsilon, \pi^\varepsilon\}$.

Let $\Delta^0 v := v$, $\Delta^1 v := \Delta v$ is a Laplasian, $\Delta^s v := \Delta(\Delta^{s-1}v)$, $s \in \mathbb{N}$,

$$\mathcal{W}_s := \big\{v \in H^{2s}(\Omega) \;\big|\; v|_{\partial\Omega} = \Delta v|_{\partial\Omega} = \ldots = \Delta^{s-1}v|_{\partial\Omega} = 0\big\}, \quad \mathcal{W}_s^* := [\mathcal{W}_s]^*. \tag{18}$$

Suppose that

$$s \in \mathbb{N}, \quad s \geq n\Big(\frac{1}{2} - \frac{1}{q^0}\Big), \quad h := \min\Big\{2, \frac{q^0}{q^0 - 1}\Big\}. \tag{19}$$

Note that if s satisfies (19), then $[\mathcal{W}_s]^n \overset{\circlearrowleft}{\subset} [H^1_0(\Omega) \cap L^{q^0}(\Omega)]^n \overset{\circlearrowleft}{\subset} Y_1$.

Theorem 2 (**existence**). *Let s is taken from* (19), *$\partial\Omega \in C^{2s}$, $\varepsilon > 0$, and conditions* **(G)**–**(W)** *hold. Then problem* (1)–(6) *has a weak solution* $\{u^\varepsilon, \pi^\varepsilon\}$.

If formally $f \equiv 0$, $\varepsilon = 0$, $u := u^0$, and $\pi := \pi^0$, then from (1)–(3) and (5) we get

$$u_t - \Delta u + g(x,t)\,|u|^{q(x)-2}u + \Phi\Big(\int\limits_{\Omega} \mathfrak{Z}(x,t,y)u(y,t)\,dy\Big) + \nabla\pi = F(x,t) \quad \text{in} \quad Q_{0,T}, \tag{20}$$

$$\operatorname{div} u = 0 \quad \text{in} \quad Q_{0,T}, \tag{21}$$

$$u|_{\partial\Omega\times(0,T)} = 0, \tag{22}$$

$$u|_{t=0} = u_0(x) \quad \text{in} \quad \Omega. \tag{23}$$

Let us consider the Sobolev space $[H^s(\Omega)]^n$ with the scalar product

$$((u,v))_s := \sum_{l=1}^{n} (u_l, v_l)_{H^s(\Omega)}, \quad u = (u_1, \dots, u_n), v = (v_1, \dots, v_n) \in [H^s(\Omega)]^n. \tag{24}$$

Let $C_{\text{div}} := \{u \in [D(\Omega)]^n \mid \operatorname{div} u = 0\}$, H is a closure of C_{div} in $[L^2(\Omega)]^n$,

$$Z_s \quad \text{is a closure of} \quad C_{\text{div}} \quad \text{in} \quad [H^s(\Omega)]^n,$$

where

$$||h; H|| := ||h; [L^2(\Omega)]^n|| = ||h_1; L^2(\Omega)|| + ... + ||h_n; L^2(\Omega)||$$

for $h = (h_1, \dots, h_n) \in H$, and $||z; Z_s|| := \sqrt{((z,z))_s}$ for $z = (z_1, \dots, z_n) \in Z_s$. By definition, put

$$V_1 := Z_1 \cap [L^{q(x)}(\Omega)]^n, \qquad U_3(Q_{0,T}) := L^2(0,T; Z_1) \cap [L^{q(x)}(Q_{0,T})]^n.$$

Note that if s and h satisfy (19), then $Z_s \overline{\circlearrowright} (Z_1 \cap [L^{q^0}(\Omega)]^n) \overline{\circlearrowright} V_1$, $H^{-1}(\Omega) \overline{\circlearrowright} W^{-1,h}(\Omega)$, and $L^{\frac{q^0}{q^0-1}}(\Omega) \overline{\circlearrowright} W^{-1,h}(\Omega)$.

Definition 2. The functions $\{u, \pi\}$ are called a *weak solution* of problem (20)–(23), if

$$u \in U_3(Q_{0,T}) \cap C([0,T]; Z_s^*), \quad u_t \in [U_3(Q_{0,T})]^*, \quad \pi \in W^{-1,\infty}(0,T; L^h(\Omega));$$

for all $w \in U_3(Q_{0,T})$ and $\tau \in (0,T]$ we have that

$$\langle u_t, \chi_{0,\tau} w \rangle_{U_3(Q_{0,T})} +$$

$$+ \int\limits_{Q_{0,\tau}} \Big[\sum_{i=1}^{n} (u_{x_i}, w_{x_i})_{\mathbb{R}^n} + (\mathtt{N}u, w)_{\mathbb{R}^n} + (\Phi(\mathtt{E}u), w)_{\mathbb{R}^n} - (F, w)_{\mathbb{R}^n} \Big] \, dxdt = 0 \tag{25}$$

holds; initial condition (23) holds; π satisfies (20) in $D^*(Q_{0,T})$.

Theorem 3. *Let s is taken from* (19), *$\partial\Omega \in C^{2s}$, and conditions* **(G)**–**(U)** *hold. Then problem* (20)–(23) *has a weak solution $\{u, \pi\}$. Moreover:*

i) u is a limit of the subsequence of $\{u^\varepsilon\}_{\varepsilon>1}$, where every u^ε is taken from Theorem 2;

ii) in the sense of $D^*(0,T)$,

we get

$$\int\limits_{\Omega} \pi(x,t) \, dx = 0 \quad \text{in} \quad (0,T). \tag{26}$$

Notice that the Oskolkov-Stokes systems with the variable exponents of the nonlinearities have not been considered yet.

3. Auxiliary statements

For the sake of convenience we remind some useful facts. First,

$$\alpha\beta \le \frac{\varepsilon\alpha^2}{2} + \frac{\beta^2}{2\varepsilon}, \quad \alpha,\beta \ge 0, \quad \varepsilon > 0, \tag{27}$$

$$|\gamma_1 + \ldots + \gamma_m|^{\mathfrak{p}} \le m^{\mathfrak{p}-1}(|\gamma_1|^{\mathfrak{p}} + \ldots + |\gamma_m|^{\mathfrak{p}}), \quad \gamma_1,\ldots,\gamma_m \in \mathbb{R}, \quad m \in \mathbb{N}, \tag{28}$$

(see, for example, [21, p. 168]). Let $\mathbb{Z}_{\ge -1} := \{s \in \mathbb{Z} \mid s \ge -1\}$. The following Propositions are needed for the sequel.

Proposition 1 (**the generalized De Rham theorem**, [16, p. 171, Proposition 7]). *Suppose that Ω be an open bounded connected and Lipschitz subset of $\mathbb{R}^n$, $T > 0$, $s_1, s_2 \in \mathbb{Z}_{\ge -1}$, $h_1, h_2 \in [1,\infty]$, and $\mathcal{F} \in W^{s_1,h_1}(0,T;[W^{s_2,h_2}(\Omega)]^n)$. Then, if*

$$\langle \mathcal{F}(\cdot), v\rangle_{[D(\Omega)]^n} = 0 \quad in \quad D^*(0,T) \tag{29}$$

for all $v \in \mathcal{V} = \{v \in [D(\Omega)]^n \mid div\, v = 0\}$, then there exists an unique

$$\pi \in W^{s_1,h_1}(0,T;W^{s_2+1,h_2}(\Omega)) \tag{30}$$

such that

$$\nabla\pi = \mathcal{F} \quad in \quad [D^*(Q_{0,T})]^n, \tag{31}$$

$$\int_\Omega \pi(\cdot)\, dx = 0 \quad in \quad D^*(0,T). \tag{32}$$

Moreover, there exists a positive number C_1 (independent of $\mathcal{F}, \pi$) such that

$$||\pi; W^{s_1,h_1}(0,T;W^{s_2+1,h_2}(\Omega))|| \le C_1 ||\mathcal{F}; W^{s_1,h_1}(0,T;[W^{s_2,h_2}(\Omega)]^n)||. \tag{33}$$

Proposition 2 (**the analogue of the Fatou Lemma** [25, p. 58, 63]). *Let B be a Banach space. If $z^j \underset{j\to\infty}{\longrightarrow} z$ slowly, or $*$-slowly in B, then we have that $||z||_B \le \varliminf_{j\to\infty} ||z^j||_B$.*

For the Banach spaces X and Y the notation $X \circlearrowleft Y$ means the continuous embedding and the notation $X \overset{K}{\subset} Y$ means the compact embedding.

Proposition 3 (**the Aubin theorem**, see [26] and [27, p. 393]). *If $s, h > 1$ are fixed numbers, $\mathcal{W}, \mathcal{L}, \mathcal{B}$ are the Banach spaces, and $\mathcal{W} \overset{K}{\subset} \mathcal{L} \circlearrowleft \mathcal{B}$, then*

$$\{u \in L^s(0,T;\mathcal{W}) \mid u_t \in L^h(0,T;\mathcal{B})\} \overset{K}{\subset} [L^s(0,T;\mathcal{L}) \cap C([0,T];\mathcal{B})].$$

Proposition 4 (Lemma 1 [21, p. 168]). *Suppose that $\mathfrak{q} \in \mathcal{B}_+(\mathcal{O})$, $\mathfrak{q}_0 > 1$, $S_{\mathfrak{q}}$ is defined by (7), and $\rho_{\mathfrak{q}}$ is defined by (9). Then for every $v \in \mathcal{M}(\mathcal{O})$ the following statements are fulfilled:*

- *(i)* $||v; L^{\mathfrak{q}(y)}(\mathcal{O})|| \le S_{1/\mathfrak{q}}(\rho_{\mathfrak{q}}(v;\mathcal{O}))$ *if* $\rho_{\mathfrak{q}}(v;\mathcal{O}) < +\infty$;
- *(ii)* $\rho_{\mathfrak{q}}(v;\mathcal{O}) \le S_{\mathfrak{q}}(||v; L^{\mathfrak{q}(y)}(\mathcal{O})||)$ *if* $||v; L^{\mathfrak{q}(y)}(\mathcal{O})|| < +\infty$.

Proposition 5 (Lemma 2 [11, p. 134]). *Suppose that the Nemytskij operators $\mathcal{N}$ and $\mathtt{N}$ are defined by (12)–(13). If condition **(G)** holds, then $\mathcal{N} : [L^{q(x)}(\Omega)]^n \to [L^{q'(x)}(\Omega)]^n$ and $\mathtt{N} : [L^{q(x)}(Q_{0,T})]^n \to [L^{q'(x)}(Q_{0,T})]^n$ are bounded and continuous operators.*

Proposition 6 (Lemma 1.18 [19, p. 39]). *For sequence $\{u^m\}_{m\in\mathbb{N}}$, $u^m \underset{m\to\infty}{\longrightarrow} u$ in $L^p(Q_{0,T})$, $p\in[1,\infty]$, there exists a subsequence $\{u^{m_k}\}_{k\in\mathbb{N}} \subset \{u_m\}_{m\in\mathbb{N}}$ such that $u^{m_k} \underset{k\to\infty}{\longrightarrow} u$ a.e. in $Q_{0,T}$.*

Proposition 7 (**the Gronwall-Bellman lemma**, [22, p. 25]). *Suppose that $A, B \in L^1(0,T)$ and $y \in C([0,T])$ are nonnegative functions. If for every $\tau\in[0,T]$ we have*

$$y(\tau) \le C + \int\limits_0^\tau \Big[A(t)y(t) + B(t)\Big]\,dt, \tag{34}$$

where C is a nonnegative number, then the following inequality is true

$$y(\tau) \le \Big(C + \int\limits_0^\tau B(t)\, e^{-\int\limits_0^t A(s)\,ds}\,dt\Big)\, e^{\int\limits_0^\tau A(t)\,dt}, \quad \tau\in[0,T]. \tag{35}$$

From Proposition 7 we obtain the following useful lemma (we omit its proof).

Lemma 1 (**the generalized Gronwall-Bellman lemma**). *Suppose that $K, K_t, L \in L^1(0,T)$ and $y \in C([0,T])$ are nonnegative functions. If for every $\tau\in[0,T]$ we have that*

$$y(\tau) \le K(\tau) + \int\limits_0^\tau L(t)y(t)\,dt \tag{36}$$

holds, then the following inequality is true: $y(\tau) \le K(\tau)\, e^{\int\limits_0^\tau L(s)\,ds}$, $\tau\in[0,T]$.

3.1. **Functional spaces and some operators.**

Let $\mathcal{H}$ be the Hilbert space with a scalar product $(\cdot,\cdot)_{\mathcal{H}}$, $\mathcal{V}$ be a reflexive separable Banach space, $\mathcal{V} \overset{-}{\circlearrowright} \mathcal{H} \cong \mathcal{H}^* \overset{-}{\circlearrowright} \mathcal{V}^*$, $\{w^j\}_{j\in\mathbb{N}}$ be an orthonormal basis for the space $\mathcal{H}$, $m\in\mathbb{N}$ be a fixed number, and $\mathfrak{M}$ be a set of all linear combinations of the elements from $\{w^1,\dots,w^m\}$. Define an unique orthogonal projection $P_m : \mathcal{H}\to\mathfrak{M}$ by the rule (see [23, p. 527])

$$P_m h := \sum_{j=1}^m (h, w^j)_{\mathcal{H}}\, w^j, \quad h\in\mathcal{H}. \tag{37}$$

This is a linear self-adjoint continuous operator (see Theorem 7.3.6 [23, p. 515]). If $\{w^j\}_{j\in\mathbb{N}} \subset \mathcal{V}$, then let us define an operator $\widehat{P}_m : \mathcal{V}\to\mathcal{V}$ (not necessarily self-adjoint) by the rule

$$\widehat{P}_m v := P_m v \quad \text{for every} \quad v\in\mathcal{V}. \tag{38}$$

For a conjugate operator $\widehat{P}_m^* : \mathcal{V}^*\to\mathcal{V}^*$ we have $\widehat{P}_m^*(\mathcal{V}^*) \subset \mathcal{V}$ (see [10, p. 865]).

Proposition 8 ([10, p. 865–866, Lemma 3.9]). *Assume that $\{w^j\}_{j\in\mathbb{N}}$ is an orthonormal basis for the space $\mathcal{H}$ such that $\{w^j\}_{j\in\mathbb{N}} \subset \mathcal{V}$, $\psi_1^m,\dots,\psi_m^m \in \mathbb{R}$ are some numbers, and*

$F \in \mathcal{V}^*$. Then $z^m := \sum_{s=1}^{m} \psi_s^m w^s \in \mathcal{V}$ satisfies

$$\begin{cases} \langle z^m, w^1 \rangle_{\mathcal{V}} = \langle F, w^1 \rangle_{\mathcal{V}}, \\ \quad \dots \\ \langle z^m, w^m \rangle_{\mathcal{V}} = \langle F, w^m \rangle_{\mathcal{V}}, \end{cases}$$

iff the following equality holds: $z^m = \widehat{P}_m^* F$ in $\mathcal{V}^*$.

Let $\mathcal{W}_s$ is taken from (18), where $s \in \mathbb{N}$. It is easy to verify that $\mathcal{W}_s$ is the reflexive Hilbert space with respect to the scalar product $(u, v)_{\mathcal{W}_s} := (\Delta^s u, \Delta^s v)_\Omega$, $u, v \in \mathcal{W}_s$, and $\mathcal{W}_s \circlearrowleft H^{2s}(\Omega)$, $\mathcal{W}_s \overline{\circlearrowleft} L^2(\Omega) \overline{\circlearrowleft} [\mathcal{W}_s]^*$.

Let $\{v^\mu\}_{\mu \in \mathbb{N}} \subset H_0^1(\Omega)$ is an orthonormal basis for the space $L^2(\Omega)$ that consists of all eigenfunctions of the problem

$$-\Delta\, v = \lambda\, v \quad \text{in} \quad \Omega, \qquad v|_{\partial\Omega} = 0, \tag{39}$$

and $\{\lambda_\mu\}_{\mu \in \mathbb{N}} \subset \mathbb{R}_{>0}$ is the set of the corresponding eigenvalues.

Proposition 9. *If $\partial\Omega \subset C^{2s}$, then the set $\{v^\mu\}_{\mu \in \mathbb{N}} \subset \mathcal{W}_s$ of all eigenfunction of problem* (39) *is a basis for the space $\mathcal{W}_s$.*

Lemma 2. *If $\partial\Omega \subset C^{2s}$, $\ell \in \mathbb{N}$, then there exists a basis $\{w^\mu\}_{\mu \in \mathbb{N}} \subset [\mathcal{W}_s]^\ell$ for the space $[\mathcal{W}_s]^\ell$. Moreover, for every $\mu \in \mathbb{N}$ all without one coordinates of the vector-function w^μ equal zero and non-zero coordinate of the w^μ is an eigenfunction of problem* (39).

Proof. For the sake of convenience we consider only the case $\ell = 2$.

Let $s \in \mathbb{N}$ and the set $\{v^\mu\}_{\mu \in \mathbb{N}} \subset \mathcal{W}_s$ is taken from Proposition 9. Then for every $v \in \mathcal{W}_s$ there exists a sequence $\{v_m\}_{m \in \mathbb{N}}$ such that $v_m \underset{m\to\infty}{\longrightarrow} v$ in $\mathcal{W}_s$ and for every $m \in \mathbb{N}$ we have

$$v_m(x) = \sum_{k=1}^{m} \gamma_k^m\, v^k(x), \quad x \in \Omega,$$

where $\gamma_1^m, \dots, \gamma_m^m \in \mathbb{R}$. Then for every $(v, z) \in [\mathcal{W}_s]^2$ we can find $\{(v_m, z_m)\}_{m \in \mathbb{N}}$ such that $(v_m, z_m) \underset{m\to\infty}{\longrightarrow} (v, z)$ in $[\mathcal{W}_s]^2$ and for every $m \in \mathbb{N}$ we have

$$\Big(v_m(x), z_m(x)\Big) = \Big(\sum_{k=1}^{m} \gamma_k^m\, v^k(x),\ \sum_{k=1}^{m} \widetilde{\gamma_k^m}\, v^k(x)\Big) =$$

$$= \sum_{k=1}^{m} \gamma_k^m (v^k(x), 0) + \sum_{k=1}^{m} \widetilde{\gamma_k^m} (0, v^k(x)), \quad x \in \Omega,$$

where $\gamma_1^m, \dots, \gamma_m^m, \widetilde{\gamma_1^m}, \dots, \widetilde{\gamma_m^m} \in \mathbb{R}$. Clearly, the set

$$\{z \ \mid\ z = (v^k, 0) \ \text{ or } \ z = (0, v^k), \ \ k \in \mathbb{N}\}$$

is countable. □

Proposition 10 ([10, p. 866, 880]). *Suppose that $\partial\Omega \subset C^{2s}$, $\ell \in \mathbb{N}$, P_m and $\widehat{P}_m$ are determined from* (37) *and* (38) *respectively, where $\mathcal{H} = [L^2(\Omega)]^\ell$, $\mathcal{V} = [\mathcal{W}_s]^\ell$, $s \in \mathbb{N}$, and*

$\{w^\mu\}_{\mu\in\mathbb{N}}$ *is an orthonormal basis for the space* $\mathcal{H}$ *that it is taken from Lemma 2. Then, for every* $w \in L^r(0,T;[\mathcal{W}_s^*]^\ell)$ *and* $r>1$*, we have the following inequality*

$$||\widehat{P}_m^* w; L^r(0,T;[\mathcal{W}_s^*]^\ell)|| \le ||w; L^r(0,T;[\mathcal{W}_s^*]^\ell)||. \tag{40}$$

3.2. Cauchy's problem for system of ordinary differential equations.

Take $\ell\in\mathbb{N}$ and $Q=(0,T)\times\mathbb{R}^\ell$. In this section we seek a weak solution $\eta:[0,T]\to\mathbb{R}^\ell$ of the problem

$$\eta'(t)+L(t,\eta(t))=M(t), \quad t\in[0,T], \qquad \eta(0)=\eta^0, \tag{41}$$

where $M:[0,T]\to\mathbb{R}^\ell$, $L:Q\to\mathbb{R}^\ell$ are some functions (for the sake of convenience we have assumed that $L(t,0)=0$ for every $t\in[0,T]$) and $\eta^0=(\eta_1^0,\dots,\eta_\ell^0)\in\mathbb{R}^\ell$.

Definition 3. A real-valued function $\eta\in W^{1,1}(0,T;\mathbb{R}^k)$ is called a *weak solution* of problem (41) if u satisfies the initial value condition and satisfies the equation almost everywhere.

Definition 4. We shall say that a function $L:Q\to\mathbb{R}^\ell$ *satisfies the Carathéodory condition* if for every $\xi\in\mathbb{R}^\ell$ the function $(0,T)\ni t\mapsto L(t,\xi)\in\mathbb{R}^\ell$ is measurable and if for a.e. $t\in(0,T)$ the function $\mathbb{R}^\ell\ni\xi\mapsto L(t,\xi)\in\mathbb{R}^\ell$ is continuous.

Definition 5 ([24, p. 241])**.** We shall say that a function $L:Q\to\mathbb{R}^\ell$ satisfies the L^p-*Carathéodory condition* if L satisfies the Carathéodory condition and for every $R>0$ there exists a function $h_R\in L^p(0,T)$ such that $|L(t,\xi)|\le h_R(t)$ for a.e. $t\in(0,T)$ and for every $\xi\in\{y\in\mathbb{R}^\ell \mid |y|\le R\}$.

Proposition 11 (**the Carathéodory-LaSalle Theorem**, [10, p. 872, Theorem 3.24])**.** *Suppose that* $p\ge 2$*, function* $L:Q\to\mathbb{R}^\ell$ *satisfies* L^p*-Carathéodory condition,* $M\in L^p(0,T;\mathbb{R}^\ell)$*, and* $\eta^0\in\mathbb{R}^\ell$*. If there exists a nonnegative functions* $\alpha,\beta\in L^1(0,T)$ *such that for every* $\xi\in\mathbb{R}^\ell$ *and for a.e.* $t\in[0,T]$ *the inequality*

$$(L(t,\xi),\xi)_{\mathbb{R}^\ell}\ge -\alpha(t)|\xi|^2-\beta(t) \tag{42}$$

holds, then problem (41) *has a global weak solution* $\eta\in W^{1,p}(0,T;\mathbb{R}^\ell)$*.*

3.3. Additional statements.

The following facts are needed for the sequel.

Remark 2. If $u=(u_1,\dots,u_n)\in[L^2(\mathcal{O})]^n$, where $\mathcal{O}=\Omega$ or $\mathcal{O}=Q_{0,T}$, then

$$||\,|u|;L^2(\mathcal{O})||^2=\int\limits_{\mathcal{O}}|u|^2\,dy=\sum_{l=1}^n||u_l;L^2(\mathcal{O})||^2\le n||u;[L^2(\mathcal{O})]^n||^2,$$

and so

$$||\,|u|;L^2(\mathcal{O})||\le\sqrt{n}||u;[L^2(\mathcal{O})]^n||. \tag{43}$$

Lemma 3. *If condition* **(E)** *holds, then the operators* $E(t):[L^2(\Omega)]^n\to[L^2(\Omega)]^n$ *and* $\mathtt{E}:[L^2(Q_{0,T})]^n\to[L^2(Q_{0,T})]^n$ *(see* (14)–(15)*) are linear bounded and continuous. Moreover, for every* $t,\tau\in(0,T)$ *the following estimates are true:*

$$||\,|E(t)z|;L^2(\Omega)||\le E^0||\,|z|;L^2(\Omega)||\le\sqrt{n}E^0||z;[L^2(\Omega)]^n||, \quad z\in[L^2(\Omega)]^n, \tag{44}$$

$$||\,|\mathtt{E}u|;L^2(Q_{0,\tau})||\le E^0||\,|u|;L^2(Q_{0,\tau})||\le\sqrt{n}E^0||u;[L^2(Q_{0,\tau})]^n||,\ u\in[L^2(Q_{0,T})]^n, \tag{45}$$

where the constant $E^0>0$ *is independent of* z,u,t,τ*.*

Proof. The Cauchy-Bunyakowski-Schwarz inequality and condition **(E)** yields that

$$
\begin{aligned}
|| \, |E(t)z|; L^2(\Omega)||^2 &= \int\limits_\Omega |(E(t)z)(x)|^2 \, dx = \\
&= \int\limits_\Omega \Big| \int\limits_\Omega \mathfrak{Z}(x,t,y) z(y) \, dy \Big|^2 \, dx \le \\
&\le \int\limits_\Omega \Big| \int\limits_\Omega ||\mathfrak{Z}(x,t,y)||_n \cdot |z(y)| \, dy \Big|^2 \, dx \le \\
&\le \int\limits_\Omega \Big(\int\limits_\Omega ||\mathfrak{Z}(x,t,y)||_n^2 \, dy \Big) \Big(\int\limits_\Omega |z(y)|^2 \, dy \Big) \, dx \le \\
&\le |E^0|^2 \int\limits_\Omega |z(y)|^2 \, dy = \\
&= |E^0|^2 \cdot || \, |z|; L^2(\Omega)||^2,
\end{aligned}
$$

where

$$
E^0 = \Big(\int\limits_\Omega \Big(\operatorname{ess} \sup_{t \in (0,T)} \int\limits_\Omega ||\mathfrak{Z}(x,t,y)||_n^2 \, dy \Big) \, dx \Big)^{1/2},
$$

$|| \cdot ||_n$ is a norm of the matrix. So, we get (44) (see also (43)). Similarly,

$$
\begin{aligned}
|| \, |\mathtt{E}u|; L^2(Q_{0,\tau})|| &= \int\limits_{Q_{0,\tau}} \Big| \int\limits_\Omega \mathfrak{Z}(x,t,y) u(y,t) \, dy \Big|^2 dxdt \le \\
&\le \int\limits_{Q_{0,\tau}} \Big| \int\limits_\Omega ||\mathfrak{Z}(x,t,y)||_n \cdot |u(y,t)| \, dy \Big|^2 dxdt \le \\
&\le \int\limits_\Omega \Big\{ \int\limits_0^\tau \Big(\int\limits_\Omega ||\mathfrak{Z}(x,t,y)||_n^2 \, dy \Big) \Big(\int\limits_\Omega |u(y,t)|^2 \, dy \Big) \, dt \Big\} \, dx \le \\
&\le |E^0|^2 \int\limits_{Q_{0,\tau}} |u(y,t)|^2 \, dydt,
\end{aligned}
$$

and (45) also holds. □

Lemma 4. *If condition **(E)** holds, then for every $r > 1$ there exists a constant $C_2 > 0$ such that for every $u_1, u_2 \in [L^r(Q_{0,T})]^n$ we have*

$$
||\Phi(\mathtt{E}u_1) - \Phi(\mathtt{E}u_2); [L^r(Q_{0,T})]^n|| \le C_2 ||u_1 - u_2; [L^r(Q_{0,T})]^n||. \tag{46}
$$

Proof. Using condition **(E)** and the Hölder inequality, we obtain

$$
\begin{aligned}
||\Phi(\mathtt{E}u_1) - \Phi(\mathtt{E}u_2); [L^r(Q_{0,T})]^n||^r &= \Big(\sum_{l=1}^{n} ||\phi_l(\mathtt{E}u_1) - \phi_l(\mathtt{E}u_2); L^r(Q_{0,T})||\Big)^r \le \\
&\le C_3 \sum_{l=1}^{n} \int\limits_{Q_{0,T}} |\phi_l(\mathtt{E}u_1) - \phi_l(\mathtt{E}u_2)|^r \, dxdt \le \\
&\le C_4 \int\limits_{Q_{0,T}} |\Phi(\mathtt{E}u_1) - \Phi(\mathtt{E}u_2)|^r \, dxdt \le \\
&\le C_4 L^r \int\limits_{Q_{0,T}} |\mathtt{E}u_1 - \mathtt{E}u_2|^r \, dxdt = \\
&= C_4 L^r \int\limits_{Q_{0,T}} \Big| \int\limits_{\Omega} \mathfrak{Z}(x,t,y)\Big(u_1(y,t) - u_2(y,t)\Big) \, dy\Big|^r \, dxdt \le \\
&\le C_5 \int\limits_{Q_{0,T}} \Big| \int\limits_{\Omega} ||\mathfrak{Z}(x,t,y)||_n \cdot |u_1(y,t) - u_2(y,t)| \, dy\Big|^r dxdt \le \\
&\le C_6 \int\limits_{Q_{0,T}} |u_1(y,t) - u_2(y,t)|^r \, dydt
\end{aligned}
$$

and (46) holds. □

Let us define the operators $A_k : Y_k \to Y_k^*$ and $\mathcal{A}_k : U_k(Q_{0,T}) \to [U_k(Q_{0,T})]^*$ by the rules:

$$
\langle A_1 z, w\rangle_{Y_1} := \int\limits_{\Omega} \sum_{i=1}^{n} \Big(z_{x_i}(x), w_{x_i}(x)\Big)_{\mathbb{R}^n} \, dx, \quad z, w \in Y_1; \tag{47}
$$

$$
\langle A_2 \xi, \eta\rangle_{Y_2} := \int\limits_{\Omega} \sum_{i=1}^{n} \xi_{x_i}(x)\eta_{x_i}(x) \, dx, \quad \xi, \eta \in Y_2; \tag{48}
$$

$$
\langle \mathcal{A}_k u, v\rangle_{U_k(Q_{0,T})} := \int\limits_0^T \langle A_k u(t), v(t)\rangle_{Y_k} \, dt, \quad u, v \in U_k(Q_{0,T}), \quad k = 1, 2. \tag{49}
$$

Lemma 5. *Let conditions* **(G)**–**(E)** *hold,* $\{w^j\}_{j\in\mathbb{N}} \subset Y_1$, $\{v^j\}_{j\in\mathbb{N}} \subset Y_2$, $m \in \mathbb{N}$, $L = col(L_1^1, \ldots, L_m^1, L_1^2, \ldots, L_m^2)$, *where* (*see also notation* (10))

$$
\begin{cases}
L_\mu^1(t,\xi) := \langle A_1 z^m, w^\mu\rangle_{Y_1} + \Big(\mathcal{N}(t) z^m, w^\mu\Big)_\Omega + \Big(\Phi(E(t)z^m), w^\mu\Big)_\Omega - \Big(s^m, div\, w^\mu\Big)_\Omega, \\
L_\mu^2(t,\xi) := \varepsilon\langle A_2 s^m, v^\mu\rangle_{Y_2} + \Big(div\, z^m, v^\mu\Big)_\Omega,
\end{cases}
$$

$\mu=\overline{1,m}$, $t\in(0,T)$, $\xi=(\widehat{\varphi}_1,\dots,\widehat{\varphi}_m,\widehat{\psi}_1,\dots,\widehat{\psi}_m)\in\mathbb{R}^m$, $z^m(x):=\sum\limits_{\mu=1}^m\widehat{\varphi}_\mu w^\mu(x)$, $s^m(x):=\sum\limits_{\mu=1}^m\widehat{\psi}_\mu v^\mu(x)$, $x\in\Omega$. *Then,*

$$\big(L(t,\xi),\xi\big)_{\mathbb{R}^{2m}}\geq\int\limits_\Omega\Big[\sum_{i=1}^n|z^m_{x_i}|^2+\varepsilon\sum_{i=1}^n|s^m_{x_i}|^2+g_0|z^m|^{q(x)}-LE^0|z^m|^2\Big]\,dx,\quad t\in(0,T),\tag{50}$$

where L is taking from **(E)**, *E^0 is taking from Lemma 3 (see also notation of type (7)).*

Proof. It's clear that

$$\Big(L(t,\xi),\xi\Big)_{\mathbb{R}^{2m}}=\sum_{\mu=1}^m L^1_\mu(t,\xi)\widehat{\varphi}_\mu+\sum_{\mu=1}^m L^2_\mu(t,\xi)\widehat{\psi}_\mu=\Big\langle A_1z^m,\sum_{\mu=1}^m\widehat{\varphi}_\mu w^\mu\Big\rangle_{Y_1}+$$
$$+\Big(\mathcal{N}(t)z^m,\sum_{\mu=1}^m\widehat{\varphi}_\mu w^\mu\Big)_\Omega+\Big(\Phi(E(t)z^m),\sum_{\mu=1}^m\widehat{\varphi}_\mu w^\mu\Big)_\Omega-\Big(s^m,\operatorname{div}\sum_{\mu=1}^m\widehat{\varphi}_\mu w^\mu\Big)_\Omega+$$
$$+\varepsilon\Big\langle A_2s^m,\sum_{\mu=1}^m\widehat{\psi}_\mu v^\mu\Big\rangle_{Y_2}+\Big(\operatorname{div}z^m,\sum_{\mu=1}^m\widehat{\psi}_\mu v^\mu\Big)_\Omega=$$
$$=\langle A_1z^m,z^m\rangle_{Y_1}+(\mathcal{N}(t)z^m,z^m)_\Omega+\Big(\Phi(E(t)z^m),z^m\Big)_\Omega+\varepsilon\langle A_2s^m,s^m\rangle_{Y_2}.$$

Then we get

$$\Big(L(t,\xi),\xi\Big)_{\mathbb{R}^{2m}}=\int\limits_\Omega\Big[\sum_{i=1}^n(z^m_{x_i}(x),z^m_{x_i}(x))_{\mathbb{R}^n}+\sum_{l=1}^n g(x,t)|z^m(x)|^{q(x)-2}|z^m_l(x)|^2+$$
$$+\Big(\Phi(E(t)z^m(x)),z^m(x)\Big)_{\mathbb{R}^n}+\varepsilon\sum_{i=1}^n s^m_{x_i}(x)s^m_{x_i}(x)\Big]\,dx\geq$$
$$\geq\int\limits_\Omega\Big[\sum_{i=1}^n|z^m_{x_i}|^2+\varepsilon\sum_{i=1}^n|s^m_{x_i}|^2+g_0|z^m|^{q(x)}\Big]\,dx-\int\limits_\Omega\Big|\Big(\Phi(E(t)z^m),z^m\Big)_{\mathbb{R}^n}\Big|\,dx.\tag{51}$$

For the next term we use condition **(E)** and estimate (44):

$$\int\limits_\Omega\Big|\Big(\Phi(E(t)z^m),z^m\Big)_{\mathbb{R}^n}\Big|\,dx\leq L\int\limits_\Omega|E(t)z^m|\cdot|z^m|\,dx\leq$$
$$\leq L||\,|E(t)z^m|;L^2(\Omega)||\cdot||\,|z^m|;L^2(\Omega)||\leq$$
$$\leq LE^0||\,|z^m|;L^2(\Omega)||\cdot||\,|z^m|;L^2(\Omega)||=LE^0\int\limits_\Omega|z^m|^2\,dx.\tag{52}$$

Using (52), from (51) we get (50). □

3.4. **Proof of main Theorems.**

Proof of Theorem 1. Let $\{u^{1,\varepsilon},\pi^{1,\varepsilon}\}$ and $\{u^{2,\varepsilon},\pi^{2,\varepsilon}\}$ be weak solutions of problem (1)–(6). Set $u^\varepsilon=u^{1,\varepsilon}-u^{2,\varepsilon}$, $\pi^\varepsilon=\pi^{1,\varepsilon}-\pi^{2,\varepsilon}$. Then from (16)–(17) we obtain

$$\langle u_t^\varepsilon,\chi_{0,\tau}w\rangle_{U_1(Q_{0,T})}+\int\limits_{Q_{0,\tau}}\Big[\sum_{i=1}^n(u_{x_i}^\varepsilon,w_{x_i})_{\mathbb{R}^n}+$$

$$+\Big(\mathtt{N}u^{1,\varepsilon}-\mathtt{N}u^{2,\varepsilon},w\Big)_{\mathbb{R}^n}+\Big(\Phi(\mathtt{E}u^{1,\varepsilon})-\Phi(\mathtt{E}u^{2,\varepsilon}),w\Big)_{\mathbb{R}^n}-\pi^\varepsilon\operatorname{div}w\Big]\,dxdt=0, \tag{53}$$

$$\varepsilon\langle\pi_t^\varepsilon,\chi_{0,\tau}\eta\rangle_{U_2(Q_{0,T})}+\int\limits_{Q_{0,\tau}}\Big[\varepsilon\sum_{i=1}^n\pi_{x_i}^\varepsilon\eta_{x_i}+\eta\operatorname{div}u^\varepsilon\Big]\,dxdt=0. \tag{54}$$

Set $w=u^\varepsilon$ in (53) and $\eta=\pi^\varepsilon$ in (54). Adding the obtained equalities, we get

$$\langle u_t^\varepsilon,\chi_{0,\tau}u^\varepsilon\rangle_{U_1(Q_{0,T})}+\varepsilon\langle\pi_t^\varepsilon,\chi_{0,\tau}\pi^\varepsilon\rangle_{U_2(Q_{0,T})}+\int\limits_{Q_{0,\tau}}\Big[\sum_{i=1}^n|u_{x_i}^\varepsilon|^2+\varepsilon\sum_{i=1}^n|\pi_{x_i}^\varepsilon|^2\Big]\,dxdt+$$

$$+I_1=I_2, \tag{55}$$

where

$$I_1:=\int\limits_{Q_{0,\tau}}g(|u^{1,\varepsilon}|^{q(x)-2}u^{1,\varepsilon}-|u^{2,\varepsilon}|^{q(x)-2}u^{2,\varepsilon},u^{1,\varepsilon}-u^{2,\varepsilon})_{\mathbb{R}^n}\,dxdt\ge0,$$

$$I_2:=-\int\limits_{Q_{0,\tau}}(\Phi(\mathtt{E}u^{1,\varepsilon})-\Phi(\mathtt{E}u^{2,\varepsilon}),u^{1,\varepsilon}-u^{2,\varepsilon})_{\mathbb{R}^n}\,dxdt.$$

From condition **(E)**, the Cauchy-Bunyakowski-Schwarz inequality, and (45) we obtain

$$|I_2|le\int\limits_{Q_{0,\tau}}|\Phi(\mathtt{E}u^{1,\varepsilon})-\Phi(\mathtt{E}u^{2,\varepsilon})|\cdot|u^{1,\varepsilon}-u^{2,\varepsilon}|\,dxdt\le$$
$$\le L\int\limits_{Q_{0,\tau}}|\mathtt{E}u^{1,\varepsilon}-\mathtt{E}u^{2,\varepsilon}|\cdot|u^{1,\varepsilon}-u^{2,\varepsilon}|\,dxdt\le$$
$$\le L\Big(\int\limits_{Q_{0,\tau}}|\mathtt{E}(u^{1,\varepsilon}-u^{2,\varepsilon})|^2\,dxdt\Big)^{\frac12}\cdot\Big(\int\limits_{Q_{0,\tau}}|u^{1,\varepsilon}-u^{2,\varepsilon}|^2\,dxdt\Big)^{\frac12}=$$
$$=L||\,|\mathtt{E}(u^{1,\varepsilon}-u^{2,\varepsilon})|;L^2(Q_{0,\tau})||\cdot||\,|u^{1,\varepsilon}-u^{2,\varepsilon}|;L^2(Q_{0,\tau})||\le$$
$$\le LE^0||\,|u^{1,\varepsilon}-u^{2,\varepsilon}|;L^2(Q_{0,\tau})||^2=$$
$$=LE^0\int\limits_{Q_{0,\tau}}|u^\varepsilon|^2\,dxdt.$$

Clearly (see [5, §1.4], [5, p. 119, Lemma 4.5], and [28, p. 46–52]),

$$\langle u_t^\varepsilon, \chi_{0,\tau} u^\varepsilon\rangle_{U_1(Q_{0,T})} + \varepsilon\langle \pi_t^\varepsilon, \chi_{0,\tau}\pi^\varepsilon\rangle_{U_2(Q_{0,T})} = \frac{1}{2}\int\limits_{\Omega_\tau} |u^\varepsilon|^2\, dx + \frac{\varepsilon}{2}\int\limits_{\Omega_\tau} |\pi^\varepsilon|^2\, dx \geq$$
$$\geq \frac{1}{2}\int\limits_{\Omega_\tau} |u^\varepsilon|^2\, dx.$$

Then, from (55) we obtain

$$\frac{1}{2}\int\limits_{\Omega_\tau} |u^\varepsilon|^2\, dx \leq LE^0 \int\limits_{Q_{0,\tau}} |u^\varepsilon|^2\, dxdt, \quad \tau\in(0,T].$$

Using Lemma 1, we get, that $\int\limits_{\Omega_\tau} |u^\varepsilon|^2\, dx \leq 0$ for $\tau\in(0,T]$. So, $u^{1,\varepsilon} = u^{2,\varepsilon}$ and from (55) we get $\frac{\varepsilon}{2}\int\limits_{\Omega_\tau} |\pi^\varepsilon|^2\, dx \leq 0$ for $\tau\in(0,T]$. Thus $\pi^{1,\varepsilon} = \pi^{2,\varepsilon}$ and Theorem 1 is proved. □

Proof of Theorem 2. The solution will be constructed via Faedo-Galerkin's method.

Step 1 (construction of approximation). Suppose that s satisfies (19), $\mathcal{W}_s$ is taken from (18), $\{w^\mu\}_{\mu\in\mathbb{N}} \subset [\mathcal{W}_s]^n$ is taken from Lemma 2 with $\ell = n$, the set $\{v^\mu\}_{\mu\in\mathbb{N}} \subset \mathcal{W}_s$ is taken from Proposition 9. For simplicity, we also assume that $\{w^\mu\}_{\mu\in\mathbb{N}}$ is orthonormal in $[L^2(\Omega)]^n$ and $\{v^\mu\}_{\mu\in\mathbb{N}}$ is orthonormal in $L^2(\Omega)$. Note that

$$[\mathcal{W}_s]^n \circlearrowright Y_1, \qquad \mathcal{W}_s \circlearrowright Y_2 \tag{56}$$

hold. By definition, put

$$u^{\varepsilon,m}(x,t) := \sum_{k=1}^m \varphi_k^{\varepsilon,m}(t) w^k(x), \quad \pi^{\varepsilon,m}(x,t) := \sum_{k=1}^m \psi_k^{\varepsilon,m}(t) v^k(x), \quad (x,t)\in Q_{0,T}, \quad m\in\mathbb{N},$$

where the unknown scalar functions $\varphi_1^{\varepsilon,m},\ldots,\varphi_{\varepsilon,m}^m$, $\psi_1^{\varepsilon,m},\ldots,\psi_{\varepsilon,m}^m$ satisfy

$$(u_t^{\varepsilon,m}(t), w^\mu)_\Omega + \langle A_1 u^{\varepsilon,m}(t), w^\mu\rangle_{Y_1} + (\mathcal{N}(t)u^{\varepsilon,m}(t), w^\mu)_\Omega + (\Phi(E(t)u^{\varepsilon,m}(t)), w^\mu)_\Omega -$$
$$- (\pi^{\varepsilon,m}(t), \operatorname{div} w^\mu)_\Omega = (F(t), w^\mu)_\Omega, \quad t\in(0,T), \quad \mu = \overline{1,m}, \tag{57}$$

$$\varepsilon(\pi_t^{\varepsilon,m}(t), v^\mu)_\Omega + \varepsilon\langle A_2\pi^{\varepsilon,m}(t), v^\mu\rangle_{Y_2} + (\operatorname{div} u^{\varepsilon,m}(t), v^\mu)_\Omega =$$
$$= (f(t), v^\mu)_\Omega, \quad t\in(0,T), \quad \mu = \overline{1,m}, \tag{58}$$

$$\varphi_1^{\varepsilon,m}(0) = \alpha_1^m, \quad \ldots, \quad \varphi_m^{\varepsilon,m}(0) = \alpha_m^m, \tag{59}$$

$$\psi_1^{\varepsilon,m}(0) = \beta_1^m, \quad \ldots, \quad \psi_m^{\varepsilon,m}(0) = \beta_m^m. \tag{60}$$

Here the numbers $\alpha_1^m,\ldots,\alpha_m^m$, $\beta_1^m,\ldots,\beta_m^m\in\mathbb{R}$ we choose so that $u_0^{\varepsilon,m} \underset{k\to\infty}{\longrightarrow} u_0$ strongly in $[L^2(\Omega)]^n$, $\pi_0^{\varepsilon,m} \underset{k\to\infty}{\longrightarrow} \pi_0$ strongly in $L^2(\Omega)$, where

$$u_0^m(x) := \sum_{\mu=1}^m \alpha_\mu^m w^\mu(x), \quad \pi_0^m(x) := \sum_{\mu=1}^m \beta_\mu^m v^\mu(x), \quad x\in\Omega.$$

It's clear that the following conditions are true: $u^{\varepsilon,m}(0)=u_0^m$, $\pi^{\varepsilon,m}(0)=\pi_0^m$.

Let us show that the mentioned functions exist. Let

$$\xi = col(\varphi_1^{\varepsilon,m},\dots,\varphi_m^{\varepsilon,m},\psi_1^{\varepsilon,m},\dots,\psi_m^{\varepsilon,m}),$$
$$\xi_0 = col(\alpha_1^m,\dots,\alpha_m^m,\beta_1^m,\dots,\beta_m^m),$$

L be a function from Lemma 5. Then problem (57)-(60) takes form (41), where

$$M(t)=col\big((F(t),w^1)_\Omega,\dots,(F(t),w^m)_\Omega,(f(t),v^1)_\Omega,\dots,(f(t),v^m)_\Omega\big).$$

Similarly as Lemmas 3.25 and 3.27 [10, p. 874-875], we prove that L satisfies the L^∞-Carathéodory condition. It follows from conditions **(U)**–**(W)** that $M\in L^2(0,T;\mathbb{R}^{2m})$. Using estimates (50), (28) and the conditions $\varepsilon, G_0>0$, we receive

$$\Big(L(t,\xi),\xi\Big)_{\mathbb{R}^{2m}} \geq -LE^0\int\limits_\Omega |u^{\varepsilon,m}|^2\,dx \geq$$
$$\geq -LE^0 m\int\limits_\Omega\sum_{k=1}^m|\varphi_k^m(t)|^2|w^k(x)|^2\,dx =$$
$$= -LE^0m(|\varphi^m(t)|^2+|\psi^m(t)|^2).$$

Then estimate (42) with $\alpha(t)\equiv LE^0m$ and $\beta(t)\equiv 0$ is true, and from the Carathéodory-LaSalle theorem (see Proposition 11) we have that $\varphi^m,\psi^m\in W^{1,2}(0,T;\mathbb{R}^m)$ satisfy (57)–(60).

Step 2. Multiply μ-th equation of (57) by $\varphi_\mu^{\varepsilon,m}(t)$ and sum over $\mu=\overline{1,m}$. Then, multiply the μ-th equation (58) by $\psi_\mu^{\varepsilon,m}(t)$ and sum over $\mu=\overline{1,m}$. We get the following

$$(u_t^{\varepsilon,m}(t),u^{\varepsilon,m}(t))_\Omega+\langle A_1u^{\varepsilon,m}(t),u^{\varepsilon,m}(t)\rangle_{Y_1}+(\mathcal{N}(t)u^{\varepsilon,m}(t),u^{\varepsilon,m}(t))_{\mathbb{R}^n}+$$
$$+(\Phi(E(t)u^{\varepsilon,m}(t)),u^{\varepsilon,m}(t))_\Omega-(\operatorname{div}u^{\varepsilon,m}(t),\pi^{\varepsilon,m}(t))_\Omega=(F(t),u^{\varepsilon,m}(t))_\Omega,$$

$$\varepsilon(\pi_t^{\varepsilon,m}(t),\pi^{\varepsilon,m}(t))_\Omega+\varepsilon\langle A_2\pi^{\varepsilon,m}(t),\pi^{\varepsilon,m}(t)\rangle_{Y_2}+(\operatorname{div}u^{\varepsilon,m}(t),\pi^{\varepsilon,m}(t))_\Omega=(f(t),\pi^{\varepsilon,m}(t))_\Omega.$$

Adding these equalities and integrating for $t\in(0,\tau)\subset(0,T)$, we get

$$\int\limits_{Q_{0,\tau}}\Big[(u_t^{\varepsilon,m},u^{\varepsilon,m})_{\mathbb{R}^n}+\varepsilon\pi_t^{\varepsilon,m}\pi^{\varepsilon,m}\Big]\,dxdt+\int\limits_0^\tau\Big(L(t,\xi(t)),\xi(t)\Big)_{\mathbb{R}^n}\,dt=$$
$$=\int\limits_{Q_{0,\tau}}\Big[(F,u^{\varepsilon,m})_{\mathbb{R}^n}+(f,\pi^{\varepsilon,m})_{\mathbb{R}^n}\Big]\,dxdt,\quad \tau\in(0,T]. \tag{61}$$

It's clear that,

$$\int\limits_{Q_{0,\tau}}(u_t^{\varepsilon,m},u^{\varepsilon,m})_{\mathbb{R}^n}\,dxdt=\int\limits_{Q_{0,\tau}}\frac12\frac{\partial}{\partial t}(|u^{\varepsilon,m}|^2)\,dxdt=$$
$$=\frac12\int\limits_{\Omega_\tau}|u^{\varepsilon,m}|^2\,dx-\frac12\int\limits_\Omega|u_0^m|^2\,dx,$$

$$\int\limits_{Q_{0,\tau}} \pi_t^{\varepsilon,m}\pi^{\varepsilon,m}\,dxdt = \int\limits_{Q_{0,\tau}} \frac{1}{2}\frac{\partial}{\partial t}(|\pi^{\varepsilon,m}|^2)\,dxdt = \\ = \frac{1}{2}\int\limits_{\Omega_\tau} |\pi^{\varepsilon,m}|^2\,dx - \frac{1}{2}\int\limits_{\Omega} |\pi_0^m|^2\,dx,$$

$$\Big|(F,u^{\varepsilon,m})_{\mathbb{R}^n} + (f,\pi^{\varepsilon,m})_{\mathbb{R}^n}\Big| \le |F|\cdot|u^{\varepsilon,m}| + |f|\cdot|\pi^{\varepsilon,m}| \le \\ \le \frac{1}{2}\Big(|F|^2 + |u^{\varepsilon,m}|^2\Big) + \frac{1}{2\varepsilon}|f|^2 + \frac{\varepsilon}{2}|\pi^{\varepsilon,m}|^2.$$

(see (27)). Then, it follows from (61) and (50) that

$$\frac{1}{2}\int\limits_{\Omega_\tau}\Big[|u^{\varepsilon,m}|^2 + \varepsilon|\pi^{\varepsilon,m}|^2\Big]\,dx + \int\limits_{Q_{0,\tau}}\Big[\sum_{i=1}^n |u_{x_i}^m|^2 + g_0|u^{\varepsilon,m}|^{q(x)} + \varepsilon\sum_{i=1}^m |\pi_{x_i}^m|^2\Big]\,dxdt \le \\ \le \frac{1}{2}\int\limits_{\Omega}\Big[|u_0^m|^2 + \varepsilon|\pi_0^m|^2\Big]\,dx + \\ + \frac{1}{2}\int\limits_{Q_{0,\tau}}\Big[|F|^2 + \frac{1}{\varepsilon}|f|^2\Big]\,dxdt + \int\limits_{Q_{0,\tau}}\Big[\Big(\frac{1}{2} + LE^0\Big)|u^{\varepsilon,m}|^2 + \frac{\varepsilon}{2}|\pi^{\varepsilon,m}|^2\Big]\,dxdt. \tag{62}$$

Take

$$y(\tau) = \int\limits_{\Omega}[|u^{\varepsilon,m}(x,\tau)|^2 + \varepsilon|\pi^{\varepsilon,m}(x,\tau)|^2]\,dx, \quad \tau\in[0,T].$$

Then, from (62), we get:

$$y(\tau) \le C_7\mathcal{F}_\varepsilon(\tau) + C_8\int\limits_0^\tau y(t)\,dt, \quad \tau\in[0,T],$$

where

$$\mathcal{F}_\varepsilon(\tau) = \int\limits_{\Omega}\big[|u_0^m|^2 + \varepsilon|\pi_0^m|^2\big]\,dx + \int\limits_{Q_{0,\tau}}\Big[|F|^2 + \frac{1}{\varepsilon}|f|^2\Big]\,dxdt, \ \ \tau\in(0,T],$$

the positive constants C_7 and C_8 are independent of m,τ,ε. Using Lemma 1, we obtain an estimate:

$$y(\tau) \le C_7\,e^{C_8T}\mathcal{F}_\varepsilon(\tau), \quad \tau\in[0,T]. \tag{63}$$

Clearly, $0\le\mathcal{F}_\varepsilon(\tau)\le\mathcal{F}_\varepsilon(T)\le C_9$, where the positive constant C_9 is independent of m,τ.

From (62) and (63) it follows that

$$\int\limits_{\Omega_\tau}\Big[|u^{\varepsilon,m}(x,\tau)|^2 + \varepsilon|\pi^{\varepsilon,m}(x,\tau)|^2\Big]\,dx + \int\limits_{Q_{0,\tau}}\Big[\sum_{i=1}^n |u_{x_i}^{\varepsilon,m}|^2 + \\ + |u^{\varepsilon,m}|^2 + |u^{\varepsilon,m}|^{q(x)} + \varepsilon\sum_{i=1}^n |\pi_{x_i}^{\varepsilon,m}|^2 + \varepsilon|\pi^{\varepsilon,m}|^2\Big]\,dxdt \le C_{10}, \quad \tau\in[0,T]. \tag{64}$$

By Proposition 5, Proposition 4, and (64), we get

$$||\mathtt{N}u^{\varepsilon,m};[L^{q'(x)}(Q_{0,T})]^n|| \le C_{11}||u^{\varepsilon,m};[L^{q(x)}(Q_{0,T})]^n|| \le C_{12}. \tag{65}$$

From **(E)**, (45), and (64) it follows the estimate

$$||\Phi(\mathtt{E}u^{\varepsilon,m});[L^2(Q_{0,T})]^n|| \le C_{13}. \tag{66}$$

Here the constants $C_{10},\dots,C_{13}$ are independent of m,τ. Notice that if $f\equiv 0$ and $\pi_0\equiv 0$, then these constants are also independent of ε.

By (64)–(66) we have existence of the subsequences $\{u^{\varepsilon,m_k}\}_{k\in\mathbb{N}}\subset\{u^{\varepsilon,m}\}_{m\in\mathbb{N}}$ and $\{\pi^{\varepsilon,m_k}\}_{k\in\mathbb{N}}\subset\{\pi^{\varepsilon,m}\}_{m\in\mathbb{N}}$ such that

$$u^{\varepsilon,m_k}\underset{k\to\infty}{\longrightarrow}u^{\varepsilon}\quad *\text{-weakly in}\quad L^\infty(0,T;[L^2(\Omega)]^n)\text{ and weakly in}\quad U_1(Q_{0,T}),$$

$$\pi^{\varepsilon,m_k}\underset{k\to\infty}{\longrightarrow}\pi^{\varepsilon}\quad *\text{-weakly in}\quad L^\infty(0,T;L^2(\Omega))\text{ and weakly in}\quad U_2(Q_{0,T}).$$

Moreover,

$$\mathtt{N}u^{\varepsilon,m_k}\underset{k\to\infty}{\longrightarrow}\chi_1^{\varepsilon}\quad\text{weakly in}\quad [L^{q'(x)}(Q_{0,T})]^n, \tag{67}$$

$$\Phi(\mathtt{E}u^{\varepsilon,m_k})\underset{k\to\infty}{\longrightarrow}\chi_2^{\varepsilon}\quad\text{weakly in}\quad [L^2(Q_{0,T})]^n. \tag{68}$$

<u>Step 3</u> (additional estimates). Let us use notation (49). From (64) it follows the inequality

$$||\mathcal{A}_1u^{\varepsilon,m};[U_1(Q_{0,T})]^*|| \le C_{14}. \tag{69}$$

From the construction of the space $U_1(Q_{0,T})$ and (56) we obtain

$$U_1(Q_{0,T})\mathrel{\bar{\circlearrowright}}L^2(0,T;[L^2(\Omega)]^n)\mathrel{\bar{\circlearrowright}}[U_1(Q_{0,T})]^*, \tag{70}$$

$$U_1(Q_{0,T})\mathrel{\bar{\circlearrowright}}[L^{q(x)}(Q_{0,T})]^n,\qquad [L^{q'(x)}(Q_{0,T})]^n\mathrel{\bar{\circlearrowright}}[U_1(Q_{0,T})]^*, \tag{71}$$

$$L^{\max\{2,q^0\}}(0,T;[\mathcal{W}_s]^n)\mathrel{\bar{\circlearrowright}}L^{\max\{2,q^0\}}(0,T;Y_1)\mathrel{\bar{\circlearrowright}}U_1(Q_{0,T})\mathrel{\bar{\circlearrowright}}L^{\min\{2,q_0\}}(0,T;Y_1). \tag{72}$$

Therefore,

$$[U_1(Q_{0,T})]^*\mathrel{\bar{\circlearrowright}}L^r(0,T;Y_1^*)\mathrel{\bar{\circlearrowright}}L^r(0,T;[\mathcal{W}_s^*]^n),\quad r=\frac{\max\{2,q^0\}}{\max\{2,q^0\}-1}. \tag{73}$$

Using (64) and (72), we obtain

$$||u^{\varepsilon,m};L^{\min\{2,q_0\}}(0,T;Y_1)|| \le C_{15}||u^{\varepsilon,m};U_1(Q_{0,T})|| \le C_{16}. \tag{74}$$

From estimate (64) it follows an inequality

$$||\mathcal{A}_2\pi^{\varepsilon,m};[U_2(Q_{0,T})]^*|| \le C_{17}(\varepsilon). \tag{75}$$

From the construction of the space $U_2(Q_{0,T})$, we obtain

$$U_2(Q_{0,T})\mathrel{\bar{\circlearrowright}}L^2(0,T;L^2(\Omega))\mathrel{\bar{\circlearrowright}}[U_2(Q_{0,T})]^*=L^2(0,T;H^{-1}(\Omega))\mathrel{\bar{\circlearrowright}}L^2(0,T;\mathcal{W}_s^*). \tag{76}$$

Using Proposition 8 and notation (37)-(38) for $\mathcal{H}=[L^2(\Omega)]^n$ and $\mathcal{V}=[\mathcal{W}_s]^n$, in same way as in [29, c. 77], we rewrite (57) as

$$u_t^{\varepsilon,m}=\widehat{P}_m^*(F-\mathcal{A}_1u^{\varepsilon,m}-\mathtt{N}u^{\varepsilon,m}-\Phi(\mathtt{E}u^{\varepsilon,m})-\nabla\pi^{\varepsilon,m}). \tag{77}$$

Thus, from (40) with $\ell=n$, embeddings (70)-(73), and estimates (64)-(66) and (69), we get:

$$||u_t^{\varepsilon,m};L^r(0,T;[\mathcal{W}_s^*]^n)||=$$

$$=||\widehat{P}_m^*(F-\mathcal{A}_1u^{\varepsilon,m}-\mathtt{N}u^{\varepsilon,m}-\Phi(\mathtt{E}u^{\varepsilon,m})-\nabla\pi^{\varepsilon,m});L^r(0,T;[\mathcal{W}_s^*]^n)|| \le$$

$$\leq ||F - \mathcal{A}_1 u^{\varepsilon,m} - \mathtt{N}u^{\varepsilon,m} - \Phi(\mathtt{E}u^{\varepsilon,m}) - \nabla\pi^{\varepsilon,m}; L^r(0,T;[\mathcal{W}_s^*]^n)|| \leq$$

$$\leq C_{18}||F - \mathcal{A}_1 u^{\varepsilon,m} - \mathtt{N}u^{\varepsilon,m} - \Phi(\mathtt{E}u^{\varepsilon,m}) - \nabla\pi^{\varepsilon,m}; [U_1(Q_{0,T})]^*|| \leq$$

$$\leq C_{19}\Big(||F;[L^2(Q_{0,T})]^n|| + ||\mathcal{A}_1 u^{\varepsilon,m};[U_1(Q_{0,T})]^*|| + ||\mathtt{N}u^{\varepsilon,m};[L^{q'(x)}(Q_{0,T})]^n|| +$$

$$+ ||\Phi(\mathtt{E}u^{\varepsilon,m});[L^2(Q_{0,T})]^n|| + ||\nabla\pi^{\varepsilon,m};[L^2(Q_{0,T})]^n||\Big) \leq C_{20}(\varepsilon). \tag{78}$$

Using Proposition 8 and notation (37)-(38) for $\mathcal{H} = L^2(\Omega)$ and $\mathcal{V} = \mathcal{W}_s$, from (58) in same way as (77), we get

$$\pi_t^{\varepsilon,m} = \widehat{P}_m^*\Big(\frac{1}{\varepsilon}f - \mathcal{A}_2\pi^{\varepsilon,m} - \frac{1}{\varepsilon}\operatorname{div} u^{\varepsilon,m}\Big). \tag{79}$$

Thus, from (40) with $\ell = 1$, embeddings (76) and estimates (64) and (75), we get:

$$||\pi_t^{\varepsilon,m}; L^2(0,T;\mathcal{W}_s^*)|| = \Big|\Big|\widehat{P}_m^*\Big(\frac{1}{\varepsilon}f - \mathcal{A}_2\pi^{\varepsilon,m} - \frac{1}{\varepsilon}\operatorname{div} u^{\varepsilon,m}\Big); L^2(0,T;\mathcal{W}_s^*)\Big|\Big| \leq$$

$$\leq \Big|\Big|\frac{1}{\varepsilon}f - \mathcal{A}_2\pi^{\varepsilon,m} - \frac{1}{\varepsilon}\operatorname{div} u^{\varepsilon,m}; L^2(0,T;\mathcal{W}_s^*)\Big|\Big| \leq$$

$$\leq C_{21}\Big|\Big|\frac{1}{\varepsilon}f - \mathcal{A}_2\pi^{\varepsilon,m} - \frac{1}{\varepsilon}\operatorname{div} u^{\varepsilon,m}; [U_2(Q_{0,T})]^*\Big|\Big| \leq$$

$$\leq C_{22}\Big(\frac{1}{\varepsilon}||f;L^2(Q_{0,T})|| + ||\mathcal{A}_2\pi^{\varepsilon,m};[U_2(Q_{0,T}]^*|| + \frac{1}{\varepsilon}||\operatorname{div} u^{\varepsilon,m};L^2(Q_{0,T})||\Big) \leq$$

$$\leq C_{23}(\varepsilon). \tag{80}$$

Since $Y_1 \overset{K}{\subset} [L^2(\Omega)]^n \circlearrowleft [\mathcal{W}_s^*]^n$, from (74), (78), the Aubin theorem (see Proposition 3) and Proposition 6 we obtain

$$u^{\varepsilon,m_k} \underset{k\to\infty}{\longrightarrow} u^\varepsilon \text{ in } L^{\min\{2,q_0\}}(0,T;[L^2(\Omega)]^n) \cap C([0,T];[\mathcal{W}_s^*]^n)$$

$$\text{and almost everywhere in } \quad Q_{0,T}.$$

Therefore, $\chi_1^\varepsilon = g|u^\varepsilon|^{q(x)-2}u^\varepsilon$, $\chi_2^\varepsilon = \Phi(\mathtt{E}u^\varepsilon)$ (see (67), (68); use (46) with $r = \min\{2, q_0\}$), and (5) holds. Since $Y_2 \overset{K}{\subset} L^2(\Omega) \circlearrowleft \mathcal{W}_s^*$, from (64), (80), the Aubin theorem (see Proposition 3) and Proposition 6 we obtain:

$$\pi^{\varepsilon,m_k} \underset{k\to\infty}{\longrightarrow} \pi^\varepsilon \quad \text{in} \quad L^2(0,T;L^2(\Omega)) \cap C([0,T];\mathcal{W}_s^*) \quad \text{and almost everywhere in} \quad Q_{0,T}$$

and (6) holds.

Step 4 (passing to the limit). From (57)-(58) and convergences above we get (16)-(17). Since $u^\varepsilon \in U_1(Q_{0,T})$ and $u_t^\varepsilon \in [U_1(Q_{0,T})]^n$, we get that $u^\varepsilon \in C([0,T];[L^2(\Omega)]^n)$ (see Lemma 4.5 [5, p. 119]). Similarly, $\pi^\varepsilon \in C([0,T];L^2(\Omega))$ and Theorem 2 is proved. □

Proof of Theorem 3. Let's use the parabolic regularization method. Suppose that the functions $\{u^{\varepsilon,m}\}_{m\in\mathbb{N}}$, $\{\pi^{\varepsilon,m}\}_{m\in\mathbb{N}}$, $\{u^{\varepsilon}\}_{\varepsilon>0}$, and $\{\pi^{\varepsilon}\}_{\varepsilon>0}$ are taking from the proof of Theorem 2, where $f\equiv 0$ and $\pi_0\equiv 0$ (see (2) and (6)). Let $\varepsilon\in(0,1)$ for simplicity.

Step 1 (estimates). From (64) we get

$$\int\limits_{\Omega} |u^{\varepsilon,m_k}(x,\tau)|^2\,dx + \int\limits_{Q_{0,\tau}} \sum_{i=1}^{n} |u^{\varepsilon,m_k}_{x_i}|^2\,dxdt \le C_{24}, \quad \tau\in(0,T], \ \ \varepsilon\in(0,1), \ \ k\in\mathbb{N}. \tag{81}$$

Remind, that C_{24} is independent of m,τ and ε. From (81) with $\tau=T$ we obtain

$$||u^{\varepsilon,m_k}; L^2(0,T;[H^1_0(\Omega)]^n)|| \le \sqrt{C_{24}}.$$

For the first term of (81) we get that

$$\operatorname*{ess\,sup}_{\tau\in(0,T)} ||u^{\varepsilon,m}(\tau);[L^2(\Omega)]^n|| \le \sqrt{C_{24}},$$

and so

$$||u^{\varepsilon,m_k}; L^{\infty}(0,T;[L^2(\Omega)]^n)|| \le \sqrt{C_{24}}.$$

Adding these enequalities, we get:

$$||u^{\varepsilon,m_k}; L^{\infty}(0,T;[L^2(\Omega)]^n)|| + ||u^{\varepsilon,m_k}; L^2(0,T;[H^1_0(\Omega)]^n|| \le 2\sqrt{C_{24}}. \tag{82}$$

From (82) and Proposition 2 we obtain:

$$2\sqrt{C_{24}} = \varliminf_{k\to\infty} 2\sqrt{C_{24}} \ge$$
$$\ge \varliminf_{k\to\infty}\Big(||u^{\varepsilon,m_k}; L^{\infty}(0,T;[L^2(\Omega)]^n)|| + ||u^{\varepsilon,m_k}; L^2(0,T;[H^1_0(\Omega)]^n||\Big)\ge$$
$$\ge ||u^{\varepsilon}; L^{\infty}(0,T;[L^2(\Omega)]^n)|| + ||u^{\varepsilon}; L^2(0,T;[H^1_0(\Omega)]^n||.$$

From this inequality it follows that (see (81) for comparison)

$$\int\limits_{\Omega} |u^{\varepsilon}(x,\tau)|^2\,dx + \int\limits_{Q_{0,\tau}} \sum_{i=1}^{n} |u^{\varepsilon}_{x_i}|^2\,dxdt \le C_{25}. \tag{83}$$

From (64)–(66), (74), and Proposition 2, similarly to (83), we get

$$\varepsilon\int\limits_{\Omega} |\pi^{\varepsilon}(x,\tau)|^2\,dx +$$
$$+\int\limits_{Q_{0,\tau}} \Big[|u^{\varepsilon}|^2 + |u^{\varepsilon}|^{q(x)} + \varepsilon\sum_{i=1}^{n}|\pi^{\varepsilon}_{x_i}|^2 + \varepsilon|\pi^{\varepsilon}|^2\Big]\,dxdt \le C_{26}, \quad \tau\in(0,T], \tag{84}$$

$$||\mathtt{N}u^{\varepsilon};[L^{q'(x)}(Q_{0,T})]^n|| \le C_{27}, \tag{85}$$

$$||\Phi(\mathtt{E}u^{\varepsilon});[L^2(Q_{0,T})]^n|| \le C_{28}, \tag{86}$$

$$||u^{\varepsilon}; L^{\min\{2,q_0\}}(0,T;Y_1)|| \le C_{29}. \tag{87}$$

Here the constants $C_{25},\dots,C_{29}$ are independent of ε,τ. Then we have the existence of the sequences $\{u^{\varepsilon_j}\}_{j\in\mathbb{N}}\subset\{u^{\varepsilon}\}_{\varepsilon\in(0,1)}$ and $\{\pi^{\varepsilon_j}\}_{j\in\mathbb{N}}\subset\{\pi^{\varepsilon}\}_{\varepsilon\in(0,1)}$ such that

$$u^{\varepsilon_j}\underset{j\to\infty}{\longrightarrow} u \quad *\text{-weakly in} \quad L^{\infty}(0,T;[L^2(\Omega)]^n) \text{ and weakly in } U_1(Q_{0,T}), \tag{88}$$

$$\sqrt{\varepsilon_j}\pi^{\varepsilon_j} \underset{j\to\infty}{\longrightarrow} \chi_0 \quad *\text{-weakly in} \quad L^\infty(0,T;L^2(\Omega)) \text{ and weakly in } U_2(Q_{0,T}), \tag{89}$$

$$\mathtt{N}u^{\varepsilon_j} \underset{j\to\infty}{\longrightarrow} \chi_1 \quad \text{weakly in} \quad [L^{q'(x)}(Q_{0,T})]^n, \tag{90}$$

$$\Phi(\mathtt{E}u^{\varepsilon_j}) \underset{j\to\infty}{\longrightarrow} \chi_2 \quad \text{weakly in} \quad [L^2(Q_{0,T})]^n. \tag{91}$$

Step 2 (additional estimates). Since $U_3(Q_{0,T}) \subset U_1(Q_{0,T})$, we have $\mathcal{A}_1 u^\varepsilon \in [U_3(Q_{0,T})]^*$,

$$\langle \mathcal{A}_1 u^\varepsilon, v\rangle_{U_3(Q_{0,T})} := \langle \mathcal{A}_1 u^\varepsilon, v\rangle_{U_1(Q_{0,T})} = \int\limits_{Q_{0,T}} \sum_{i=1}^n (u^\varepsilon_{x_i}, v_{x_i})_{\mathbb{R}^n}\, dxdt \le C_{30}||v;U_3(Q_{0,T})||,$$

for every $v \in U_3(Q_{0,T})$, where C_{30} is independent of ε, and so

$$||\mathcal{A}_1 u^\varepsilon; [U_3(Q_{0,T})]^*|| \le C_{30}. \tag{92}$$

Take $\psi \in D(0,T)$, multiply μ-th equaltion of (58) with $m = m_k$ and $\varepsilon = \varepsilon_j$ by $\psi(t)$, integrate for $t \in (0,T)$, and the first term integrate by parts. We obtain

$$\int\limits_{Q_{0,T}} \Big[-\varepsilon_j\, \pi^{\varepsilon_j, m_k} v^\mu \psi_t + \varepsilon_j \sum_{i=1}^n \pi^{\varepsilon_j, m_k}_{x_i} v^\mu_{x_i}\psi + v^\mu \psi \operatorname{div} u^{\varepsilon_j, m_k}\Big]\, dxdt = 0$$

(remind that $f \equiv 0$). If we tend $k \to \infty$, then, after some transformation, we get

$$\sqrt{\varepsilon_j} \int\limits_{Q_{0,T}} \Big[-(\sqrt{\varepsilon_j}\, \pi^{\varepsilon_j}) v^\mu \psi_t + \sum_{i=1}^n (\sqrt{\varepsilon_j}\pi^{\varepsilon_j})_{x_i} v^\mu_{x_i}\psi\Big]\, dxdt + \int\limits_{Q_{0,T}} v^\mu \psi \operatorname{div} u^{\varepsilon_j}\, dxdt = 0.$$

If we tend $j \to \infty$, then $\int_{Q_{0,T}} v^\mu \psi \operatorname{div} u\, dxdt = 0$, $\mu \in \mathbb{N}$, $\psi \in D(0,T)$, and so (21) holds.

Take $\varphi \in D(0,T)$, multiply μ-th equaltion of (57) with $m = m_k$ and $\varepsilon = \varepsilon_j$ by $\varphi(t)$, integrate for $t \in (0,T)$, and the first term integrate by parts. We obtain

$$\int\limits_{Q_{0,T}} \Big[-(u^{\varepsilon_j, m_k}, w^\mu)_{\mathbb{R}^n}\varphi_t + \\ + \sum_{i=1}^n (u^{\varepsilon_j, m_k}_{x_i}, w^\mu_{x_i})_{\mathbb{R}^n}\varphi + (\mathtt{N}u^{\varepsilon_j, m_k}, w^\mu)_{\mathbb{R}^n}\varphi + (\Phi(\mathtt{E}u^{\varepsilon_j, m_k}), w^\mu)_{\mathbb{R}^n}\varphi - \\ - \pi^{\varepsilon_j, m}\varphi \operatorname{div} w^\mu\Big]\, dxdt = \int\limits_{Q_{0,T}} (F, w^\mu)_{\mathbb{R}^n}\varphi\, dxdt.$$

If we tend $k \to \infty$, then we get

$$\int\limits_{Q_{0,T}} \Big[-(u^{\varepsilon_j}, w^\mu)_{\mathbb{R}^n}\varphi_t + \sum_{i=1}^n (u^{\varepsilon_j}_{x_i}, w^\mu_{x_i})_{\mathbb{R}^n}\varphi + (\mathtt{N}u^{\varepsilon_j}, w^\mu)_{\mathbb{R}^n}\varphi + (\Phi(\mathtt{E}u^{\varepsilon_j}), w^\mu)_{\mathbb{R}^n}\varphi - \\ - \pi^{\varepsilon_j}\varphi \operatorname{div} w^\mu\Big]\, dxdt = \int\limits_{Q_{0,T}} (F, w^\mu)_{\mathbb{R}^n}\varphi\, dxdt.$$

Since $Z_1 \subset H_0^1(\Omega)$, we easily get this equality for every $z \in Z_1$ instead of w^μ. But $\operatorname{div} z = 0$ and so, after some transformation, we have

$$\int\limits_{Q_{0,T}} -(u^{\varepsilon_j}, z)_{\mathbb{R}^n} \varphi_t \, dxdt = \int\limits_{Q_{0,T}} \Big[-\sum_{i=1}^n (u^{\varepsilon_j}_{x_i}, z_{x_i})_{\mathbb{R}^n} \varphi + (F - \mathtt{N} u^{\varepsilon_j} - \Phi(\mathtt{E} u^{\varepsilon_j}), z)_{\mathbb{R}^n} \varphi \Big] \, dxdt \quad (93)$$

for every $\varphi \in D(0,T)$ and $z \in Z_1$. In the same way, we obtain (93) for every $z \in Z_s$, where s is taken from (19). Then, in the meaning of spaces $[U_3(Q_{0,T})]^*$ and $D^*(0,T;Z_s^*)$, we get

$$u_t^{\varepsilon_j} = F - \mathcal{A}_1 u^{\varepsilon_j} - \mathtt{N} u^{\varepsilon_j} - \Phi(\mathtt{E} u^{\varepsilon_j}). \quad (94)$$

From the construction of the space $U_3(Q_{0,T})$ and choosing of s from (19), we obtain

$$U_3(Q_{0,T}) \mathrel{\bar{\circlearrowleft}} [L^2(Q_{0,T})]^n \mathrel{\bar{\circlearrowleft}} [U_3(Q_{0,T})]^*, \quad (95)$$

$$L^{\max\{2,q^0\}}(0,T;Z_s) \mathrel{\bar{\circlearrowleft}} L^{\max\{2,q^0\}}(0,T;V_1) \mathrel{\bar{\circlearrowleft}} U_3(Q_{0,T}). \quad (96)$$

Therefore, for $r > 1$ that it is taken from (73), we obtain

$$[U_3(Q_{0,T})]^* \mathrel{\bar{\circlearrowleft}} L^r(0,T;V_1^*) \mathrel{\bar{\circlearrowleft}} L^r(0,T;Z_s^*). \quad (97)$$

Using (85)-(86), (92), and (94)-(97), we obtain

$$||u_t^{\varepsilon_j}; L^r(0,T;Z_s^*)|| = ||F - \mathcal{A}_1 u^{\varepsilon_j} - \mathtt{N} u^{\varepsilon_j} - \Phi(\mathtt{E} u^{\varepsilon_j}); L^r(0,T;Z_s^*)|| \le$$

$$\le C_{31} ||F - \mathcal{A}_1 u^{\varepsilon_j} - \mathtt{N} u^{\varepsilon_j} - \Phi(\mathtt{E} u^{\varepsilon_j}); [U_3(Q_{0,T})]^*|| \le C_{32} (||F; [L^2(Q_{0,T})]^n|| +$$

$$+ ||\mathcal{A}_1 u^{\varepsilon_j}; [U_3(Q_{0,T})]^*|| + ||\mathtt{N} u^{\varepsilon_j}; [L^{q'(x)}(Q_{0,T})]^n|| + ||\Phi(\mathtt{E} u^{\varepsilon_j}); [L^2(Q_{0,T})]^n||) \le C_{33}, \quad (98)$$

where $C_{33} > 0$ is independent of ε_j (see (78) for comparison).

Step 4 (passing to the limit). Since $Y_1 \overset{K}{\subset} [L^2(\Omega)]^n \circlearrowleft Z_s^*$, from (87), (98), the Aubin theorem (see Proposition 3), and Proposition 6 we obtain:

$$u^{\varepsilon_j} \underset{j \to \infty}{\longrightarrow} u \quad \text{in} \quad L^{\min\{2,q_0\}}(0,T;[L^2(\Omega)]^n) \cap C([0,T];Z_s^*)$$

$$\text{and almost everywhere in} \quad Q_{0,T}.$$

Therefore, $\chi_1 = g|u|^{q(x)-2}u$ and $\chi_2 = \Phi(\mathtt{E}u)$ (see (90), (91); use (46) with $r = \min\{2, q_0\}$) and (23) holds. Thus, form (93) and convergences above, we deduce that u satisfies (25).

Step 5 (finding the π). Note that we can not prove the convergence of $\{\pi^{\varepsilon_j}\}_{j \in \mathbb{N}}$, for example, to the solution π of problem (20)-(23). Let us find π from the generalized De Rham theorem. From (93), we obtain

$$\int\limits_0^T \langle \mathcal{F}(t), w \rangle_{[D(\Omega)]^n} \varphi(t) \, dt = 0, \qquad w \in [D(\Omega)]^n, \quad \varphi \in D(0,T),$$

where $\mathcal{F} := u_t + \mathcal{A}_1 u + \mathtt{N}u + \Phi(\mathtt{E}u) - F$, $\operatorname{div} w = 0$, so (29) holds.

Note that, if X is the reflexive Banach space and $\frac{1}{h} + \frac{1}{h'} = 1$, then

$$W_0^{1,1}(0,T;X^*) \mathrel{\bar{\circlearrowleft}} C([0,T];X^*) \mathrel{\bar{\circlearrowleft}} L^{h'}(0,T;X^*), \quad L^h(0,T;X) \mathrel{\bar{\circlearrowleft}} W^{-1,\infty}(0,T;X).$$

Since $u \in L^\infty(0,T;[L^2(\Omega)]^n)$, we have $u_t \in W^{-1,\infty}(0,T;[L^2(\Omega)]^n)$ by definition (see [2, p. 1099]). Then

$$\mathcal{A}_1 u + \mathtt{N}u + \Phi(\mathtt{E}u) - F \in L^2(0,T;[H^{-1}(\Omega)]^n) + [L^{\frac{q(x)}{q(x)-1}}(Q_{0,T})]^n + [L^2(Q_{0,T})]^n \subset$$

$$\subset L^2(0,T;[H^{-1}(\Omega)]^n) + [L^{\frac{q^0}{q^0-1}}(Q_{0,T})]^n \subset L^h\Big(0,T;[H^{-1}(\Omega) + L^{\frac{q^0}{q^0-1}}(\Omega)]^n\Big) \subset$$

$$\subset L^h(0,T;[W^{-1,h}(\Omega)]^n) \subset W^{-1,\infty}(0,T;[W^{-1,h}(\Omega)]^n),$$

where h is taken from (19). Thus $\mathcal{F} \in W^{-1,\infty}(0,T;[W^{-1,h}(\Omega)]^n)$ and the generalized De Rham theorem (see Proposition 1) yields that there exists a distribution

$$\pi \in W^{-1,\infty}(0,T;W^{0,h}(\Omega)) = W^{-1,\infty}(0,T;L^h(\Omega))$$

such that (31)-(32) hold. Thus, π satisfies (20) in $[D^*(Q_{0,T})]^n$ and satisfies (26) in $D^*(0,T)$. Theorem 3 is proved. □

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ПРО НЕЛІНІЙНУ ІНТЕГРО-ДИФЕРЕНЦІАЛЬНУ СИСТЕМУ ОСКОЛКОВА-СТОКСА ЗІ ЗМІННИМ ПОКАЗНИКОМ НЕЛІНІЙНОСТІ ТА МАЛИМ ПАРАМЕТРОМ

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Розглянуто нелінійну інтегро-диференціальну систему рівнянь Осколкова-Стокса зі змінним показником нелінійності та малим параметром. Доведено існування та єдиність узагальненого розв'язку мішаної задачі для цієї системи. Також отримано певну граничну поведінку розв'язків, коли параметр прямує до нуля.

Ключові слова: еволюційна система Осколкова-Стокса, інтегро-диференціальне рівняння, мішана задача, змінний показник нелінійності.