ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2023. Випуск 95. С. 67–72
Visnyk of the Lviv Univ. Series Mech. Math. 2023. Issue 95. P. 67–72
http://publications.lnu.edu.ua/bulletins/index.php/mmf
doi: http://dx.doi.org/10.30970/vmm.2023.95.067-072

УДК 512.628.2

INTERRELATIONS BETWEEN SOME TYPES OF SEMIRING IDEALS

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The paper is focused on further study of some types of ideals of differential semirings, not necessarily commutative, i. e. prime differential ideals, differentially prime ideals and quasi-prime ideals. Some interrelation between these types of semiring ideals is established.

Key words: Semiring, semigroup, semiring ideal, differential ideal.

In 1935 a notion of semiring was introduced as a generalization of associative rings and distributive lattices. Semiring derivations, differential semirings and their differential ideals were considered by Golan [1], where he gave few simple examples and properties.

This paper is devoted to the investigation of different types of ideals in differential semirings, not necessarily commutative, and interrelations between them.

For the sake of completeness some definitions and properties used in the paper will be given here. For more information on semirings, see [1], [2], [3], [4], [5], [7], [6].

Let R be a nonempty set and let $+$ and $\cdot$ be binary operations on R named addition and multiplication, respectively. An algebraic system $(R, +, \cdot)$ is called a *semiring* if $(R, +)$ is a commutative semigroup and $(R, \cdot)$ is a semigroup such that multiplication distributes over addition from each side. A semiring which is not a ring is called a *proper semiring*. A semiring $(R, +, \cdot)$ is called *commutative* if multiplication is commutative.

An element $0 \in R$ is called *zero* if $a + 0 = 0 + a = a$ for all $a \in R$. An element $1 \in R$ is called *identity* if $a \cdot 1 = 1 \cdot a = a$ for all $a \in R$. Zero $0 \in R$ is called (*multiplicatively*) *absorbing* if $a \cdot 0 = 0 \cdot a = 0$ for all $a \in R$.

A non-empty subset of R, closed under addition and multiplication, is called a *subsemiring* of R. A nonempty subset $I \neq \emptyset$ of R is called a *(semiring) ideal* of R, if it is closed under addition and both $ra \in I$ and $ar \in I$ hold for any $r \in R$ and $a \in I$.

2020 Mathematics Subject Classification: 16Y60, 13N99.

An ideal I of R is called a *subtractive ideal* (or *k-ideal*) if $a+b \in I$ and $a \in I$ imply that $b \in I$. The *k-closure* $\mathrm{cl}(I)$ of an ideal I is defined as the set

$$\mathrm{cl}(I) = \{a \in R \mid a+b \in I \text{ for some } b \in I\}.$$

It is an ideal of R satisfying $I \subseteq \mathrm{cl}(I)$ and $\mathrm{cl}(\mathrm{cl}(I)) = \mathrm{cl}(I)$. An ideal I of R is subtractive if and only if $I = \mathrm{cl}(I)$.

A proper ideal I of R is called *maximal* if $I \subsetneqq J$ for any ideal J of R implies $J = R$.

In a commutative semiring R the *radical* of an ideal I is denoted by $\sqrt{I}$ and defined to be the set $\sqrt{I} = \{r \in R \mid r^n \in I \text{ for some } n \in \mathbb{N}_0\}$. $I \subseteq \sqrt{I}$. If I is a subtractive ideal of R, then so is $\sqrt{I}$. Moreover, $\sqrt{I}$ is an intersection of all the prime ideals of R containing I, whenever $1 \in R$. An ideal I of R is said to be *radical* (or *perfect*) if $I = \sqrt{I}$.

Throughout the paper R denotes a semiring in the above sense with identity 1 and absorbing zero $0 \neq 1$, unless stated otherwise. $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$ the set of non-negative integers.

Let R be a semiring, not necessarily commutative. A map $\delta \colon R \to R$ is called a *derivation on* R if $\delta(a+b) = \delta(a) + \delta(b)$ and $\delta(ab) = \delta(a)b + a\delta(b)$ for any $a, b \in R$ [1]. A semiring R equipped with a derivation δ is called *differential* with respect to the derivation δ, or a *δ-semiring*, and denoted by (R, δ).

An ideal I of the δ-semiring R is called *differential* [1] if $\delta(a) \in I$ whenever $a \in I$. A subsemiring S of the δ-semiring R is called *differential* if $a \in S$ follows $\delta(a) \in S$.

In a differential semiring R with absorbing zero the set $V(R)$ of all additively invertible elements of R is a differential ideal. The set $I^+(R)$ of all additively idempotent elements of R is a differential ideal of R. Every multiplicatively idempotent two-sided ideal I of a differential semiring R is differential. [10] $\{0\}$ is a differential k-ideal of any differential semiring R. If I is a differential ideal of R, then its k-closure $\mathrm{cl}(I)$ is a differential k-ideal of R [12]. The zeroid $Zr(R)$ of a differential δ-semiring R is a differential ideal of R [9]. If R is an additively cancellative differential semiring, then its center $Z(R)$ is a differential subsemiring of R.

A semiring R is called *ideally differential* if all of its ideals are differential. Every additively idempotent differential semiring is ideally differential. Every multiplicatively idempotent commutative differential semiring is ideally differential. A subset A of R is called *differentially closed*, if $a \in A$ implies $\delta(a) \in A$. Differential ideals are differentially closed.

For an element $a \in R$ denote $a^{(0)} = a$, $a' = \delta(a)$, $a'' = \delta(\delta(a))$, …, $a^{(n)} = \delta(a^{(n-1)})$, $n \in \mathbb{N}_0$, and $a^{(\infty)} = \{a^{(n)} \mid n \in \mathbb{N}_0\}$. The set $a^{(\infty)}$ of all derivatives of $a \in R$ is differentially closed in R, so we have the following result.

Let $A \subseteq R$ be a non-empty subset of a semiring R. The *annihilator ideal of* A is defined as the set $(0 : A) = \{r \in R \mid ra = 0 \text{ for all } a \in A\}$.

Let (R, δ) be a differential semiring. For a subset A of R we define its *differential* $A_\#$ to be the set

$$A_\# = \left\{a \in R \mid a^{(n)} \in A \,\text{for all}\, n \in \mathbb{N}_0\right\}.$$

A non-empty subset S of the semiring R is called an *m-system* [1] of R if for every $a, b \in S$ there exists an element $r \in R$ such that $arb \in S$. An ideal I of R is prime if and only if $R \setminus I$ is an m-system [1]. Any maximal ideal of a semiring is prime [1].

A differential ideal Q of R is called *quasi-prime* if it is maximal among differential ideals of R disjoint from some m-system of R [12], [11].

Since for any prime ideal P of R the complement $R \setminus P = S$ is an m-system [1], then any prime differential ideal of R is quasi-prime.

Every maximal differential ideal of R is quasi-prime. Indeed, if Q is a maximal among differential ideals of R, $S = U(R)$ is the set of units of R, then S is an m-system and no differential ideal I contains a unit of R, so $Q \cap U(R) = \emptyset$.

In any differential semiring R for any prime ideal P of R the differential ideal $P_{\#}$ is quasi-prime. Indeed, the set $S = R \setminus P$ is an m-system and $S \cap P = \varnothing$, so by Propositions 10 and 11 from [9], $P_{\#}$ is a differential ideal of R disjoint from S. If I is any differential ideal disjoint from S, then $I \subseteq P$, which implies $I = I_{\#} \subseteq P_{\#}$.

Theorem 1. *For a differential semiring R the following conditions are equivalent:*

(1) *Any quasi-prime ideal I in R is prime.*
(2) *If I is a prime ideal of R, then $I_{\#}$ is a prime differential ideal of R.*
(3) *If $S \subseteq R$ is an m-system of R ($0 \notin S$) and I is a differential ideal of R disjoint from S, then every differential ideal of R which is maximal among differential ideals containing I and not meeting S is prime.*

Proof. (1) $\Longrightarrow$ (2) If I is prime then $I_{\#}$ is quasi-prime. Therefore, $I_{\#}$ is a prime differential ideal.

(2) $\Longrightarrow$ (1) Obvious. (2) $\Longrightarrow$ (3) Obvious.

(3) $\Longrightarrow$ (2) Suppose $S \subseteq R$ is an m-system of R ($0 \notin S$), I is a differential ideal of R such that $I \cap S = \varnothing$, and every differential ideal K of R, maximal among those containing I and not meeting S is prime. Let P be any prime ideal. Under given conditions $S = R \setminus P$ is an m-system of R and $\{0\}$ is a differential ideal disjoint from S. Moreover, $P_{\#} \subseteq P$ follows $S \cap P_{\#} = \varnothing$. Thus $P_{\#}$ is a differential ideal of R disjoint from S. If I is an arbitrary differential ideal of R such that $P_{\#} \subseteq I$ and $I \cap S = \varnothing$, then $I \subseteq P$. It follows that $I = I_{\#} \subseteq P_{\#}$. Thus $P_{\#}$ is prime. □

A differential k-ideal P of a semiring R is called *differentially prime* if for any differential ideals I and J of R the inclusion $IJ \subseteq P$ follows $I \subseteq P$ or $J \subseteq P$.

Let $S \neq \varnothing$ be a subset of R. A subset S is called dm-*system* if for any $a, b \in S$ there exist $n \in \mathbb{N}_0$ and $r \in R$ such that $arb^{(n)} \in S$. If a subset is dm-system, then it is an m-system.

Proposition 1. *An ideal I of R is differentially prime if and only if $R \setminus I$ is a* dm-*system.*

Proof. Suppose I is a differentially prime ideal of R and there exist $a, b \notin I$ such that $aRb^{(n)} \subseteq I$ for all $n \in \mathbb{N}_0$. Then $a \in I$ or $b \in I$, which contradicts the condition $a, b \in R \setminus I$. Conversely, suppose $R \setminus I$ is dm-system, and for all $a, b \in R$ and all $n \in \mathbb{N}_0$, $arb^{(n)} \subseteq I$, $a, b \notin I$. Then $arb^{(k)} \notin I$ for some $k \in \mathbb{N}_0$, which is a contradiction. □

Theorem 2. *For a proper differential ideal P of R, the following conditions are equivalent:*

(1) *For any differential ideals I and J of R, $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$;*
(2) *For any $a, b \in R$, $[a] \cdot [b] \subseteq P$ implies $a \in P$ or $b \in P$;*
(3) *For any $a, b \in R$, $k, l \in \mathbb{N}_0$, $a^{(l)}Rb^{(k)} \subseteq P$ implies $a \in P$ or $b \in P$;*

(4) *For any* $a, b \in R$, $k \in \mathbb{N}_0$, $aRb^{(k)} \subseteq P$ *implies* $a \in P$ *or* $b \in P$.

Proof. The implication (1) $\Longrightarrow$ (2) is obvious.

(2) $\Longrightarrow$ (1). Suppose $I \nsubseteq P$ and $J \nsubseteq P$. There exists $a \in I$, $a \notin P$, and $b \in J$, $b \notin P$. Clearly, $[a] \cdot [b] \subseteq IJ \subseteq P$. Therefore, $a \in P$ or $b \in P$, which is a contradiction.

(2) $\Longrightarrow$ (3) Note that $[a] = \sum_{l \in \mathbb{N}_0} Ra^{(l)}$, $[b] = \sum_{k \in \mathbb{N}_0} Rb^{(k)}$, and so $[a] \cdot [b] = \sum_{k,l \in \mathbb{N}_0} a^{(l)} Rb^{(k)}$. If $a^{(l)}Rb^{(k)} \subseteq P$, then $\sum_{k,l \in \mathbb{N}_0} a^{(l)}Rb^{(k)} \subseteq P$. Therefore, $[a] \cdot [b] \subseteq P$, which follows $a \in P$ or $b \in P$.

(3) $\Longrightarrow$ (2) Conversely, if $[a] \cdot [b] \subseteq P$ then $\sum_{k,l \in \mathbb{N}_0} a^{(l)}Rb^{(k)} \subseteq P$, in particular $a^{(l)}Rb^{(k)} \subseteq P$. Hence, $a \in P$ or $b \in P$.

(3) $\Longrightarrow$ (4) Obvious when $l = 0$.

(4) $\Longrightarrow$ (3) Suppose $aRb^{(k)} \subseteq P$ for any $k \in \mathbb{N}_0$. Prove that $a^{(l)}Rb^{(k)} \subseteq P$ for any $k, l \in \mathbb{N}_0$. Denote $t = l + k$. For $t = 0$ we have $a^{(0)}Rb^{(0)} = aRb \subseteq P$. Therefore, $(aRb)' \subseteq P$. For a k-ideal P, $(aRb)' = a'Rb + aRb'b \subseteq P$, $aRb'b \subseteq P$ imply $a'Rb \subseteq P$.

Assume that $a^{(l)}Rb^{(k)} \subseteq P$ for any $k, l \in \mathbb{N}_0$, $l + k \leqslant t$.

Consider

$$(aRb^{(k)})' = a'Rb^{(k)} + aRb^{(k+1)} + aR'b(k) \subseteq P.$$

Similarly, $(aRb^{(k)})' \subseteq P$, $aRb^{(k+1)} \subseteq P$ implies $a'Rb^{(k)} \subseteq P$. Then from $(a'Rb^{(k-1)})' = a''Rb^{(k-1)} + a'Rb^{(k)} \subseteq P$, $a'Rb^{(k)} \subseteq P$ and subtractiveness of P we obtain $a''Rb^{(k)} \subseteq P$, etc. □

Theorem 3. *Let S be* dm-*system of R. If the ideal I is d-maximal in $R \setminus S$, then I is a differentially prime ideal of R.*

Proof. Suppose that there exist $a, b \in R$ and $n \in \mathbb{N}_0$ such that $aRb^{(n)} \subseteq P$, $a, b \notin P$. It is clear that $P \subset P + [a]$ and $P \subset P + [b]$. Since P is maximal among the differential ideals not meeting some dm-system S, $(P + [a]) \cap S \neq \varnothing$, $(P + [b]) \cap S \neq \varnothing$. Therefore there exist $a, b \in S$ such that $a \in P + [a]$ and $b \in P + [b]$. On the other hand, since S is a dm-system, then $a, b \in S$ follows the existence of $n \in \mathbb{N}_0$ such that $aRb^{(n)} \subseteq S$. Therefore $b^{(n)} \in (P + [a]) \cap S$. Then

$$aRb^{(n)} \subseteq (P + [a]) \cdot R \cdot (P + [b]) \subseteq P.$$

Therefore, $aRb^{(n)} \subseteq P \bigcap S \neq \varnothing$, but it contradicts the assumption that $S \bigcap P = \varnothing$. Hence P is a differentially prime ideal. □

Proposition 2. *Let $a \in R$. There exists $n \in \mathbb{N}_0$ such that $(I : a^n R)$ is a differential ideal and $(I : a^n R) = (I : a^k R)$ for any $k \geqslant n$.*

Proof. Denote $U = \bigcup_{l=0}^{\infty} (I : a^l R) \subseteq R$. For any $b \in U$ there exists $l \in \mathbb{N}_0$ such that $b \in (I : a^l R)$. Then $a^l bR \subseteq I$. Since I is a differential ideal of R, then $\delta(a^l b) \in I$. From $\delta(a^l b) = l a^{l-1} \delta ab + a^l \delta b \in I$ and subtractiveness of I we have that $a^l \delta(b) \in I$. Thus, $\delta(b) \in (I : a^l R)$, which follows $\delta(b) \in U$.

The ideals $(I : a^n R)$, $n \in \mathbb{N}_0$, form a chain, therefore there exists $n \in \mathbb{N}_0$ such that $(I : a^n R) = (I : a^k R)$ for any $k \geqslant n$. □

Theorem 4. *Let R be differentially Noetherian semiring. If the ideal P of R is differentially prime, then P is a primary ideal of R.*

Proof. For an ideal P we have that $a^n Rb \subseteq P$, which follows $bR \subseteq (P : a^n R)$. By Proposition 2, $[b] \subseteq (P : a^n R)$. From $a^n Rb \subseteq P$ we also have that $a^n R \subseteq (P : [b])$. Therefore, $[a^n] \subseteq (P : [b])$. Then $[a^n][b] \subseteq P$. By Theorem 2, we have $[a^n] \subseteq P$ or $[b] \subseteq P$. Hence, $a^n R \subseteq P$ or $bR \subseteq P$, i. e. P is a primary ideal. □

Theorem 5. *For every differential ideal I of the differentially Noetherian semiring R the following are equivalent:*

(1) *I is a quasi-prime ideal;*
(2) *I is a differentially prime ideal;*
(3) *$I = P_\#$ for some prime ideal P of R.*

Proof. (1) $\Longrightarrow$ (2) Clear.

(2) $\Longrightarrow$ (1) Let I be some differentially prime ideal of R. Then, by Proposition 1, the set $R \setminus I$ is a dm-system of the semiring R. Since I is maximal differential ideal disjoint from $R \setminus I$, then, by definition, it is quasi-prime.

(1) $\Longrightarrow$ (3) Let I be a quasi-prime ideal of R, i.e., maximal among differential ideals disjoint from the multiplicatively closed subset S, and let K be maximal among ordinary ideals disjoint from S and containing I. Then K is prime ideal. Show that $I = K_\#$. Since I is a differential ideal of R, then $I \subseteq K_\#$. The converse inclusion implies due to maximality of the differential ideal I among those disjoint from S, because $K_\#$ is disjoint from S and it is differential ideal of R.

(3) $\Longrightarrow$ (2) Let $I = P_\#$ for some prime ideal of R of M. Then I is maximal amongst differential ideals disjoint from P. Let $S = R \setminus P$. Assuming that all the derivations are trivial, we see that S is a multiplicatively closed subset of R. Denote by K the intersection of all dm-systems of R, which contain S. Then S is the least dm-system of those containing S. Hence I is a differentially prime ideal of R because of 1. It remains to verify that $I = R \setminus K$. Since $R \setminus K$ is disjoint from S, then $R \setminus K \subseteq P$, and due to the fact that $R \setminus K$ is a differential ideal of R, we have the inclusion $R \setminus K \subseteq I$. Taking into consideration the minimality of the set K, we obtain that the set $R \setminus K$ is a maximal ideal among the differential ideals of I. Hereby $R \setminus K = N$. □

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Стаття: надійшла до редколегії 03.01.2023
доопрацьована 25.02.2023
прийнята до друку 15.05.2023

ВЗАЄМОЗВ'ЯЗКИ МІЖ ДЕЯКИМИ ТИПАМИ ІДЕАЛІВ НАПІВКІЛЕЦЬ

Іванна МЕЛЬНИК

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Виконано подальше дослідження деяких типів ідеалів дифернціальних напівкілець, не обов'язково комутативних: первинних диференціальних ідеалів, диференціально-первинних ідеалів та квазіпервинних ідеалів. Визначено деякі взаємозв'язки між цими типами ідеалів.

Ключові слова: Напівкільце, напівгрупа, ідеал напівкільця, диференціальний ідеал.