УДК 512.53

ON THE SEMIGROUP OF INJECTIVE MONOID ENDOMORPHISMS OF THE MONOID $B_\omega^{\mathscr{F}}$ WITH A TWO-ELEMENT FAMILY $\mathscr{F}$ OF INDUCTIVE NONEMPTY SUBSETS OF ω

Oleg GUTIK, Inna POZDNIAKOVA

Ivan Franko National University of Lviv,
Universytetska Str., 1, 79000, Lviv, UKRAINE
e-mails: oleg.gutik@lnu.edu.ua, pozdnyakova.inna@gmail.com

We study injective endomorphisms of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ with the two-elements family $\mathscr{F}$ of inductive nonempty subsets of ω. We describe the elements of the semigroup $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ of all injective monoid endomorphisms of the monoid $\boldsymbol{B}_\omega^{\mathscr{F}}$, and show that Green's relations $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$, $\mathscr{D}$, and $\mathscr{J}$ on $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ coincide with the relation of equality.

Key words: bicyclic monoid, inverse semigroup, bicyclic extension, injective endomorphism, Green's relations.

1. Introduction, motivation and main definitions

We shall follow the terminology of [1, 2, 12]. By ω we denote the set of all non-negative integers, by $\mathbb{N}$ the set of all positive integers, and by $\mathbb{Z}$ the set of all integers.

Let $\mathscr{P}(\omega)$ be the family of all subsets of ω. For any $F \in \mathscr{P}(\omega)$ and $n \in \mathbb{Z}$ we put $nF = \{nk \colon k \in F\}$ if $F \neq \varnothing$ and $n\varnothing = \varnothing$. A subfamily $\mathscr{F} \subseteq \mathscr{P}(\omega)$ is called ω-*closed* if $F_1 \cap (-n + F_2) \in \mathscr{F}$ for all $n \in \omega$ and $F_1, F_2 \in \mathscr{F}$. For any $a \in \omega$ we denote

$$[a) = \{x \in \omega \colon x \geqslant a\}.$$

A subset A of ω is said to be *inductive*, if $i \in A$ implies $i + 1 \in A$. In particular the empty set $\varnothing$ is an inductive subset of ω.

Remark 1 ([5]).   (1) By Lemma 6 from [4], a nonempty subset $F \subseteq \omega$ is inductive in ω if and only $(-1 + F) \cap F = F$.

(2) Since the set ω with the usual order is well-ordered, for any nonempty inductive subset F in ω there exists nonnegative integer $n_F \in \omega$ such that $[n_F) = F$.

(3) Statement (2) implies that the intersection of an arbitrary finite family of nonempty inductive subsets in ω is a nonempty inductive subset of ω.

A semigroup S is called *inverse* if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element x^{-1} is called the *inverse of* $x \in S$. If S is an inverse semigroup, then the function inv: $S \to S$ which assigns to every element x of S its inverse element x^{-1} is called the *inversion*.

If S is a semigroup, then we shall denote the subset of all idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as a *band* (or the *band of* S). Then the semigroup operation on S determines the following partial order $\preccurlyeq$ on $E(S)$:

$$e \preccurlyeq f \quad \text{if and only if} \quad ef = fe = e.$$

This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents.

If S is an inverse semigroup then the semigroup operation on S determines the following partial order $\preccurlyeq$ on S:

$$s \preccurlyeq t \quad \text{if and only if there exists} \quad e \in E(S) \quad \text{such that} \quad s = te.$$

This order is called the *natural partial order* on S [16].

The bicyclic monoid $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq = 1$. The semigroup operation on $\mathscr{C}(p,q)$ is determined as follows:

$$q^k p^l \cdot q^m p^n = q^{k+m-\min\{l,m\}} p^{l+n-\min\{l,m\}}.$$

It is well known that the bicyclic monoid $\mathscr{C}(p,q)$ is a bisimple (and hence simple) combinatorial E-unitary inverse semigroup and every non-trivial congruence on $\mathscr{C}(p,q)$ is a group congruence [1].

On the set $\boldsymbol{B}_\omega = \omega \times \omega$ we define the semigroup operation "$\cdot$" in the following way

$$(1) \qquad (i_1,j_1) \cdot (i_2,j_2) = \begin{cases} (i_1 - j_1 + i_2, j_2), & \text{if } j_1 \leqslant i_2; \\ (i_1, j_1 - i_2 + j_2), & \text{if } j_1 \geqslant i_2. \end{cases}$$

It is well known that the bicyclic monoid $\mathscr{C}(p,q)$ is isomorphic to the semigroup $\boldsymbol{B}_\omega$ and the mapping

$$\mathfrak{h} \colon \mathscr{C}(p,q) \to \boldsymbol{B}_\omega, \quad q^k p^l \mapsto (k,l)$$

is an isomorphism (see: [1, Section 1.12] or [14, Exercise IV.1.11(ii)]).

Next, we shall describe the construction which is introduced in [4].

Let $\boldsymbol{B}_\omega$ be the bicyclic monoid and $\mathscr{F}$ be an ω-closed subfamily of $\mathscr{P}(\omega)$. On the set $\boldsymbol{B}_\omega \times \mathscr{F}$ we define the semigroup operation "$\cdot$" in the following way

$$(2) \qquad (i_1,j_1,F_1) \cdot (i_2,j_2,F_2) = \begin{cases} (i_1 - j_1 + i_2, j_2, (j_1 - i_2 + F_1) \cap F_2), & \text{if } j_1 \leqslant i_2; \\ (i_1, j_1 - i_2 + j_2, F_1 \cap (i_2 - j_1 + F_2)), & \text{if } j_1 \geqslant i_2. \end{cases}$$

In [4] is proved that if the family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ is ω-closed then $(\boldsymbol{B}_\omega \times \mathscr{F}, \cdot)$ is a semigroup. Moreover, if an ω-closed family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ contains the empty set $\varnothing$, then the set

34
    Oleg GUTIK, Inna POZDNIAKOVA

ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2022. Випуск 94

$I = \{(i, j, \varnothing) \colon i, j \in \omega\}$ is an ideal of the semigroup $(\boldsymbol{B}_\omega \times \mathscr{F}, \cdot)$. For any ω-closed family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ the following semigroup

$$\boldsymbol{B}_\omega^{\mathscr{F}} = \begin{cases} (\boldsymbol{B}_\omega \times \mathscr{F}, \cdot)/\boldsymbol{I}, & \text{if } \varnothing \in \mathscr{F}; \\ (\boldsymbol{B}_\omega \times \mathscr{F}, \cdot), & \text{if } \varnothing \notin \mathscr{F} \end{cases}$$

is defined in [4]. The semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ generalizes the bicyclic monoid and the countable semigroup of matrix units. It is proven in [4] that $\boldsymbol{B}_\omega^{\mathscr{F}}$ is a combinatorial inverse semigroup and Green's relations, the natural partial order on $\boldsymbol{B}_\omega^{\mathscr{F}}$ and its set of idempotents are described. Here, the criteria when the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ is simple, 0-simple, bisimple, 0-bisimple, or it has the identity, are given. In particularly in [4] it is proved that the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ is isomorphic to the semigrpoup of $\omega \times \omega$-matrix units if and only if $\mathscr{F}$ consists of a singleton set and the empty set, and $\boldsymbol{B}_\omega^{\mathscr{F}}$ is isomorphic to the bicyclic monoid if and only if $\mathscr{F}$ consists of a non-empty inductive subset of ω.

Group congruences on the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ and its homomorphic retracts in the case when an ω-closed family $\mathscr{F}$ consists of inductive non-empty subsets of ω are studied in [5]. It is proven that a congruence $\mathfrak{C}$ on $\boldsymbol{B}_\omega^{\mathscr{F}}$ is a group congruence if and only if its restriction on a subsemigroup of $\boldsymbol{B}_\omega^{\mathscr{F}}$, which is isomorphic to the bicyclic semigroup, is not the identity relation. Also in [5], all non-trivial homomorphic retracts and isomorphisms of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ are described. In [6] it is proved that an injective endomorphism ε of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ is the indentity transformation if and only if ε has three distinct fixed points, which is equivalent to existence non-idempotent element $(i, j, [p]) \in \boldsymbol{B}_\omega^{\mathscr{F}}$ such that $(i, j, [p])\varepsilon = (i, j, [p])$.

The semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ and its group of automorphisms in the case when an ω-closed family $\mathscr{F}$ consists of inductive non-empty subsets of the set of integers $\mathbb{Z}$ are studied in [9, 10].

In [3, 13] the algebraic structure of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ is established in the case when ω-closed family $\mathscr{F}$ consists of atomic subsets of ω. Also, in [7, 8, 15] the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}_n}$ and its semigroup of endomorphisms are described in the case when the ω-closed family $\mathscr{F}_n$ is generated by an n-elements interval of ω.

It is well-known that every automorphism of the bicyclic monoid $\boldsymbol{B}_\omega$ is the identity self-map of $\boldsymbol{B}_\omega$ [1], and hence the group $\mathbf{Aut}(\boldsymbol{B}_\omega)$ of automorphisms of $\boldsymbol{B}_\omega$ is trivial. In [11] it is proved that the semigroups $\mathbf{End}(\boldsymbol{B}_\omega)$ of the endomorphisms of the bicyclic semigroup $\boldsymbol{B}_\omega$ is isomorphic to the semidirect products $(\omega, +) \rtimes_\varphi (\omega, *)$, where $+$ and $*$ are the usual addition and the usual multiplication on ω.

Later we assume that an ω-closed family $\mathscr{F}$ consists of nonempty inductive nonempty subsets of ω.

In this paper we study injective endomorphisms of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ for the family $\mathscr{F} = \{[0), [1)\}$. We describe the elements of the monoid $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ of all injective monoid endomorphisms of the monoid $\boldsymbol{B}_\omega^{\mathscr{F}}$, and show that Green's relations $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$, $\mathscr{D}$, and $\mathscr{J}$ on $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ coincide with the relation of equality.

2. ON INJECTIVE ENDOMORPHISMS OF THE MONOID $\boldsymbol{B}_\omega^{\mathscr{F}}$

Remark 2. By Proposition 1 of [5] for any ω-closed family $\mathscr{F}$ of inductive subsets in $\mathscr{P}(\omega)$ there exists an ω-closed family $\mathscr{F}^*$ of inductive subsets in $\mathscr{P}(\omega)$ such that $[0) \in \mathscr{F}^*$ and the semigroups $\boldsymbol{B}_\omega^{\mathscr{F}}$ and $\boldsymbol{B}_\omega^{\mathscr{F}^*}$ are isomorphic. Hence without loss of generality we may assume that the family $\mathscr{F}$ contains the set $[0)$.

If $\mathscr{F}$ is an arbitrary ω-closed family $\mathscr{F}$ of inductive subsets in $\mathscr{P}(\omega)$ and $[s) \in \mathscr{F}$ for some $s \in \omega$ then

$$\boldsymbol{B}_\omega^{\{[s)\}} = \{(i, j, [s)) \colon i, j \in \omega\}$$

is a subsemigroup of $\boldsymbol{B}_\omega^{\mathscr{F}}$ [5] and by Proposition 3 of [4] the semigroup $\boldsymbol{B}_\omega^{\{[s)\}}$ is isomorphic to the bicyclic semigroup.

Lemma 1. *Let $\mathscr{F}$ be an arbitrary ω-closed family $\mathscr{F}$ of inductive subsets in $\mathscr{P}(\omega)$ and α be an endomorphism of the monoid $\boldsymbol{B}_\omega^{\mathscr{F}}$. Then there exists a positive integer k such that*

$$(i, j, [0))\alpha = (ki, kj, [0))$$

for all $i, j \in \omega$.

Proof. By Proposition 3 of [4] the submonoid $\boldsymbol{B}_\omega^{\{[0)\}}$ is isomorphic to the bicyclic monoid. Since $(0, 0, [0))\alpha = (0, 0, [0))$, Proposition 4 of [5] implies that

$$(\boldsymbol{B}_\omega^{\{[0)\}})\alpha \subseteq \boldsymbol{B}_\omega^{\{[0)\}}.$$

Then Lemma 1 of [11] there exists a positive integer k such that

$$(i, j, [0))\alpha = (ki, kj, [0))$$

for all $i, j \in \omega$. $\qquad\qquad\square$

Example 1. Let $\mathscr{F} = \{[0), [1)\}$. Fix an arbitrary positive integer k and any $p \in \{0, \ldots, k-1\}$. For all $i, j \in \omega$ we define the transformation $\alpha_{k,p}$ of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ in the following way

$$(i, j, [0))\alpha_{k,p} = (ki, kj, [0)),$$
$$(i, j, [1))\alpha_{k,p} = (p + ki, p + kj, [1)).$$

It is obvious that $\alpha_{k,p}$ is an injective transformation of the monoid $\boldsymbol{B}_\omega^{\mathscr{F}}$.

Lemma 2. *Let $\mathscr{F} = \{[0), [1)\}$. Then for an arbitrary positive integer k and any $p \in \{0, \ldots, k-1\}$ the map $\alpha_{k,p}$ is an injective endomorphism of the monoid $\boldsymbol{B}_\omega^{\mathscr{F}}$.*

Proof. By Proposition 3 of [4] the subsemigroups $\boldsymbol{B}_\omega^{\{[0)\}}$ and $\boldsymbol{B}_\omega^{\{[1)\}}$ are isomorphic to the bicyclic semigroup. By Lemma 1 of [11] the restrictions of $\alpha_{k,p}$ onto the subsemigroups $\boldsymbol{B}_\omega^{\{[0)\}}$ and $\boldsymbol{B}_\omega^{\{[1)\}}$ are endomorphisms of $\boldsymbol{B}_\omega^{\{[0)\}}$ and $\boldsymbol{B}_\omega^{\{[1)\}}$, respectively. This implies that for all $i, j, s, t \in \omega$ the following equalities hold

$$((i, j, [0)) \cdot (s, t, [0)))\alpha_{k,p} = (i, j, [0))\alpha_{k,p} \cdot (s, t, [0))\alpha_{k,p},$$
$$((i, j, [1)) \cdot (s, t, [1)))\alpha_{k,p} = (i, j, [1))\alpha_{k,p} \cdot (s, t, [1))\alpha_{k,p}.$$

Oleg GUTIK, Inna POZDNIAKOVA

36   *ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2022. Випуск 94*

For any $i, j, p, q \in \omega$ we have that

$$((i,j,[0])) \cdot (s,t,[1]))\alpha_{k,p} = \begin{cases} (i+s-j,t,(j-s+[0])\cap[1])\alpha_{k,p}, & \text{if } j < s; \\ (i,t,[0]\cap[1])\alpha_{k,p}, & \text{if } j = s; \quad = \\ (i,j+t-s,[0]\cap(s-j+[1]))\alpha_{k,p}, & \text{if } j > s \end{cases}$$

$$= \begin{cases} (i+s-j,t,[1])\alpha_{k,p}, & \text{if } j < s; \\ (i,t,[1])\alpha_{k,p}, & \text{if } j = s; \quad = \\ (i,j+t-s,[0])\alpha_{k,p}, & \text{if } j > s \end{cases}$$

$$= \begin{cases} (p+k(i+s-j),p+kt,[1]), & \text{if } j < s; \\ (p+ki,p+kt,[1]), & \text{if } j = s; \\ (ki,k(j+t-s),[0]), & \text{if } j > s, \end{cases}$$

$$(i,j,[0])\alpha_{k,p} \cdot (s,t,[1])\alpha_{k,p} = (ki,kj,[0]) \cdot (p+ks,p+kt,[1]) =$$

$$= \begin{cases} (ki+p+ks-kj,p+kt,(kj-p-ks+[0])\cap[1]), & \text{if } kj < p+ks; \\ (ki,p+kt,[0]\cap[1]), & \text{if } kj = p+ks; \quad = \\ (ki,kj+p+kt-p-ks,[0]\cap(p+ks-kj+[1])), & \text{if } kj > p+ks \end{cases}$$

$$= \begin{cases} (p+k(i+s-j),p+kt,[1]), & \text{if } kj < p+ks; \\ (ki,p+kt,[1]), & \text{if } kj = p+ks; \quad = \\ (ki,k(j+t-s),[0]), & \text{if } kj > p+ks \end{cases}$$

$$= \begin{cases} (p+k(i+s-j),p+kt,[1]), & \text{if } kj < ks; \\ (p+ki,p+kt,[1]), & \text{if } kj = ks; \\ (ki,kt,[1]), & \text{if } kj = p+ks \text{ and } p = 0; \quad = \\ \texttt{vagueness}, & \text{if } kj = p+ks \text{ and } p \neq 0; \\ (ki,k(j+t-s),[0]), & \text{if } kj > ks \end{cases}$$

$$= \begin{cases} (p+k(i+s-j),p+kt,[1]), & \text{if } j < s; \\ (p+ki,p+kt,[1]), & \text{if } j = s; \\ (ki,k(j+t-s),[0]), & \text{if } j > s, \end{cases}$$

and

$$((i,j,[1])) \cdot (s,t,[0]))\alpha_{k,p} = \begin{cases} (i+s-j,t,(j-s+[1])\cap[0])\alpha_{k,p}, & \text{if } j < s; \\ (i,t,[1]\cap[0])\alpha_{k,p}, & \text{if } j = s; \quad = \\ (i,j+t-s,[1]\cap(s-j+[0]))\alpha_{k,p}, & \text{if } j > s \end{cases}$$

$$= \begin{cases} (i+s-j,t,[0])\alpha_{k,p}, & \text{if } j < s; \\ (i,t,[1])\alpha_{k,p}, & \text{if } j = s; \quad = \\ (i,j+t-s,[1])\alpha_{k,p}, & \text{if } j > s \end{cases}$$

$$= \begin{cases} (k(i+s-j),kt,[0]), & \text{if } j < s; \\ (p+ki,p+kt,[1]), & \text{if } j = s; \\ (p+ki,p+k(j+t-s),[1]), & \text{if } j > s, \end{cases}$$

$(i, j, [1))\alpha_{k,p} \cdot (s, t, [0))\alpha_{k,p} = (p + ki, p + kj, [1)) \cdot (ks, kt, [0)) =$

$$= \begin{cases} (p + ki + ks - p - kj, kt, (p + kj - ks + [1)) \cap [0)), & \text{if } p + kj < ks; \\ (p + ki, kt, [1) \cap [0)), & \text{if } p + kj = ks; \\ (p + ki, p + kj + kt - ks, [1) \cap (ks - p - kj + [0))), & \text{if } p + kj > ks \end{cases} =$$

$$= \begin{cases} (k(i + s - j), kt, [0)), & \text{if } p + kj < ks; \\ (p + ki, kt, [1)), & \text{if } p + kj = ks; \\ (p + ki, p + k(j + t - s), [1)), & \text{if } p + kj > ks \end{cases} =$$

$$= \begin{cases} (k(i + s - j), kt, [0)), & \text{if } kj < ks; \\ (ki, kt, [1)), & \text{if } p + kj = ks \text{ and } p = 0; \\ \texttt{vagueness}, & \text{if } p + kj = ks \text{ and } p \neq 0; \\ (p + ki, p + kt, [1)), & \text{if } kj = ks; \\ (p + ki, p + k(j + t - s), [1)), & \text{if } kj > ks \end{cases} =$$

$$= \begin{cases} (k(i + s - j), kt, [0)), & \text{if } j < s; \\ (p + ki, p + kt, [1)), & \text{if } j = s; \\ (p + ki, p + k(j + t - s), [1)), & \text{if } j > s, \end{cases}$$

because $p \in \{0, \dots, k - 1\}$. Thus, $\alpha_{k,p}$ is an endomorphism of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$. $\qquad\square$

Proposition 1. *Let $\mathscr{F} = \{[0), [1)\}$ and α be an injective endomorphism of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ such that*

$$(i, j, [0))\alpha = (ki, kj, [0))$$

for all $i, j \in \omega$ and for some positive integer k. If $(0, 0, [1))\alpha \in \boldsymbol{B}_{\{[1)\}}^{\mathscr{F}}$ then there exists $p \in \{0, \dots, k - 1\}$ such that

$$(i, j, [1))\alpha = (p + ki, p + kj, [1))$$

for all $i, j \in \omega$, i.e., $\alpha = \alpha_{k,p}$.

Proof. Suppose that $(0, 0, [1))\alpha = (p, p, [1))$ for some $p \in \omega$. Since

$$(1, 1, [0))\alpha = (k, k, [0)), \qquad (1, 1, [0)) \preccurlyeq (0, 0, [1))$$

and by Proposition 1.4.21(6) of [12] the endomorphism α preserves the natural partial order, we have that $(k, k, [0)) \preccurlyeq (p, p, [1))$. Hence the definition of the natural partial order on $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ (see [5, Proposition 3]) implies that $p \leqslant k - 1$.

Suppose that

$$(1, 1, [1))\alpha = (p + s, p + s, [1))$$

for some positive integer s. Since by Proposition 3 of [4] the subsemigroup $\boldsymbol{B}_{\omega}^{\{[1)\}}$ is isomorphic to the bicyclic semigroup, the injectivity of α implies that the image $(\boldsymbol{B}_{\omega}^{\{[1)\}})\alpha$ is isomorphic to the bicyclic semigroup. Put $(0, 1, [1))\alpha = (x, y, [1))$. By Proposition 1.4.21 from [12] and Lemma 4 of [4] we get that

$$(1, 0, [1))\alpha = ((0, 1, [1))^{-1})\alpha =$$
$$= ((0, 1, [1))\alpha)^{-1} =$$
$$= (x, y, [1))^{-1} =$$
$$= (y, x, [1)).$$

38   Oleg GUTIK, Inna POZDNIAKOVA

ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2022. Випуск 94

This implies that

$$
\begin{aligned}
(p, p, [1)) = (0, 0, [1))\alpha &= ((0, 1, [1)) \cdot (1, 0, [1)))\alpha = \\
&= (0, 1, [1))\alpha \cdot (1, 0, [1))\alpha = \\
&= (x, y, [1)) \cdot (y, x, [1)) = \\
&= (x, x, [1))
\end{aligned}
$$

and

$$
\begin{aligned}
(p + s, p + s, [1)) = (1, 1, [1))\alpha &= \\
&= ((1, 0, [1)) \cdot (0, 1, [1)))\alpha = \\
&= (1, 0, [1))\alpha \cdot (0, 1, [1))\alpha = \\
&= (y, x, [1)) \cdot (x, y, [1)) = \\
&= (y, y, [1)),
\end{aligned}
$$

and hence by the definition of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ we get that

$$
(0, 1, [1))\alpha = (p, p + s, [1)) \qquad \text{and} \qquad (1, 0, [1))\alpha = (p + s, p, [1)).
$$

Then for any $i, j \in \omega$ we have that

$$
\begin{aligned}
(i, j, [1))\alpha &= ((i, 0, [1)) \cdot (0, j, [1)))\alpha = \\
&= ((1, 0, [1))^i \cdot (0, 1, [1))^j)\alpha = \\
&= ((1, 0, [1))\alpha)^i \cdot ((0, 1, [1))\alpha)^j = \\
&= (p + s, p, [1))^i \cdot (p, p + s, [1))^j = \\
&= (p + si, p, [1)) \cdot (p, p + sj, [1)) = \\
&= (p + si, p + sj, [1)).
\end{aligned}
$$

The definition of the natural partial order on $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ (see [5, Proposition 2]) implies that for any positive integer t we have that

$$
(t + 1, t + 1, [1)) \preccurlyeq (t + 1, t + 1, [0)) \preccurlyeq (t, t, [1)).
$$

Since by Proposition 1.4.21 of [12] the endomorphism α preserves the natural partial order, we conclude that

$$
(t + 1, t + 1, [1))\alpha \preccurlyeq (t + 1, t + 1, [0))\alpha \preccurlyeq (t, t, [1))\alpha.
$$

This and the definition of α imply that

$$
(p + s(t + 1), p + s(t + 1), [1)) \preccurlyeq (k(t + 1), k(t + 1), [0)) \preccurlyeq (p + st, p + st, [1)).
$$

Then by the definition of the natural partial order on $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ (see [5, Proposition 3]) we obtain that

$$
(3) \qquad\qquad p + s(t + 1) \geqslant k(t + 1) \qquad \text{and} \qquad k(t + 1) - 1 \geqslant p + st
$$

for any positive integer t.

We claim that $s = k$. Indeed, if $s < k$ then the first inequality of (3) implies that

$$
p + (s - k)(t + 1) \geqslant 0 \qquad \text{for all positive integers} \quad t.
$$

But the last inequality is incorrect because $s - k < 0$. Similarly, if $s > k$ then the second inequality of (3) implies that

$$(k - s)t + k - 1 - p \geqslant 0 \qquad \text{for all positive integers} \quad t.$$

But the last inequality is incorrect because $k - s < 0$. Then $\alpha = \alpha_{k,p}$. By Lemma 2, the map $\alpha_{k,p}$ is an endomorphism of the monoid $\boldsymbol{B}_\omega^{\mathscr{F}}$. $\qquad \square$

Example 2. Let $\mathscr{F} = \{[0], [1]\}$. Fix an arbitrary positive integer $k \geqslant 2$ and any $p \in \{1, \ldots, k-1\}$. For all $i, j \in \omega$ we define the transformation $\beta_{k,p}$ of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ in the following way

$$(i, j, [0))\beta_{k,p} = (ki, kj, [0)),$$
$$(i, j, [1))\beta_{k,p} = (p + ki, p + kj, [0)).$$

It is obvious that $\beta_{k,p}$ is an injective transformation of the monoid $\boldsymbol{B}_\omega^{\mathscr{F}}$.

Lemma 3. *Let $\mathscr{F} = \{[0], [1]\}$. Then for an arbitrary positive integer $k \geqslant 2$ and any $p \in \{1, \ldots, k-1\}$ the map $\beta_{k,p}$ is an injective endomorphism of the monoid $\boldsymbol{B}_\omega^{\mathscr{F}}$.*

Proof. By Proposition 3 of [4] the subsemigroups $\boldsymbol{B}_\omega^{\{[0)\}}$ and $\boldsymbol{B}_\omega^{\{[1)\}}$ are isomorphic to the bicyclic semigroup. By Lemma 1 of [11], the restriction of $\beta_{k,p}$ onto the subsemigroup $\boldsymbol{B}_\omega^{\{[0)\}}$ is an endomorphisms of $\boldsymbol{B}_\omega^{\{[0)\}}$, and the restriction of $\beta_{k,p}$ onto the subsemigroup $\boldsymbol{B}_\omega^{\{[1)\}}$ is a homomorphisms of $\boldsymbol{B}_\omega^{\{[1)\}}$ into $\boldsymbol{B}_\omega^{\{[0)\}}$. This implies that for all $i, j, s, t \in \omega$ the following equalities hold

$$((i, j, [0)) \cdot (s, t, [0)))\beta_{k,p} = (i, j, [0))\beta_{k,p} \cdot (s, t, [0))\beta_{k,p},$$
$$((i, j, [1)) \cdot (s, t, [1)))\beta_{k,p} = (i, j, [1))\beta_{k,p} \cdot (s, t, [1))\beta_{k,p}.$$

For any $i, j, p, q \in \omega$ we have that

$$((i, j, [0)) \cdot (s, t, [1)))\beta_{k,p} = \begin{cases} (i + s - j, t, (j - s + [0)) \cap [1))\beta_{k,p}, & \text{if } j < s; \\ (i, t, [0) \cap [1))\beta_{k,p}, & \text{if } j = s; \\ (i, j + t - s, [0) \cap (s - j + [1)))\beta_{k,p}, & \text{if } j > s \end{cases} =$$

$$= \begin{cases} (i + s - j, t, [1))\beta_{k,p}, & \text{if } j < s; \\ (i, t, [1))\beta_{k,p}, & \text{if } j = s; \\ (i, j + t - s, [0))\beta_{k,p}, & \text{if } j > s \end{cases} =$$

$$= \begin{cases} (p + k(i + s - j), p + kt, [0)), & \text{if } j < s; \\ (p + ki, p + kt, [0)), & \text{if } j = s; \\ (ki, k(j + t - s), [0)), & \text{if } j > s, \end{cases}$$

$$(i, j, [0))\beta_{k,p} \cdot (s, t, [1))\beta_{k,p} = (ki, kj, [0)) \cdot (p + ks, p + kt, [0)) =$$

$$= \begin{cases} (ki + p + ks - kj, p + kt, (kj - p - ks + [0)) \cap [0)), & \text{if } kj < p + ks; \\ (ki, p + kt, [0) \cap [0)), & \text{if } kj = p + ks; \\ (ki, kj + p + kt - p - ks, [0) \cap (p + ks - kj + [0))), & \text{if } kj > p + ks \end{cases} =$$

Oleg GUTIK, Inna POZDNIAKOVA

40   *ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2022. Випуск 94*

$$= \begin{cases} (p + k(i + s - j), p + kt, [0)), & \text{if } kj < p + ks; \\ (ki, p + kt, [0)), & \text{if } kj = p + ks; \quad = \\ (ki, k(j + t - s), [0)), & \text{if } kj > p + ks \end{cases}$$

$$= \begin{cases} (p + k(i + s - j), p + kt, [0)), & \text{if } kj < ks; \\ (p + ki, p + kt, [0)), & \text{if } kj = ks; \\ (ki, kt, [0)), & \text{if } kj = p + ks \text{ and } p = 0; \quad = \\ \texttt{vagueness}, & \text{if } kj = p + ks \text{ and } p \neq 0; \\ (ki, k(j + t - s), [0))), & \text{if } kj > ks \end{cases}$$

$$= \begin{cases} (p + k(i + s - j), p + kt, [0)), & \text{if } j < s; \\ (p + ki, p + kt, [0)), & \text{if } j = s; \\ (ki, k(j + t - s), [0)), & \text{if } j > s, \end{cases}$$

and

$$((i, j, [1)) \cdot (s, t, [0)))\beta_{k,p} = \begin{cases} (i + s - j, t, (j - s + [1) \cap [0))\beta_{k,p}, & \text{if } j < s; \\ (i, t, [1) \cap [0))\beta_{k,p}, & \text{if } j = s; \quad = \\ (i, j + t - s, [1) \cap (s - j + [0))\beta_{k,p}, & \text{if } j > s \end{cases}$$

$$= \begin{cases} (i + s - j, t, [0))\beta_{k,p}, & \text{if } j < s; \\ (i, t, [1))\beta_{k,p}, & \text{if } j = s; \quad = \\ (i, j + t - s, [1))\beta_{k,p}, & \text{if } j > s \end{cases}$$

$$= \begin{cases} (k(i + s - j), kt, [0)), & \text{if } j < s; \\ (p + ki, p + kt, [0)), & \text{if } j = s; \\ (p + ki, p + k(j + t - s), [0)), & \text{if } j > s, \end{cases}$$

$$(i, j, [1))\beta_{k,p} \cdot (s, t, [0))\beta_{k,p} = (p + ki, p + kj, [0)) \cdot (ks, kt, [0)) =$$

$$= \begin{cases} (p + ki + ks - p - kj, kt, (p + kj - ks + [0)) \cap [0)), & \text{if } p + kj < ks; \\ (p + ki, kt, [0) \cap [0)), & \text{if } p + kj = ks; \quad = \\ (p + ki, p + kj + kt - ks, [0) \cap (ks - p - kj + [0))), & \text{if } p + kj > ks \end{cases}$$

$$= \begin{cases} (k(i + s - j), kt, [0)), & \text{if } p + kj < ks; \\ (p + ki, kt, [0)), & \text{if } p + kj = ks; \quad = \\ (p + ki, p + k(j + t - s), [0)), & \text{if } p + kj > ks \end{cases}$$

$$= \begin{cases} (k(i + s - j), kt, [0)), & \text{if } kj < ks; \\ (ki, kt, [0)), & \text{if } p + kj = ks \text{ and } p = 0; \\ \texttt{vagueness}, & \text{if } p + kj = ks \text{ and } p \neq 0; \quad = \\ (p + ki, p + kt, [0)), & \text{if } kj = ks; \\ (p + ki, p + k(j + t - s), [0)), & \text{if } kj > ks \end{cases}$$

$$= \begin{cases} (k(i + s - j), kt, [0)), & \text{if } j < s; \\ (p + ki, p + kt, [0)), & \text{if } j = s; \\ (p + ki, p + k(j + t - s), [0)), & \text{if } j > s, \end{cases}$$

because $p \in \{1, \ldots, k - 1\}$. Thus, $\beta_{k,p}$ is an endomorphism of the monoid $\boldsymbol{B}_\omega^{\mathscr{F}}$. $\qquad\square$

Proposition 2. *Let* $\mathscr{F} = \{[0), [1)\}$ *and* β *be an injective endomorphism of the monoid* $\boldsymbol{B}_\omega^{\mathscr{F}}$ *such that*

$$(i, j, [0))\beta = (ki, kj, [0))$$

for all $i, j \in \omega$ and for some positive integer $k \geqslant 2$. If $(0, 0, [1))\beta \in \boldsymbol{B}_\omega^{\{[0)\}}$ then there exists $p \in \{1, \ldots, k-1\}$ such that

$$(i, j, [1))\beta = (p + ki, p + kj, [0))$$

for all $i, j \in \omega$, i.e., $\beta = \beta_{k,p}$.

Proof. Suppose that $(0, 0, [1))\beta = (p, p, [0))$ for some $p \in \omega$. By assumption

$$(0, 0, [0))\beta = (0, 0, [0))$$

and hence $p \neq 1$. Since

$$(1, 1, [0))\beta = (k, k, [0)), \qquad (1, 1, [0)) \preccurlyeq (0, 0, [1))$$

and by Proposition 1.4.21(6) of [12] the endomorphism β preserves the natural partial order, we conclude that $(k, k, [0)) \preccurlyeq (p, p, [0))$. Hence the definition of the natural partial order on the monoid $\boldsymbol{B}_\omega^{\mathscr{F}}$ (see [5, Proposition 3]) implies that $p \leqslant k - 1$.

Suppose that

$$(1, 1, [1))\beta = (p + s, p + s, [0))$$

for some positive integer s. Since by Proposition 3 of [4] the subsemigroup $\boldsymbol{B}_\omega^{\{[1)\}}$ is isomorphic to the bicyclic semigroup, the injectivity of β implies that the image $(\boldsymbol{B}_\omega^{\{[1)\}})\beta$ is isomorphic to the bicyclic semigroup. Put $(0, 1, [1))\beta = (x, y, [0))$. By Proposition 1.4.21 from [12] and Lemma 4 of [4] we get that

$$
\begin{aligned}
(1, 0, [1))\beta &= ((0, 1, [1))^{-1})\beta = \\
&= ((0, 1, [1))\beta)^{-1} = \\
&= (x, y, [0))^{-1} = \\
&= (y, x, [0)).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
(p, p, [0)) = (0, 0, [1))\beta &= ((0, 1, [1)) \cdot (1, 0, [1)))\beta = \\
&= (0, 1, [1))\beta \cdot (1, 0, [1))\beta = \\
&= (x, y, [0)) \cdot (y, x, [0)) = \\
&= (x, x, [0))
\end{aligned}
$$

and

$$
\begin{aligned}
(p + s, p + s, [0)) = (1, 1, [1))\beta &= \\
&= ((1, 0, [1)) \cdot (0, 1, [1)))\beta = \\
&= (1, 0, [1))\beta \cdot (0, 1, [1))\beta = \\
&= (y, x, [0)) \cdot (x, y, [0)) = \\
&= (y, y, [0)),
\end{aligned}
$$

and hence by the definition of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ we get that

$$(0, 1, [1))\beta = (p, p + s, [0)) \qquad \text{and} \qquad (1, 0, [1))\beta = (p + s, p, [0)).$$

Then for any $i, j \in \omega$ we have that

$$
\begin{aligned}
(i, j, [1))\beta &= ((i, 0, [1)) \cdot (0, j, [1)))\beta = \\
&= ((1, 0, [1))^i \cdot (0, 1, [1))^j)\beta = \\
&= ((1, 0, [1))\beta)^i \cdot ((0, 1, [1))\beta)^j = \\
&= (p+s, p, [0))^i \cdot (p, p+s, [0))^j = \\
&= (p+si, p, [0)) \cdot (p, p+sj, [0)) = \\
&= (p+si, p+sj, [0)).
\end{aligned}
$$

The definition of the natural partial order on $\boldsymbol{B}_\omega^{\mathscr{F}}$ (see [5, Proposition 2]) implies that for any positive integer t we have that

$$
(t+1, t+1, [1)) \preccurlyeq (t+1, t+1, [0)) \preccurlyeq (t, t, [1)).
$$

Since by Proposition 1.4.21 of [12] the endomorphism α preserves the natural partial order, we conclude that

$$
(t+1, t+1, [1))\beta \preccurlyeq (t+1, t+1, [0))\beta \preccurlyeq (t, t, [1))\beta,
$$

which implies that

$$
(p+s(t+1), p+s(t+1), [0)) \preccurlyeq (k(t+1), k(t+1), [0)) \preccurlyeq (p+st, p+st, [0)).
$$

Then by the definition of the natural partial order on $\boldsymbol{B}_\omega^{\mathscr{F}}$ (see [5, Proposition 3]) we obtain that

$$
(4) \qquad p+s(t+1) \geqslant k(t+1) \geqslant p+st
$$

for any positive integer t.

We claim that $s = k$. Indeed, if $s < k$ then the first inequality of (4) implies that

$$
p + (s-k)(t+1) \geqslant 0 \qquad \text{for all positive integers} \quad t.
$$

But the last inequality is incorrect because $s - k < 0$. Similar, if $s > k$ then the second inequality of (4) implies that

$$
(k-s)t + k - p \geqslant 0 \qquad \text{for all positive integers} \quad t.
$$

But the last inequality is incorrect because $k - s < 0$. Hence $\beta = \beta_{k,p}$. By Lemma 3, the map $\beta_{k,p}$ is an endomorphism of the monoid $\boldsymbol{B}_\omega^{\mathscr{F}}$. $\qquad \square$

In the following theorem we describe the elements of the semigroup $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ of all injective monoid endomorphisms of the monoid $\boldsymbol{B}_\omega^{\mathscr{F}}$ for the family $\mathscr{F} = \{[0), [1)\}$.

Theorem 1. *Let $\mathscr{F} = \{[0), [1)\}$ and ε be an injective monoid endomorphism of $\boldsymbol{B}_\omega^{\mathscr{F}}$. Then either there exist a positive integer k and $p \in \{0, \ldots, k-1\}$ such that $\varepsilon = \alpha_{k,p}$ or there exist a positive integer $k \geqslant 2$ and $p \in \{1, \ldots, k-1\}$ such that $\varepsilon = \beta_{k,p}$. Moreover. the following statements hold:*

(1) *$(i, j, [0))\alpha_{k,p} = (i, j, [0))\beta_{k,p}$ for all $i, j \in \omega$ and any positive integer $k \geqslant 2$ and $p \in \{1, \ldots, k-1\}$;*

(2) *if $k = 1$ then $\varepsilon = \alpha_{1,0}$ is an automorphism of the monoid $\boldsymbol{B}_\omega^{\mathscr{F}}$ which is the identity selfmap of $\boldsymbol{B}_\omega^{\mathscr{F}}$;*

(3) $\alpha_{k_1,p_1}\alpha_{k_2,p_2} = \alpha_{k_1k_2,p_2+k_2p_1}$ *for all positive integers* k_1, k_2, *any* $p_1 \in \{0, \ldots, k_1-1\}$, *and any* $p_2 \in \{0, \ldots, k_2-1\}$;

(4) $\alpha_{k_1,p_1}\beta_{k_2,p_2} = \beta_{k_1k_2,p_2+k_2p_1}$ *for all positive integers* k_1 *and* $k_2 \geqslant 2$, *any* $p_1 \in \{0, \ldots, k_1-1\}$, *and any* $p_2 \in \{1, \ldots, k_2-1\}$;

(5) $\beta_{k_1,p_1}\beta_{k_2,p_2} = \beta_{k_1k_2,k_2p_1}$ *for all positive integers* $k_1, k_2 \geqslant 2$, *any* $p_1 \in \{1, \ldots, k_1-1\}$, *and any* $p_2 \in \{1, \ldots, k_2-1\}$;

(6) $\beta_{k_1,p_1}\alpha_{k_2,p_2} = \beta_{k_1k_2,k_2p_1}$ *for all positive integers* $k_1 \geqslant 2$ *and* k_2, *any* $p_1 \in \{1, \ldots, k_1-1\}$, *and any* $p_2 \in \{0, \ldots, k_2-1\}$;

(7) *if* α_{k_2,p_2}, β_{k_1,p_1}, *and* β_{k_2,p_2} *are well defined elements of* $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ *then* $\beta_{k_1,p_1}\alpha_{k_2,p_2} = \beta_{k_1,p_1}\beta_{k_2,p_2}$;

(8) $\alpha_{1,0}$ *is the unique idempotent of* $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$, *and moreover* $\alpha_{1,0}$ *is the identity of* $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$;

(9) $S_\alpha = \langle \alpha_{k,p} \mid k \in \mathbb{N}, p \in \{0, \ldots, k-1\} \rangle$ *is a cancellative submonoid of* $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$;

(10) $S_\beta = \langle \beta_{k,p} \mid k = 2, 3, \ldots, p \in \{1, \ldots, k-1\} \rangle$ *is an ideal in* $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$.

Proof. Since by Proposition 3 of [4] the subsemigroup $\boldsymbol{B}_\omega^{\{[0)\}}$ is isomorphic to the bicyclic semigroup, the injectivity of ε implies that the image $(\boldsymbol{B}_\omega^{\{[0)\}})\varepsilon$ is isomorphic to the bicyclic semigroup. The condition $(0,0,[0))\varepsilon = (0,0,[0))$ and Proposition 4 of [5] imply that $(\boldsymbol{B}_\omega^{\{[0)\}})\varepsilon \subseteq \boldsymbol{B}_\omega^{\{[0)\}}$. Similar arguments imply that either $(\boldsymbol{B}_\omega^{\{[1)\}})\varepsilon \subseteq \boldsymbol{B}_\omega^{\{[1)\}}$ or $(\boldsymbol{B}_\omega^{\{[1)\}})\varepsilon \subseteq \boldsymbol{B}_\omega^{\{[0)\}}$. Then there exists a positive integer k such that $(1,1,[0))\varepsilon = (k,k,[0))$. Next we apply either Proposition 1 or Proposition 2. This completes the first statement of the theorem.

(1) follows from the definitions of endomorphisms $\alpha_{k,p}$ and $\beta_{k,p}$.

(2) If $k = 1$ then $(1,1,[0))\varepsilon = (1,1,[0))$ and $(0,0,[1))\varepsilon = (0,0,[1))$. By Theorem 3 of [6], ε is the identity selfmap of $\boldsymbol{B}_\omega^{\mathscr{F}}$.

(3) For any $i, j \in \omega$ we have that

$$(i,j,[1))\alpha_{k_1,p_1}\alpha_{k_2,p_2} = ((i,j,[1))\alpha_{k_1,p_1})\alpha_{k_2,p_2} =$$
$$= (p_1 + k_1i, p_1 + k_1j, [1))\alpha_{k_2,p_2} =$$
$$= (p_2 + k_2(p_1 + k_1i), p_2 + k_2(p_1 + k_1j), [1)) =$$
$$= ((p_2 + k_2p_1) + k_2k_1i, (p_2 + k_2p_1) + k_2k_1j, [1)),$$

and hence by (1) we get that $\alpha_{k_1,p_1}\alpha_{k_2,p_2} = \alpha_{k_1k_2,p_2+k_2p_1}$.

(4) For any $i, j \in \omega$ we have that

$$(i,j,[1))\alpha_{k_1,p_1}\beta_{k_2,p_2} = ((i,j,[1))\alpha_{k_1,p_1})\beta_{k_2,p_2} =$$
$$= (p_1 + k_1i, p_1 + k_1j, [1))\beta_{k_2,p_2} =$$
$$= (p_2 + k_2(p_1 + k_1i), p_2 + k_2(p_1 + k_1j), [0)) =$$
$$= ((p_2 + k_2p_1) + k_2k_1i, (p_2 + k_2p_1) + k_2k_1j, [0)),$$

and by (1) we obtain that $\alpha_{k_1,p_1}\alpha_{k_2,p_2} = \beta_{k_1k_2,p_2+k_2p_1}$.

44   Oleg GUTIK, Inna POZDNIAKOVA

ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2022. Випуск 94

(5) For any $i, j \in \omega$ we have that

$$
\begin{aligned}
(i, j, [1))\beta_{k_1,p_1}\beta_{k_2,p_2} &= ((i, j, [1))\beta_{k_1,p_1})\beta_{k_2,p_2} = \\
&= (p_1 + k_1 i, p_1 + k_1 j, [0))\beta_{k_2,p_2} = \\
&= (k_2(p_1 + k_1 i), k_2(p_1 + k_1 j), [0)) = \\
&= (k_2 p_1 + k_2 k_1 i, k_2 p_1 + k_2 k_1 j, [0)).
\end{aligned}
$$

Hence by (1) we get that $\beta_{k_1,p_1}\beta_{k_2,p_2} = \beta_{k_1 k_2, k_2 p_1}$.

(6) For any $i, j \in \omega$ we have that

$$
\begin{aligned}
(i, j, [1))\beta_{k_1,p_1}\beta_{k_2,p_2} &= ((i, j, [1))\beta_{k_1,p_1})\alpha_{k_2,p_2} = \\
&= (p_1 + k_1 i, p_1 + k_1 j, [0))\alpha_{k_2,p_2} = \\
&= (k_2(p_1 + k_1 i), k_2(p_1 + k_1 j), [0)) = \\
&= (k_2 p_1 + k_2 k_1 i, k_2 p_1 + k_2 k_1 j, [0)).
\end{aligned}
$$

Hence by (1) we get that $\beta_{k_1,p_1}\alpha_{k_2,p_2} = \beta_{k_1 k_2, k_2 p_1}$.

(7) follows from statements (5) and (6).

(8) The first part follows from the first statement and statements (3) and (5). Next we apply (2).

(9) Statement (3) implies that S_α is a subsemigroup of $\boldsymbol{End}^1_*(\boldsymbol{B}^{\mathscr{F}}_\omega)$. Since $\alpha_{1,0}$ is the identity of $\boldsymbol{End}^1_*(\boldsymbol{B}^{\mathscr{F}}_\omega)$ and $\alpha_{1,0} \in S_\alpha$, S_α is a submonoid of $\boldsymbol{End}^1_*(\boldsymbol{B}^{\mathscr{F}}_\omega)$.

Suppose that $\alpha_{k,p}\alpha_{k_1,p_1} = \alpha_{k,p}\alpha_{k_2,p_2}$ for some $\alpha_{k,p}, \alpha_{k_1,p_1}, \alpha_{k_2,p_2}$. Then by (3) we have that $\alpha_{kk_1,p_1+k_1 p} = \alpha_{kk_2,p_2+k_2 p}$ and the definition of the monoid $\boldsymbol{B}^{\mathscr{F}}_\omega$ implies that $kk_1 = kk_2$ and $p_1 + k_1 p = p_2 + k_2 p$. The first equality implies that $k_1 = k_2$ because $k \geqslant 1$. The equality $p_1 + k_1 p = p_2 + k_2 p$ implies that $p_1 - p_2 = p(k_2 - k_1)$, and since $k_1 = k_2$ we get that $p_1 = p_2$. Hence S_α is a left cancellative semigroup.

Suppose that $\alpha_{k_1,p_1}\alpha_{k,p} = \alpha_{k_2,p_2}\alpha_{k,p}$ for some $\alpha_{k,p}, \alpha_{k_1,p_1}, \alpha_{k_2,p_2} \in S_\alpha$. Then by (3) we have that $\alpha_{k_1 k, p+k p_1} = \alpha_{k_2 k, p+k p_2}$ and the definition of the monoid $\boldsymbol{B}^{\mathscr{F}}_\omega$ implies that $kk_1 = kk_2$ and $p + kp_1 = p + kp_2$. This implies that $k_1 = k_2$ and $p_1 = p_2$ because $k \geqslant 1$. Hence S_α is a right cancellative semigroup.

(10) follows from statements (4), (5), and (6). $\qquad\qquad \square$

3. Green's relations on the semigroup $\boldsymbol{End}^1_*(\boldsymbol{B}^{\mathscr{F}}_\omega)$

If S is a semigroup, then we shall denote the Green relations on S by $\mathscr{R}, \mathscr{L}, \mathscr{J}, \mathscr{D}$ and $\mathscr{H}$ (see [1, Section 2.1]):

$$
\begin{aligned}
&a\mathscr{R}b \text{ if and only if } aS^1 = bS^1; \\
&a\mathscr{L}b \text{ if and only if } S^1 a = S^1 b; \\
&a\mathscr{J}b \text{ if and only if } S^1 a S^1 = S^1 b S^1; \\
&\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}; \\
&\mathscr{H} = \mathscr{L} \cap \mathscr{R}.
\end{aligned}
$$

In this section we describe Green's relations on the semigroup $\boldsymbol{End}^1_*(\boldsymbol{B}^{\mathscr{F}}_\omega)$ for the family $\mathscr{F} = \{[0), [1)\}$.

ON THE SEMIGROUP OF INJECTIVE MONOID ENDOMORPHISMS OF $\boldsymbol{B}_\omega^{\mathscr{F}}$

ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2022. Випуск 94 **45**

Statements (3)–(6) of Theorem 1 imply the following two lemmas.

Lemma 4. *Let $\mathscr{F} = \{[0), [1)\}$. Then neither $\alpha_{k_1,p_1}\mathscr{R}\beta_{k_2,p_2}$ nor $\alpha_{k_1,p_1}\mathscr{L}\beta_{k_2,p_2}$ holds in the monoid $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ for any $\alpha_{k_1,p_1} \in S_\alpha$ and any $\beta_{k_2,p_2} \in S_\beta$.*

Lemma 5. *Let $\mathscr{F} = \{[0), [1)\}$. Then the relation $\alpha_{k_1,p_1}\mathscr{J}\beta_{k_2,p_2}$ in $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ does not holds for any $\alpha_{k_1,p_1} \in S_\alpha$ and any $\beta_{k_2,p_2} \in S_\beta$.*

Proposition 3. *Let $\mathscr{F} = \{[0), [1)\}$. Then the following statements hold:*

(1) $\alpha_{k_1,p_1}\mathscr{R}\alpha_{k_2,p_2}$ in $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ if and only if $k_1 = k_2$ and $p_1 = p_2$;
(2) $\alpha_{k_1,p_1}\mathscr{L}\alpha_{k_2,p_2}$ in $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ if and only if $k_1 = k_2$ and $p_1 = p_2$;
(3) $\beta_{k_1,p_1}\mathscr{R}\beta_{k_2,p_2}$ in $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ if and only if $k_1 = k_2$ and $p_1 = p_2$;
(4) $\beta_{k_1,p_1}\mathscr{L}\beta_{k_2,p_2}$ in $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ if and only if $k_1 = k_2$ and $p_1 = p_2$;
(5) $\alpha_{k_1,p_1}\mathscr{J}\alpha_{k_2,p_2}$ in $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ if and only if $k_1 = k_2$ and $p_1 = p_2$;
(6) $\beta_{k_1,p_1}\mathscr{J}\beta_{k_2,p_2}$ in $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ if and only if $k_1 = k_2$ and $p_1 = p_2$;

Proof. (1) Implication ($\Leftarrow$) is trivial.

($\Rightarrow$) Suppose that $\alpha_{k_1,p_1}\mathscr{R}\alpha_{k_2,p_2}$ in $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$. The definition of the Green relation $\mathscr{R}$ implies that there exist $\varepsilon_1, \varepsilon_2 \in \boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ such that

$$\alpha_{k_1,p_1} = \alpha_{k_2,p_2}\varepsilon_1 \qquad \text{and} \qquad \alpha_{k_2,p_2} = \alpha_{k_1,p_1}\varepsilon_2.$$

By statements (3) and (4) of Theorem 1 we have that $\varepsilon_1, \varepsilon_2 \in S_\alpha$, i.e., $\varepsilon_1 = \alpha_{k_1',p_1'}$ and $\varepsilon_2 = \alpha_{k_2',p_2'}$ for some positive integers k_1' and k_2', $p_1' \in \{0, \ldots, k_1' - 1\}$, and $p_2' \in \{0, \ldots, k_2' - 1\}$.

We claim that $\varepsilon_1 = \varepsilon_2 = \alpha_{1,0}$. Indeed, by Theorem 1(3) we get that

$$\alpha_{k_1,p_1} = \alpha_{k_2,p_2}\alpha_{k_1',p_1'} = \alpha_{k_2 k_1', p_1' + k_1' p_2}$$

and

$$\alpha_{k_2,p_2} = \alpha_{k_1,p_1}\alpha_{k_2',p_2'} = \alpha_{k_1 k_2', p_2' + k_2' p_1},$$

which implies the following equalities

$$\begin{aligned} k_1 &= k_2 k_1', \\ p_1 &= p_1' + k_1' p_2, \\ k_2 &= k_1 k_2', \\ p_2 &= p_2' + k_2' p_1. \end{aligned}$$

Since k_1, k_2, k_1', k_2' are positive integers, the equalities $k_1 = k_2 k_1'$ and $k_2 = k_1 k_2'$ imply that $k_1' = k_2' = 1$. Then we have that $p_1 = p_1' + p_2$ and $p_2 = p_2' + p_1$ which implies that $p_1' = p_2' = 0$. Then we have that $\varepsilon_1 = \varepsilon_2 = \alpha_{1,0}$, and hence $k_1 = k_2$ and $p_1 = p_2$.

The proof of statement (2) is similar to (1).

(3) Implication ($\Leftarrow$) is trivial.

($\Rightarrow$) Suppose that $\beta_{k_1,p_1}\mathscr{R}\beta_{k_2,p_2}$ in $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$. The definition of the Green relation $\mathscr{R}$ implies that there exist $\varepsilon_1, \varepsilon_2 \in \boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ such that

$$\beta_{k_1,p_1} = \beta_{k_2,p_2}\varepsilon_1 \qquad \text{and} \qquad \beta_{k_2,p_2} = \beta_{k_1,p_1}\varepsilon_2.$$

Then we have that

(a) either $\varepsilon_1 = \alpha_{k_1',p_1'}$ or $\varepsilon_1 = \beta_{k_1',p_1'}$; and

Oleg GUTIK, Inna POZDNIAKOVA

46   *ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2022. Випуск 94*

(b) either $\varepsilon_2 = \alpha_{k_2',p_2'}$ or $\varepsilon_2 = \beta_{k_2',p_2'}$.

By Theorem 1(7) in any above case we get that

$$\beta_{k_1,p_1} = \beta_{k_2,p_2}\varepsilon_1 = \beta_{k_2 k_1',k_1'p_2};$$
$$\beta_{k_2,p_2} = \beta_{k_1,p_1}\varepsilon_2 = \beta_{k_1 k_2',k_2'p_1},$$

and hence

$$k_1 = k_2 k_1';$$
$$p_1 = k_1' p_2;$$
$$k_2 = k_1 k_2';$$
$$p_2 = k_2' p_1.$$

This implies that $p_1 = k_1' k_2' p_1$ and $p_2 = k_2' k_1' p_2$, and hence $k_1' k_2' = 1$. Since k_1' and k_2' are positive integers, $k_1' = k_2' = 1$, and hence $k_1 = k_2$. This implies that $\varepsilon_1 = \alpha_{k_1',p_1'}$ and $\varepsilon_2 = \alpha_{k_2',p_2'}$. Also the equalities $p_1 = k_1' p_2$ and $p_2 = k_2' p_1$ imply that $p_1 = p_2$.

(4) Implication ($\Leftarrow$) is trivial.

($\Rightarrow$) Suppose that $\beta_{k_1,p_1}\mathscr{L}\beta_{k_2,p_2}$ in $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$. The definition of the Green relation $\mathscr{L}$ implies that there exist $\varepsilon_1, \varepsilon_2 \in \boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ such that

$$\beta_{k_1,p_1} = \varepsilon_1\beta_{k_2,p_2} \qquad \text{and} \qquad \beta_{k_2,p_2} = \varepsilon_2\beta_{k_1,p_1}.$$

We consider the possible cases:

(a) $\varepsilon_1 = \alpha_{k_1',p_1'}$ and $\varepsilon_2 = \alpha_{k_2',p_2'}$;
(b) $\varepsilon_1 = \alpha_{k_1',p_1'}$ and $\varepsilon_2 = \beta_{k_2',p_2'}$;
(c) $\varepsilon_1 = \beta_{k_1',p_1'}$ and $\varepsilon_2 = \alpha_{k_2',p_2'}$;
(d) $\varepsilon_1 = \beta_{k_1',p_1'}$ and $\varepsilon_2 = \beta_{k_2',p_2'}$.

In case (a) by Theorem 1(4) we have that

$$\beta_{k_1,p_1} = \alpha_{k_1',p_1'}\beta_{k_2,p_2} = \beta_{k_1' k_2,p_2+k_2 p_1'};$$
$$\beta_{k_2,p_2} = \alpha_{k_2',p_2'}\beta_{k_1,p_1} = \beta_{k_2' k_1,p_1+k_1 p_2'},$$

which implies the following equalities

$$k_1 = k_1' k_2;$$
$$p_1 = p_2 + k_2 p_1';$$
$$k_2 = k_2' k_1;$$
$$p_2 = p_1 + k_1 p_2'.$$

Since k_1, k_2, k_1', k_2' are positive integers, the equalities $k_1 = k_1' k_2$ and $k_2 = k_2' k_1$ imply that $k_1' k_2' = 1$, and hence $k_1' = k_2' = 1$. This implies that $k_1 = k_2$. Next, by the equalities $p_1 = p_2 + k_2 p_1'$, $p_2 = p_1 + k_1 p_2'$, and $k_1 = k_2$ we get that $p_1' = -p_2'$. Since p_1' and p_2' are nonnegative integers, we obtain that $p_1' = p_2' = 0$, which implies that

$$\varepsilon_1 = \alpha_{k_1',p_1'} = \varepsilon_2 = \alpha_{k_2',p_2'} = \alpha_{1,0}$$

is the unit element of the monoid $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$. Then $\beta_{k_1,p_1} = \beta_{k_2,p_2}$, and the definition of elements of S_β implies that $k_1 = k_2$ and $p_1 = p_2$.

Suppose case (b) holds. By statements (4) and (5) of Theorem 1 we get that

$$\beta_{k_1,p_1} = \alpha_{k'_1,p'_1}\beta_{k_2,p_2} = \beta_{k'_1 k_2, p_2 + k_2 p'_1};$$
$$\beta_{k_2,p_2} = \beta_{k'_2,p'_2}\beta_{k_1,p_1} = \beta_{k'_2 k_1, k_1 p'_2},$$

which implies the following equalities

$$k_1 = k'_1 k_2;$$
$$p_1 = p_2 + k_2 p'_1;$$
$$k_2 = k'_2 k_1;$$
$$p_2 = k_1 p'_2.$$

Since k_1, k_2, k'_1, k'_2 are positive integers, the equalities $k_1 = k'_1 k_2$ and $k_2 = k'_2 k_1$ imply that $k'_1 k'_2 = 1$, and hence $k'_1 = k'_2 = 1$. But $k'_2 \geqslant 2$, a contradiction. The obtained contradiction implies that case (b) is impossible.

Suppose case (c) holds. By statements (4) and (5) of Theorem 1(4) we get that

$$\beta_{k_1,p_1} = \beta_{k'_1,p'_1}\beta_{k_2,p_2} = \beta_{k'_1 k_2, k_2 p'_1};$$
$$\beta_{k_2,p_2} = \alpha_{k'_2,p'_2}\beta_{k_1,p_1} = \beta_{k'_2 k_1, p_1 + k_1 p'_2},$$

which implies the following equalities

$$k_1 = k'_1 k_2;$$
$$p_1 = k_2 p'_1;$$
$$k_2 = k'_2 k_1;$$
$$p_2 = p_1 + k_1 p'_2.$$

Since k_1, k_2, k'_1, k'_2 are positive integers, the equalities $k_1 = k'_1 k_2$ and $k_2 = k'_2 k_1$ imply that $k'_1 k'_2 = 1$, and hence $k'_1 = k'_2 = 1$. But $k'_1 \geqslant 2$, a contradiction. The obtained contradiction implies that case (c) does not hold.

Suppose case (d) holds. By Theorem 1(5) we have that

$$\beta_{k_1,p_1} = \beta_{k'_1,p'_1}\beta_{k_2,p_2} = \beta_{k'_1 k_2, k_2 p'_1};$$
$$\beta_{k_2,p_2} = \beta_{k'_2,p'_2}\beta_{k_1,p_1} = \beta_{k'_2 k_1, k_1 p'_2},$$

which implies the following equalities

$$k_1 = k'_1 k_2;$$
$$p_1 = k_2 p'_1;$$
$$k_2 = k'_2 k_1;$$
$$p_2 = k_1 p'_2.$$

Since k_1, k_2, k'_1, k'_2 are positive integers, the equalities $k_1 = k'_1 k_2$ and $k_2 = k'_2 k_1$ imply that $k'_1 k'_2 = 1$, and hence $k'_1 = k'_2 = 1$. But $k'_1 \geqslant 2$ and $k'_2 \geqslant 2$, a contradiction. The obtained contradiction implies that case (d) does not hold.

(5) Implication ($\Leftarrow$) is trivial.

Oleg GUTIK, Inna POZDNIAKOVA

48   *ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2022. Випуск 94*

$(\Rightarrow)$ Suppose that $\alpha_{k_1,p_1} \,\mathscr{J}\, \alpha_{k_2,p_2}$ in $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$. The definition of the Green relation $\mathscr{J}$ implies that there exist $\varepsilon_1', \varepsilon_1'', \varepsilon_2', \varepsilon_2'' \in \boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ such that

$$\alpha_{k_1,p_1} = \varepsilon_1' \alpha_{k_2,p_2} \varepsilon_1'' \qquad \text{and} \qquad \alpha_{k_2,p_2} = \varepsilon_2' \alpha_{k_1,p_1} \varepsilon_2''.$$

By statements (3) and (4) of Theorem 1 we have that $\varepsilon_1', \varepsilon_1'', \varepsilon_2', \varepsilon_2'' \in S_\alpha$, i.e., $\varepsilon_1' = \alpha_{k_1',p_1'}$, $\varepsilon_1'' = \alpha_{k_1'',p_1''}$, $\varepsilon_2' = \alpha_{k_2',p_2'}$, and $\varepsilon_2'' = \alpha_{k_2'',p_2''}$ for some positive integers k_1', k_1'', k_2' and k_2'', $p_1' \in \{0,\ldots,k_1'-1\}$, $p_1'' \in \{0,\ldots,k_1''-1\}$, $p_2' \in \{0,\ldots,k_2'-1\}$, and $p_2'' \in \{0,\ldots,k_2''-1\}$.

By Theorem 1(3) we have that

$$\alpha_{k_1,p_1} = \alpha_{k_1',p_1'} \alpha_{k_2,p_2} \alpha_{k_1'',p_1''} =$$
$$= \alpha_{k_1'k_2,p_2+k_2p_1'} \alpha_{k_1'',p_1''} =$$
$$= \alpha_{k_1'k_2k_1'',p_1''+k_1''p_2+k_1''k_2p_1'};$$
$$\alpha_{k_2,p_2} = \alpha_{k_2',p_2'} \alpha_{k_1,p_1} \alpha_{k_2'',p_2''} =$$
$$= \alpha_{k_2'k_1,p_1+k_1p_2'} \alpha_{k_2'',p_2''} =$$
$$= \alpha_{k_2'k_1k_2'',p_2''+k_2''p_1+k_2''k_1p_2'},$$

which implies the following equalities

$$k_1 = k_1'k_2k_1'';$$
$$p_1 = p_1'' + k_1''p_2 + k_1''k_2p_1';$$
$$k_2 = k_2'k_1k_2'';$$
$$p_2 = p_2'' + k_2''p_1 + k_2''k_1p_2'.$$

Since $k_1, k_2, k_1', k_2', k_1'', k_2''$ are positive integers, the equalities $k_1 = k_1'k_2k_1''$ and $k_2 = k_2'k_1k_2''$ imply that $k_1'k_2'k_1''k_2'' = 1$, and hence we have

$$k_1' = k_2' = k_1'' = k_2'' = 1.$$

This implies that $k_1 = k_2$. Hence the equalities $k_1' = k_2' = k_1'' = k_2'' = 1$, $k_1 = k_2$ and the equalities

$$p_1 = p_1'' + k_1''p_2 + k_1''k_2p_1';$$
$$p_2 = p_2'' + k_2''p_1 + k_2''k_1p_2'$$

imply that

$$p_1 = p_1'' + p_2 + k_1p_1';$$
$$p_2 = p_2'' + p_1 + k_1p_2'.$$

Hence we get that

$$p_1'' + p_2'' + k_1p_2' + k_1p_1' = 0.$$

Since k_1 is a positive integer, by the above equality we have that

$$p_1'' = p_2'' = p_2' = p_1' = 0.$$

This implies that

$$\varepsilon_1' = \varepsilon_1'' = \varepsilon_2' = \varepsilon_2'' = \alpha_{1,0},$$

and hence we have that $\alpha_{k_1,p_1} = \alpha_{k_2,p_2}$, which implies the requested equalities $k_1 = k_2$ and $p_1 = p_2$.

(6) Implication $(\Leftarrow)$ is trivial.

($\Rightarrow$) Suppose that $\beta_{k_1,p_1} \mathscr{J} \beta_{k_2,p_2}$ in $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$. The definition of the Green relation $\mathscr{J}$ implies that there exist $\varepsilon_1', \varepsilon_1'', \varepsilon_2', \varepsilon_2'' \in \boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ such that

$$\beta_{k_1,p_1} = \varepsilon_1' \beta_{k_2,p_2} \varepsilon_1'' \qquad \text{and} \qquad \beta_{k_2,p_2} = \varepsilon_2' \beta_{k_1,p_1} \varepsilon_2''.$$

We consider the possible cases:

(a) $\varepsilon_1' = \alpha_{k_1',p_1'}$, $\varepsilon_1'' = \alpha_{k_1'',p_1''}$, $\varepsilon_2' = \alpha_{k_2',p_2'}$, and $\varepsilon_2'' = \alpha_{k_2'',p_2''}$;
(b) $\varepsilon_1' = \alpha_{k_1',p_1'}$, $\varepsilon_1'' = \alpha_{k_1'',p_1''}$, $\varepsilon_2' = \alpha_{k_2',p_2'}$, and $\varepsilon_2'' = \beta_{k_2'',p_2''}$;
(c) $\varepsilon_1' = \alpha_{k_1',p_1'}$, $\varepsilon_1'' = \alpha_{k_1'',p_1''}$, $\varepsilon_2' = \beta_{k_2',p_2'}$, and $\varepsilon_2'' = \alpha_{k_2'',p_2''}$;
(d) $\varepsilon_1' = \alpha_{k_1',p_1'}$, $\varepsilon_1'' = \alpha_{k_1'',p_1''}$, $\varepsilon_2' = \beta_{k_2',p_2'}$, and $\varepsilon_2'' = \beta_{k_2'',p_2''}$;
(e) $\varepsilon_1' = \alpha_{k_1',p_1'}$, $\varepsilon_1'' = \beta_{k_1'',p_1''}$, $\varepsilon_2' = \alpha_{k_2',p_2'}$, and $\varepsilon_2'' = \alpha_{k_2'',p_2''}$;
(f) $\varepsilon_1' = \alpha_{k_1',p_1'}$, $\varepsilon_1'' = \beta_{k_1'',p_1''}$, $\varepsilon_2' = \alpha_{k_2',p_2'}$, and $\varepsilon_2'' = \beta_{k_2'',p_2''}$;
(g) $\varepsilon_1' = \alpha_{k_1',p_1'}$, $\varepsilon_1'' = \beta_{k_1'',p_1''}$, $\varepsilon_2' = \beta_{k_2',p_2'}$, and $\varepsilon_2'' = \alpha_{k_2'',p_2''}$;
(h) $\varepsilon_1' = \alpha_{k_1',p_1'}$, $\varepsilon_1'' = \beta_{k_1'',p_1''}$, $\varepsilon_2' = \beta_{k_2',p_2'}$, and $\varepsilon_2'' = \beta_{k_2'',p_2''}$;
(i) $\varepsilon_1' = \beta_{k_1',p_1'}$, $\varepsilon_1'' = \alpha_{k_1'',p_1''}$, $\varepsilon_2' = \alpha_{k_2',p_2'}$, and $\varepsilon_2'' = \alpha_{k_2'',p_2''}$;
(j) $\varepsilon_1' = \beta_{k_1',p_1'}$, $\varepsilon_1'' = \alpha_{k_1'',p_1''}$, $\varepsilon_2' = \alpha_{k_2',p_2'}$, and $\varepsilon_2'' = \beta_{k_2'',p_2''}$;
(k) $\varepsilon_1' = \beta_{k_1',p_1'}$, $\varepsilon_1'' = \alpha_{k_1'',p_1''}$, $\varepsilon_2' = \beta_{k_2',p_2'}$, and $\varepsilon_2'' = \alpha_{k_2'',p_2''}$;
(l) $\varepsilon_1' = \beta_{k_1',p_1'}$, $\varepsilon_1'' = \alpha_{k_1'',p_1''}$, $\varepsilon_2' = \beta_{k_2',p_2'}$, and $\varepsilon_2'' = \beta_{k_2'',p_2''}$;
(m) $\varepsilon_1' = \beta_{k_1',p_1'}$, $\varepsilon_1'' = \beta_{k_1'',p_1''}$, $\varepsilon_2' = \alpha_{k_2',p_2'}$, and $\varepsilon_2'' = \alpha_{k_2'',p_2''}$;
(n) $\varepsilon_1' = \beta_{k_1',p_1'}$, $\varepsilon_1'' = \beta_{k_1'',p_1''}$, $\varepsilon_2' = \alpha_{k_2',p_2'}$, and $\varepsilon_2'' = \beta_{k_2'',p_2''}$;
(o) $\varepsilon_1' = \beta_{k_1',p_1'}$, $\varepsilon_1'' = \beta_{k_1'',p_1''}$, $\varepsilon_2' = \beta_{k_2',p_2'}$, and $\varepsilon_2'' = \alpha_{k_2'',p_2''}$.

In case (a) by statements (6) and (4) of Theorem 1 we have that

$$\beta_{k_1,p_1} = \alpha_{k_1',p_1'} \beta_{k_2,p_2} \alpha_{k_1'',p_1''} =$$
$$= \alpha_{k_1',p_1'} \beta_{k_2 k_1'',p_2 k_1''} =$$
$$= \beta_{k_1' k_2 k_1'',p_2 k_1''+k_2 k_1'' p_1'};$$
$$\beta_{k_2,p_2} = \alpha_{k_2',p_1'} \beta_{k_1,p_1} \alpha_{k_2'',p_1''} =$$
$$= \alpha_{k_2',p_2'} \beta_{k_1 k_2'',p_1 k_2''} =$$
$$= \beta_{k_2' k_1 k_2'',p_1 k_2''+k_1 k_2'' p_2'},$$

which implies the following equalities

$$k_1 = k_1' k_2 k_1'';$$
$$p_1 = p_2 k_1'' + k_2 k_1'' p_1';$$
$$k_2 = k_2' k_1 k_2'';$$
$$p_2 = p_1 k_2'' + k_1 k_2'' p_2'.$$

Since $k_1, k_2, k_1', k_2', k_1'', k_2''$ are positive integers, the equalities $k_1 = k_1' k_2 k_1''$ and $k_2 = k_2' k_1 k_2''$ imply that $k_1' k_2' k_1'' k_2'' = 1$, and hence we have

$$k_1' = k_2' = k_1'' = k_2'' = 1.$$

This implies that $k_1 = k_2$. Hence the equalities $k_1' = k_2' = k_1'' = k_2'' = 1$, $k_1 = k_2$ and the equalities

$$p_1 = p_2 k_1'' + k_2 k_1'' p_1';$$
$$p_2 = p_1 k_2'' + k_1 k_2'' p_2'$$

Oleg GUTIK, Inna POZDNIAKOVA

50   *ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2022. Випуск 94*

imply that

$$p_1 = p_2 + k_1 p_1';$$
$$p_2 = p_1 + k_1 p_2'.$$

Then we have that

$$p_1 + p_2 = p_1 + p_2 + k_1 p_1' + k_1 p_2',$$

and hence $k_1(p_1' + p_2') = 0$. Since k_1 is a positive integer, we get that $p_1' = -p_2'$. Next, the inequalities $p_1' \geqslant 0$ and $p_2' \geqslant 0$ imply that $p_1' = p_2' = 0$, and by the equality $p_1 = p_2 + k_1 p_1'$ we have that $p_1 = p_2$. This completes the proof in case (a).

In case (b) by statements (6) and (5) of Theorem 1 we have that

$$\beta_{k_1,p_1} = \alpha_{k_1',p_1'} \beta_{k_2,p_2} \alpha_{k_1'',p_1''} =$$
$$= \alpha_{k_1',p_1'} \beta_{k_2 k_1'',p_2 k_1''} =$$
$$= \beta_{k_1' k_2 k_1'', p_2 k_1'' + k_2 k_1'' p_1'};$$
$$\beta_{k_2,p_2} = \alpha_{k_2',p_1'} \beta_{k_1,p_1} \beta_{k_2'',p_2''} =$$
$$= \alpha_{k_2',p_2'} \beta_{k_1 k_2'', p_1 k_2''} =$$
$$= \beta_{k_2' k_1 k_2'', p_1 k_2'' + k_1 k_2'' p_2'},$$

which implies the following equalities

$$k_1 = k_1' k_2 k_1'';$$
$$p_1 = p_2 k_1'' + k_2 k_1'' p_1';$$
$$k_2 = k_2' k_1 k_2'';$$
$$p_2 = p_1 k_2'' + k_1 k_2'' p_2'.$$

By similar arguments as in case (a) we obtain that $k_1 = k_2$ and $p_1 = p_2$.

In case (c) by statements (6), (4) and (5) of Theorem 1 we have that

$$\beta_{k_1,p_1} = \alpha_{k_1',p_1'} \beta_{k_2,p_2} \alpha_{k_1'',p_1''} =$$
$$= \alpha_{k_1',p_1'} \beta_{k_2 k_1'', p_2 k_1''} =$$
$$= \beta_{k_1' k_2 k_1'', p_2 k_1'' + k_2 k_1'' p_1'};$$
$$\beta_{k_2,p_2} = \beta_{k_2',p_1'} \beta_{k_1,p_1} \alpha_{k_2'',p_2''} =$$
$$= \beta_{k_2',p_2'} \beta_{k_1 k_2'', p_1 k_2''} =$$
$$= \beta_{k_2' k_1 k_2'', k_1 k_2'' p_2'},$$

which implies the following equalities

$$k_1 = k_1' k_2 k_1'';$$
$$p_1 = p_2 k_1'' + k_2 k_1'' p_1';$$
$$k_2 = k_2' k_1 k_2'';$$
$$p_2 = k_1 k_2'' p_2'.$$

Then we have that $k_1 = k_1' k_2' k_1 k_2'' k_1''$. The conditions that $k_1, k_2, k_1', k_2', k_1'', k_2''$ are positive integers and $k_2' \geqslant 2$ contradict the equality $k_1 = k_1' k_2' k_1 k_2'' k_1''$, and hence case (c) does not hold.

In case (d) by statements (6), (4) and (5) of Theorem 1 we have that

$$
\begin{aligned}
\beta_{k_1,p_1} &= \alpha_{k_1',p_1'}\beta_{k_2,p_2}\alpha_{k_1'',p_1''} = \\
&= \alpha_{k_1',p_1'}\beta_{k_2 k_1'',p_2 k_1''} = \\
&= \beta_{k_1' k_2 k_1'',p_2 k_1''+k_2 k_1'' p_1'}; \\
\beta_{k_2,p_2} &= \beta_{k_2',p_1'}\beta_{k_1,p_1}\beta_{k_2'',p_1''} = \\
&= \beta_{k_2',p_2'}\beta_{k_1 k_2'',p_1 k_2''} = \\
&= \beta_{k_2' k_1 k_2'',k_1 k_2'' p_2'},
\end{aligned}
$$

which implies the following equalities

$$
\begin{aligned}
k_1 &= k_1' k_2 k_1''; \\
p_1 &= p_2 k_1'' + k_2 k_1'' p_1'; \\
k_2 &= k_2' k_1 k_2''; \\
p_2 &= k_1 k_2'' p_2'.
\end{aligned}
$$

Next, similar arguments as in case (c) show that case (d) does not hold.

In case (e) the proof is similar to case (b).

In case (f) by statements (5) and (4) of Theorem 1 we have that

$$
\begin{aligned}
\beta_{k_1,p_1} &= \alpha_{k_1',p_1'}\beta_{k_2,p_2}\beta_{k_1'',p_1''} = \\
&= \alpha_{k_1',p_1'}\beta_{k_2 k_1'',p_2 k_1''} = \\
&= \beta_{k_1' k_2 k_1'',p_2 k_1''+k_2 k_1'' p_1'}; \\
\beta_{k_2,p_2} &= \alpha_{k_2',p_1'}\beta_{k_1,p_1}\beta_{k_2'',p_1''} = \\
&= \alpha_{k_2',p_2'}\beta_{k_1 k_2'',p_1 k_2''} = \\
&= \beta_{k_2' k_1 k_2'',p_1 k_2''+k_1 k_2'' p_2'},
\end{aligned}
$$

which implies the following equalities

$$
\begin{aligned}
k_1 &= k_1' k_2 k_1''; \\
p_1 &= p_2 k_1'' + k_2 k_1'' p_1'; \\
k_2 &= k_2' k_1 k_2''; \\
p_2 &= p_1 k_2'' + k_1 k_2'' p_2'.
\end{aligned}
$$

By similar arguments as in case (a) we obtain that $k_1 = k_2$ and $p_1 = p_2$.

In case (g) by statements (5), (4), and (6) of Theorem 1 we have that

$$
\begin{aligned}
\beta_{k_1,p_1} &= \alpha_{k_1',p_1'}\beta_{k_2,p_2}\beta_{k_1'',p_1''} = \\
&= \alpha_{k_1',p_1'}\beta_{k_2 k_1'',p_2 k_1''} = \\
&= \beta_{k_1' k_2 k_1'',p_2 k_1''+k_2 k_1'' p_1'}; \\
\beta_{k_2,p_2} &= \beta_{k_2',p_1'}\beta_{k_1,p_1}\alpha_{k_2'',p_1''} = \\
&= \beta_{k_2',p_2'}\beta_{k_1 k_2'',p_1 k_2''} = \\
&= \beta_{k_2' k_1 k_2'',p_2' p_1 k_2''},
\end{aligned}
$$

52   Oleg GUTIK, Inna POZDNIAKOVA

ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2022. Випуск 94

which implies the following equalities

$$k_1 = k_1' k_2 k_1'';$$
$$p_1 = p_2 k_1'' + k_2 k_1'' p_1';$$
$$k_2 = k_2' k_1 k_2'';$$
$$p_2 = p_2' p_1 k_2''.$$

The conditions that $k_1, k_2, k_1', k_2', k_1'', k_2''$ are positive integers, $k_2' \geqslant 2$, and $k_1'' \geqslant 2$ contradict the equalities

$$k_1 = k_1' k_2 k_1'' = k_1' k_2' k_1 k_2'' k_1''.$$

Hence case (g) does not hold.

In case (h) by statements (5) and (4) of Theorem 1 we have that

$$\beta_{k_1,p_1} = \alpha_{k_1',p_1'} \beta_{k_2,p_2} \beta_{k_1'',p_1''} =$$
$$= \alpha_{k_1',p_1'} \beta_{k_2 k_1'',p_2 k_1''} =$$
$$= \beta_{k_1' k_2 k_1'', p_2 k_1'' + k_2 k_1'' p_1'};$$
$$\beta_{k_2,p_2} = \beta_{k_2',p_1'} \beta_{k_1,p_1} \beta_{k_2'',p_1''} =$$
$$= \beta_{k_2',p_2'} \beta_{k_1 k_2'',p_1 k_2''} =$$
$$= \beta_{k_2' k_1 k_2'', p_2' p_1 k_2''},$$

which implies the following equalities

$$k_1 = k_1' k_2 k_1'';$$
$$p_1 = p_2 k_1'' + k_2 k_1'' p_1';$$
$$k_2 = k_2' k_1 k_2'';$$
$$p_2 = p_2' p_1 k_2''.$$

Similar arguments as in case (g) show that case (h) does not hold.

The proofs in cases (i) and (j) are similar to cases (c) and (g), respectively.

In case (k) by statements (5), (4), and (6) of Theorem 1 we have that

$$\beta_{k_1,p_1} = \beta_{k_1',p_1'} \beta_{k_2,p_2} \alpha_{k_1'',p_1''} =$$
$$= \beta_{k_1',p_1'} \beta_{k_2 k_1'', p_2 k_1''} =$$
$$= \beta_{k_1' k_2 k_1'', p_1' p_2 k_1''};$$
$$\beta_{k_2,p_2} = \beta_{k_2',p_1'} \beta_{k_1,p_1} \alpha_{k_2'',p_1''} =$$
$$= \beta_{k_2',p_2'} \beta_{k_1 k_2'', p_1 k_2''} =$$
$$= \beta_{k_2' k_1 k_2'', p_2' p_1 k_2''},$$

which implies the following equalities

$$k_1 = k_1' k_2 k_1'';$$
$$p_1 = p_1' p_2 k_1'';$$
$$k_2 = k_2' k_1 k_2'';$$
$$p_2 = p_2' p_1 k_2''.$$

The conditions that $k_1, k_2, k_1', k_2', k_1'', k_2''$ are positive integers, $k_1' \geqslant 2$, and $k_1' \geqslant 2$ contradict the equalities

$$k_1 = k_1' k_2 k_1'' = k_1' k_2' k_1 k_1'' k_1''.$$

Hence case (k) does not hold.

In case (l) by statements (5) and (6) of Theorem 1 we have that

$$
\begin{aligned}
\beta_{k_1,p_1} &= \beta_{k_1',p_1'} \beta_{k_2,p_2} \alpha_{k_1'',p_1''} = \\
&= \beta_{k_1',p_1'} \beta_{k_2 k_1'',p_2 k_1''} = \\
&= \beta_{k_1' k_2 k_1'',p_1' p_2 k_1''}; \\
\beta_{k_2,p_2} &= \beta_{k_2',p_1'} \beta_{k_1,p_1} \beta_{k_1'',p_1''} = \\
&= \beta_{k_2',p_2'} \beta_{k_1 k_2'',p_1 k_2''} = \\
&= \beta_{k_2' k_1 k_2'',p_2' p_1 k_2''},
\end{aligned}
$$

which implies the following equalities

$$
\begin{aligned}
k_1 &= k_1' k_2 k_1''; \\
p_1 &= p_1' p_2 k_1''; \\
k_2 &= k_2' k_1 k_2''; \\
p_2 &= p_2' p_1 k_2''.
\end{aligned}
$$

By similar arguments as in case (k) we get that case (l) does not hold.

The proofs in cases (m) and (n) are similar to cases (d) and (h), respectively.

In case (o) by Theorem 1(5) we have that

$$
\begin{aligned}
\beta_{k_1,p_1} &= \beta_{k_1',p_1'} \beta_{k_2,p_2} \beta_{k_1'',p_1''} = \\
&= \beta_{k_1',p_1'} \beta_{k_2 k_1'',p_2 k_1''} = \\
&= \beta_{k_1' k_2 k_1'',p_1' p_2 k_1''}; \\
\beta_{k_2,p_2} &= \beta_{k_2',p_1'} \beta_{k_1,p_1} \beta_{k_2'',p_1''} = \\
&= \beta_{k_2',p_2'} \beta_{k_1 k_2'',p_1 k_2''} = \\
&= \beta_{k_2' k_1 k_2'',p_2' p_1 k_2''},
\end{aligned}
$$

which implies the following equalities

$$
\begin{aligned}
k_1 &= k_1' k_2 k_1''; \\
p_1 &= p_1' p_2 k_1''; \\
k_2 &= k_2' k_1 k_2''; \\
p_2 &= p_2' p_1 k_2''.
\end{aligned}
$$

By similar arguments as in case (k) we get that case (o) does not hold. $\qquad\square$

Proposition 3 implies the following theorem.

Theorem 2. *If $\mathscr{F} = \{[0), [1)\}$ then Green's relations $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$, $\mathscr{D}$, and $\mathscr{J}$ on the monoid $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ coincide with the relation of equality.*

Oleg GUTIK, Inna POZDNIAKOVA

54 *ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2022. Випуск 94*

REFERENCES

1. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol. I., Amer. Math. Soc. Surveys 7, Providence, R.I., 1961.
2. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol. II., Amer. Math. Soc. Surveys 7, Providence, R.I., 1967.
3. O. Gutik and O. Lysetska, *On the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ which is generated by the family $\mathscr{F}$ of atomic subsets of ω*, Visn. L'viv. Univ., Ser. Mekh.-Mat. **92** (2021) 34–50. DOI: 10.30970/vmm.2021.92.034-050
4. O. Gutik and M. Mykhalenych, *On some generalization of the bicyclic monoid*, Visnyk Lviv. Univ. Ser. Mech.-Mat. **90** (2020), 5–19 (in Ukrainian). DOI: 10.30970/vmm.2020.90.005-019
5. O. Gutik and M. Mykhalenych, *On group congruences on the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ and its homomorphic retracts in the case when the family $\mathscr{F}$ consists of inductive non-empty subsets of ω*, Visnyk Lviv. Univ. Ser. Mech.-Mat. **91** (2021), 5–27 (in Ukrainian). DOI: 10.30970/vmm.2021.91.005-027
6. O. Gutik and M. Mykhalenych, *On automorphisms of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ in the case when the family $\mathscr{F}$ consists of nonempty inductive subsets of ω*, Visnyk Lviv. Univ. Ser. Mech.-Mat. **93** (2022), 54–65. (in Ukrainian). DOI: 10.30970/vmm.2022.93.054-065
7. O. Gutik and O. Popadiuk, *On the semigroup of injective endomorphisms of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}_n}$ which is generated by the family $\mathscr{F}_n$ of finite bounded intervals of ω*, Mat. Metody Fiz.-Mekh. Polya **65** (2022), no. 1–2, 42–57 (arXiv:2209.08377).
8. O. Gutik and O. Popadiuk, *On the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}_n}$ which is generated by the family $\mathscr{F}_n$ of finite bounded intervals of ω*, Carpatian Math. Publ. **15** (2023), no. 2, 331–355. DOI: 10.15330/cmp.15.2.331-355
9. O. V. Gutik and I. V. Pozdniakova, *On the semigroup generating by extended bicyclic semigroup and an ω-closed family*, Mat. Metody Fiz.-Mekh. Polya **64** (2021), no. 1, 21–34 (in Ukrainian).
10. O. Gutik and I. Pozdniakova, *On the group of automorphisms of the semigroup $\boldsymbol{B}_{\mathbb{Z}}^{\mathscr{F}}$ with the family $\mathscr{F}$ of inductive nonempty subsets of ω*, Algebra Discrete Math. **35** (2023), no. 1, 42–61. DOI: 10.12958/adm2010
11. O. Gutik, O. Prokhorenkova, and D. Sekh, *On endomorphisms of the bicyclic semigroup and the extended bicyclic semigroup*, Visn. L'viv. Univ., Ser. Mekh.-Mat. **92** (2021) 5–16 (in Ukrainian). DOI: 10.30970/vmm.2022.93.042-053
12. M. Lawson, *Inverse semigroups. The theory of partial symmetries*, World Scientific, Singapore, 1998.
13. O. Lysetska, *On feebly compact topologies on the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}_1}$*, Visnyk Lviv. Univ. Ser. Mech.-Mat. **90** (2020), 48–56. DOI: 10.30970/vmm.2020.90.048-056
14. M. Petrich, *Inverse semigroups*, John Wiley & Sons, New York, 1984.
15. O. Popadiuk, *On endomorphisms of the inverse semigroup of convex order isomorphisms of a bounded rank which are generated by Rees congruences*, Visnyk Lviv. Univ. Ser. Mech.-Mat. **93** (2022), 34–41. DOI: 10.30970/vmm.2022.93.034-041
16. V. V. Wagner, *Generalized groups*, Dokl. Akad. Nauk SSSR **84** (1952), 1119–1122 (in Russian).

ON THE SEMIGROUP OF INJECTIVE MONOID ENDOMORPHISMS OF $B_\omega^{\mathscr{F}}$

ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2022. Випуск 94   **55**

ПРО НАПІВГРУПУ ІН'ЄКТИВНИХ ЕНДОМОРФІЗМІВ МОНОЇДА $B_\omega^{\mathscr{F}}$ З ДВОЕЛЕМЕНТНОЮ СІМ'ЄЮ $\mathscr{F}$ ІНДУКТИВНИХ НЕПОРОЖНІХ ПІДМНОЖИН У ω

Олег ГУТІК, Інна ПОЗДНЯКОВА

Львівський національний університет імені Івана Франка,
вул. Університетська 1, 79000, м. Львів
e-mails: oleg.gutik@lnu.edu.ua, pozdnyakova.inna@gmail.com

Досліджено ін'єктивні ендоморфізми напівгрупи $B_\omega^{\mathscr{F}}$ з двоелементною сім'єю $\mathscr{F}$ індуктивних непорожніх підмножин у ω. Описано елементи напівгрупи $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ усіх ін'єктивних енодоморфізмів, які зберігають одиницю моноїда $\boldsymbol{B}_\omega^{\mathscr{F}}$, і доводимо, що відношення Ґріна $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$, $\mathscr{D}$ і $\mathscr{J}$ на напівгрупі $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ збігаються з відношенням рівності.

Ключові слова: біциклічнй моноїд, інверсна напівгрупа, біциклічне розширення, ін'єктивний ендоморфізм, відношення Ґріна.