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ELEMENT-WISE EXPONENTIAL A POSTERIORI ERROR ESTIMATOR FOR SINGULARLY PERTURBED CONVECTION-DIFFUSION BOUNDARY VALUE PROBLEMS

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This paper is devoted to the development of an a posteriori error estimator (AEE) based on exponential basis functions for piecewise-linear finite element approximations on triangular elements. The proposed AEE reliably computes an upper bound for the norm of the approximation error for convection-diffusion boundary value problems. The indicator function of the AEE for each finite element is derived as an analytical solution to an auxiliary boundary value problem, using averaged values of the coefficients of the original differential equation and Dirichlet boundary conditions defined at the nodal values of the computed approximation. Due to its element-wise nature, the computations of the AEE can be efficiently parallelized and adapted for GPGPU computing. The upper error bounds calculated by the proposed AEE in model problems with known exact solutions on uniformly refined meshes demonstrate better efficiency indices compared to a residual-type estimator constructed by polynomial basis function.

Key words: a posteriori error estimation, exponential a posteriori error estimator, finite element method, variational formulation, bilinear form, triangular finite elements, linear basis functions, barycentric coordinates, diffusion-advection, efficiency index of the error estimator.

1. Introduction

Convection-diffusion boundary value problems play a crucial role in modern scientific modeling, particularly for processes such as the spread of pollutants in the atmosphere and surface waters, tissue mass transfer, and population migration, among others. In real-world problems, the structure of solutions to the convection-diffusion model is profoundly influenced by the relative rates of the underlying processes: diffusion and convection. For instance, when convection significantly dominates over diffusion, the second-order differential equation effectively degenerates into a first-order equation. This, combined with two boundary conditions, leads to the emergence of regions with very large gradients – boundary or internal layers within the solutions. These problems are referred to as singular perturbations, and their approximate solutions often exhibit significant oscillations near these layers, preventing an accurate approximation of the exact solution. In such cases, the Finite Element Method (FEM) approximation error [12] is unevenly distributed across the nodes of the grid, most of it concentrated around the boundary or internal layer [5]. To avoid excessive computational costs, h-adaptive FEM schemes are currently widely used for solving such singularly perturbed problems. Owing to reliable and efficient a posteriori error estimators, these schemes are able to accurately detect the aforementioned layers and enable local refinement of the mesh structure in these regions; see, for example, the monographs [1], [2], [8], [11], [12] and the recent survey [3]. In the

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present paper, we develop the ideas of [9] and complement the polynomial a posteriori error estimators proposed in [6], [7], [10] by constructing an estimator based on exponential basis functions.

In recent years, the development of general-purpose computing on graphics processing units (GPGPU) has significantly advanced the implementation of parallel numerical algorithms. Algorithms and computational procedures that can be parallelized, including those used in FEM and AEE computations, have demonstrated remarkable performance improvements when executed on modern GPUs. The proposed element-wise computation scheme for the AEE is particularly well-suited for GPU acceleration, making it a promising direction for further research.

The paper is organized as follows. In Section 2, the boundary value problem of convection-diffusion and its corresponding variational formulation are presented. In Section 3, a brief description of the FEM scheme for obtaining an approximate solution of the variational problem is given. Section 4 introduces linear FEM approximations on triangular finite elements. Section 5 describes the application of the exponential a posteriori error estimator for two-dimensional convection-diffusion problems. In Section 6, a residual-type estimator is presented. Section 7 is devoted to the analysis of the results of numerical experiments for a model boundary value problem whose solution is complicated by the presence of singular perturbation.

2. Problem statement

Consider the boundary value problem of convection-diffusion in a bounded domain $\Omega \subset \mathbb{R}^2$ with a Lipschitz boundary $\Gamma = \partial\Omega$.

$$\begin{cases} \text{find } u = u(x) \text{ such that} \\ -\nabla.(\mu\nabla u) + \beta.\nabla u = f \text{ in } \Omega, \\ u = 0 \text{ on } \Gamma_u \subset \Gamma, \ mes(\Gamma_u) > 0, \\ -(\mu\nabla u).n = \alpha u + g \text{ on } \Gamma_q := \Gamma \setminus \Gamma_u, \end{cases} \tag{1}$$

and its corresponding variational formulation

$$\begin{cases} \text{find } u \in V := \{v \in H^1(\Omega) : \ v = 0 \text{ on } \Gamma_u\} \text{ such that} \\ a(u,v) = \langle l, v\rangle \quad \forall v \in V, \end{cases} \tag{2}$$

where

$$\begin{cases} a(u,v) \ := \int_\Omega [(\mu\nabla u).\nabla v + v\beta.\nabla u]dx + \int_{\Gamma_q} \alpha uv d\gamma, \\ \langle l, v\rangle := \int_\Omega fv dx + \int_{\Gamma_q} gv d\gamma \quad \forall v \in V \end{cases} \tag{3}$$

Throughout the paper we assume that $\mu = \mu(x) > 0$, $\beta = \{\beta_i(x)\}_{i=1}^2$, $f = f(x)$, $\alpha = \alpha(x) \geq 0$ and $g = g(x)$ $x \in \Omega$ are sufficiently regular functions that guarantee the conditions of the Lax-Milgram theorem [11] regarding the well-posedness of the variational problem (2)–(3).

3. Galerkin Discretization

We cover the domain Ω by a mesh $\mathcal{T}_h = \{K\}$ of finite elements K with the following properties: (i) $\Omega = \cup_{K\in\mathcal{T}_h} K$;, (ii) $K_1 \cap K_2 = \varnothing \ \forall K_1, K_2 \in \mathcal{T}_h,\ K_1 \neq K_2$;, (iii)

$$\text{(iii)}\quad \overline{K}_1 \cap \overline{K}_2 = \begin{cases} S := \{\text{common edge of elements } K_1 \text{ and } K_2\}, \\ A := \{\text{common vertex of elements } K_1 \text{ and } K_2\}, \\ \varnothing. \end{cases} \tag{4}$$

and transfer problem (2) to a finite-dimensional approximation subspace $V_h \subset V$, whose basis functions $\{\varphi_i(x)\}_{i=1}^{N_h}$ are constructed as follows. Define $I_h = \{A_i\}_{i=1}^{N_h}$ as the set of all vertices $A_i = (x_{1i}, x_{2i})$, $\quad A_i \notin \Gamma_u$, of the triangles from triangulation $\mathcal{T}_h$, and associate with each point (node) A_i a piecewise-defined continuous function $\varphi_i = \varphi_i(x_1, x_2)$ such that

$$\begin{cases} \Omega_i \equiv \operatorname{supp}\varphi_i := \bigcup_{K\in\mathcal{T}_h, A_i \in \overline{K}} K, \\ \varphi_i(A_j) = \delta_{ij} \quad \forall A_j \in I_h, \\ \varphi_i \in P_1(K) \quad \forall K \in \Omega_i. \end{cases} \tag{5}$$

where $P_1(K)$ denotes the space of first-order polynomials defined on the triangle K. Finally, we define the approximation subspace V_h as

$$V_h := \operatorname{span}\{\varphi_1, \dots, \varphi_{N_h}\}, \quad h = \max_{K\in\mathcal{T}_h} h_K, \quad h_K = \operatorname{diam}(K). \tag{6}$$

Then the Galerkin procedure for transferring the solution of problem(2) to the approximation subspace $V_h \subset V$ leads to the following problem:

$$\begin{cases} \text{given } \mathcal{T}_h = \{K\} \text{ and } V_h = \operatorname{span}\{\varphi_i\}_{i=1}^{N_h} \subset V, \quad \dim V_h = N_h < +\infty, \\ \text{find } u_h = \sum_{i=1}^{N_h} q_i \varphi_i \in V_h \text{ such that} \\ a(u_h, v) = \langle l, v\rangle \quad \forall v \in V_h. \end{cases} \tag{7}$$

A more detailed analysis of this procedure shows that, in fact, problem (7) is a system of linear algebraic equations to determine the coefficients of the following expansion

$$u_h(x) = \sum_{j=1}^{N_h} q_j \varphi_j(x), \quad \forall x \in \Omega. \tag{8}$$

which is written in expanded form as follows:

$$\sum_{j=1}^{N_h} a(\varphi_j, \varphi_i) q_j = \langle l, \varphi_i\rangle, \quad i = 1, 2, \dots, N_h. \tag{9}$$

The matrix $M = \{a(\varphi_j, \varphi_i)\}_{i,j=1}^{N_h}$ of this system is sparse and, under the assumptions of the Lax-Milgram theorem, is positive definite, which guarantees unique solvability of the system (9).

It is well known that, under an additional assumption on the choice of approximation spaces,

$$\dim V_h = N_h = N(h) \to \infty \quad \text{as} \quad h \to 0, \quad \bigcup_{h>0} V_h = V, \tag{10}$$

the sequence of finite element approximations $\{u_h\}_{h>0} \subset V$ converges to the exact solution $u \in V$ of the original variational problem (2)–(3) as $h \to 0$. Moreover, the convergence of the approximate solutions is characterized by the following a priori estimate: there exists a constant $C = C(\Omega) > 0$ such that

$$\|u - u_h\|_{H^1(\Omega)} \le C(\Omega)\, h\, |u|_{H^2(\Omega)}. \tag{11}$$

4. Piecewise Linear Finite Element Approximations on Triangular Mesh Elements

Considering the approximation $u_h \in V_h$ on an arbitrary triangle $K = \triangle A_i A_j A_m \in \mathcal{T}_h$, we introduce the following convenient local representation:

$$u_h(x)\big|_K \equiv u_K(x) := \sum\nolimits_{k=i,j,m} q_k \varphi_k(x) \equiv \sum\nolimits_{k=i,j,m} q_k L_k(x). \tag{12}$$

Here $L_i(x), L_j(x), L_m(x)$ denote the system of barycentric coordinates on the triangle K.

5. Exponential A Posteriori Error Estimator

The a priori error estimate (11) with a constant $C = C(\Omega) > 0$ (the value of which is difficult or impossible to determine) provides only a general characterization of the entire class of finite element approximations and serves merely as a guideline for assessing the quality of a particular representative of this class. To obtain a meaningful error estimate,

$$e_h := u - u_h \in E := \{w \in V : a(w, v_h) = 0 \quad \forall v_h \in V_h\}, \quad \dim E = +\infty, \tag{13}$$

consider its approximations, so-called *a posteriori error estimators* (AEE), constructed by various approaches.

The first of them is based on the following definition.

Definition 1. *Elementwise-defined exponential-type AEE.*

Let $u_K(x) = \sum_{k=i,j,m} q_k \varphi_k(x)$ be the local representation of the computed piecewise linear approximation $u_h \in V_h$ to the triangle $K = \triangle A_i A_j A_m \in \mathcal{T}_h$. We call the function $\varepsilon_K^{\mathrm{Exp}} = \varepsilon_K^{\mathrm{Exp}}(x)$ the element-wise component of the exponential a posteriori error estimator $\varepsilon_h^{\mathrm{Exp}}$ to the finite element triangle $K \in \mathcal{T}_h$ if and only if it possesses the following properties:

$$\text{(i)} \quad \varepsilon_K^{\mathrm{Exp}}(x) := u_K(x) - \tilde{u}_K(x), \quad \forall\, x \in K; \tag{14}$$

$$\begin{aligned} \text{(ii)} \quad \tilde{u}_K(x) = C_0 + \sum\nolimits_{i=1}^{2} C_i \exp\left[\mu^{-1}(x^K)\beta_i(x^K)(x_i - x_i^K)\right] \\ + f(x^K)\frac{\beta_1(x^K)x_1 + \beta_2(x^K)x_2}{\beta_1^2(x^K) + \beta_2^2(x^K)}; \end{aligned} \tag{15}$$

where x^K denotes the barycenter of the triangle $K = \triangle A_i A_j A_m \in \mathcal{T}_h$;

(iii) $\{C_i\}_{i=0}^2$ are the solutions of a system of linear algebraic equations of the form

$$\tilde{u}_K(A_k) = u_K(A_k), \quad k = i, j, m, \quad \forall\, K = \triangle A_i A_j A_m \in \mathcal{T}_h. \tag{16}$$

The construction of the proposed a posteriori error estimator consists in the sequential computation of its indicators on each finite element K of the partition $\mathcal{T}_h = \{K\}$ as follows.

On a finite element K we compute

$$\|\varepsilon_K^{\rm Exp}\|_{1,K} = \sqrt{\int_K \left((\varepsilon_K^{\rm Exp})^2 + |\nabla \varepsilon_K^{\rm Exp}|^2\right)\, dx}. \tag{17}$$

The global error indicator is defined by

$$\|\varepsilon_h^{\rm Exp}\|_{1,\Omega} = \sqrt{\sum\nolimits_{K\in\mathcal{T}_h} \|\varepsilon_K^{\rm Exp}\|^2_{1,K}}. \tag{18}$$

6. Residual-Type a Posteriori Error Estimator

An alternative approach to constructing a posteriori error estimators is based on an approximate solution on each finite element K of the triangulation $\mathcal{T}_h$ of the discretized error problem:

$$\begin{cases} \text{given } \mathcal{T}_h,\ u_h \in V_h \text{ and } E_h \subset E,\ \dim E_h = M(h) < +\infty, \\ \text{find the error } e_h \in E_h \text{ such that} \\ a(e_h, v) = \langle \rho(u_h), v\rangle \quad \forall\, v \in E_h. \end{cases} \tag{19}$$

Here the linear functional

$$\langle \rho(u_h), v\rangle = \langle l, v\rangle - a(u_h, v), \qquad \forall\, v \in V, \tag{20}$$

is called the residual functional of the approximation.

Definition 2. *Elementwise-defined polynomial-type AEE.*

Let $u_K(x) = \sum_{k=i,j,m} q_k\varphi_k(x)$ be the local representation of the computed piecewise linear approximation $u_h \in V_h$ on the triangle $K = \triangle A_iA_jA_m \in \mathcal{T}_h$. We call the function $\varepsilon_K^{\rm Pol} = \varepsilon_K^{\rm Pol}(x)$ the localization of the polynomial-type a posteriori error estimator $\varepsilon_h^{\rm Pol} = \varepsilon_h^{\rm Pol}(x)$ to the finite element $K \in \mathcal{T}_h$ if and only if it is defined as follows:

$$\varepsilon_K^{\rm Pol}(x) := \lambda_K \psi_K(x), \qquad \forall\, K \in \mathcal{T}_h. \tag{21}$$

where

$$\lambda_K = \langle \rho(u_h), \psi_K\rangle / a(\psi_K, \psi_K), \tag{22}$$

$$\begin{cases} \operatorname{supp}\psi_K := K = \triangle A_iA_jA_m, \\ \psi_K(x) := 3\big[L_i(x)L_j(x) + L_j(x)L_m(x) + L_m(x)L_i(x)\big],\ \forall\, K \in \mathcal{T}_h. \end{cases} \tag{23}$$

Remark 1. *In view of definition (22), the computation of the coefficient λ_K requires integration only on a single finite element K.*

Remark 2. *By choosing different basis functions $\psi_K(x)$ with local support* $\operatorname{supp}\psi_K = K,\ \forall\, K \in \mathcal{T}_h$, *one can construct various element-wise finite element approximation error estimators providing both lower and upper bounds for the true error. For example, the estimator (21) with basis functions*

$$\psi_K(x) := 27\, L_i(x)L_j(x)L_m(x), \quad \forall\, K = \triangle A_iA_jA_m \in \mathcal{T}_h, \tag{24}$$

calculates a lower bound for the energy norm of the true error.

More detailed results on two-sided error estimates for the finite element method can be found in [4]. We shall choose $\psi_K(x)$ in the form given in [6].

7. Numerical results

Below we present the numerical results of the computational experiments we performed, which allow us to draw conclusions regarding the reliability and efficiency of the exponential error estimator for finite element approximations computed on a sequence of uniformly refined meshes of triangular finite elements.

Example 1. *Consider the following boundary value problem of convection-diffusion [13]:*

$$\begin{gathered} -\mu\Delta u + (\beta_1,\beta_2).\nabla u = f \quad \text{in } \Omega=(0,1)^2, \quad u=0 \quad \text{on } \Gamma, \\ \mu = 10^{-2}, \quad \beta=(1,1), \\ f(x) = (x_1+x_2)\big[1-\chi_1(x)\chi_2(x)\big] - (x_1-x_2)\big[\chi_1(x)-\chi_2(x)\big], \\ \chi_i(x) = \exp(-(1-x_i)/\mu)\,. \end{gathered} \tag{25}$$

The exact solution of this problem is the function

$$u(x) = x_1x_2\big[1-\chi_1(x)\big]\big[1-\chi_2(x)\big],$$

whose graph is shown in Fig. 1.

In the neighborhood of the boundary parts $x_1 = 1$ and $x_2 = 1$, boundary layers can be observed in the solution caused by the singularly perturbed nature of the problem with the Peclet number $Pe \approx 140$. A more detailed characterization of these layers is illustrated in Fig. 2 by the graph of $|\nabla u(x)|$.

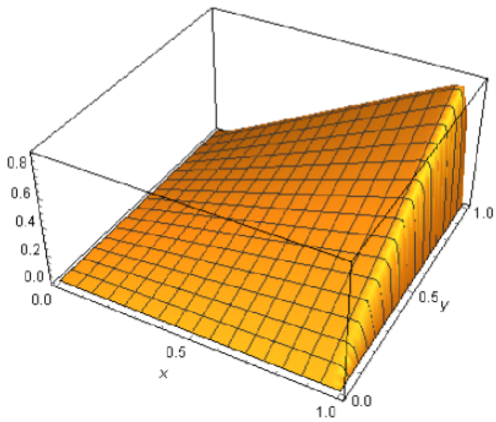

Fig. 1. Graph of the exact solution $u = u(x)$

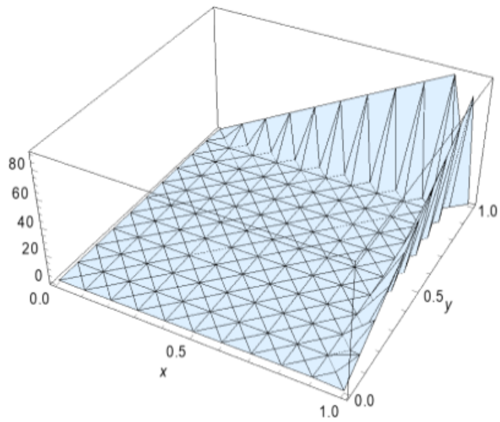

Fig. 2. Graph of the gradient magnitude $|\nabla u(x)|$

Consider several numerical characteristics of the finite element approximations computed on a sequence of uniformly refined meshes until the prescribed accuracy with tolerance $TOL = 3\%$ is achieved, as presented in Tabl. 1 and 2. Here $\mathcal{T}_n$ denotes the triangulation used at the n-th refinement step; $N_h(\mathcal{T}_n)$ is the number of nodes of the triangulation $\mathcal{T}_n$; $\mathrm{Card}_n = \mathrm{Card}(\mathcal{T}_n)$ is the number of finite elements in $\mathcal{T}_n$; $\delta_n^{AEE} = \|\varepsilon_{h,n}^{AEE}\|\|u_n\|^{-1} \cdot 100\%$ are the relative approximation errors computed in the

Table 1

Convergence of piecewise linear FEM approximations on uniformly refined meshes

n	$N_h(\mathcal{T}_n)$	Card$(\mathcal{T}_n)$	$\delta_n, \%$	$\delta_n^{\mathrm{Exp}}, \%$	$\delta_n^{\mathrm{Pol}}, \%$	p	p^{Exp}	p^{Pol}
1	221	400	95.9	123.6	314.2			
2	841	1 600	75.9	85.2	157.2	0.3	0.5	1.0
3	3 281	6 400	50.8	55.2	78.6	0.6	0.6	1.0
4	12 961	25 600	28.6	30.4	39.3	0.8	0.9	1.0
5	51 521	102 400	14.8	15.6	19.6	1.0	1.0	1.0
6	205 441	409 600	7.5	7.9	9.8	1.0	1.0	1.0
7	820 481	1 638 400	3.8	3.9	4.9	1.0	1.0	1.0
8	3 279 361	6 553 600	1.9	2.0	2.5	1.0	1.0	1.0

Table 2

Convergence of error norms of FEM approximations and their a posteriori estimators on uniformly refined meshes

i	$N_h(\mathcal{T}_n)$	Card$(\mathcal{T}_n)$	$\|e_n\|$	$\|\varepsilon_n^{\mathrm{Exp}}\|$	$\|\varepsilon_n^{\mathrm{Pol}}\|$	k^{Exp}	k^{Pol}	$\|u_n\|$
1	221	400	5.396	6.950	17.667	1.29	3.27	5.624
2	841	1 600	4.272	4.795	8.843	1.12	2.07	5.627
3	3 281	6 400	2.860	3.104	4.422	1.09	1.55	5.627
4	12 961	25 600	1.612	1.709	2.210	1.06	1.37	5.626
5	51 521	102 400	0.835	0.878	1.105	1.05	1.32	5.626
6	205 441	409 600	0.422	0.442	0.552	1.05	1.31	5.626
7	820 481	1 638 400	0.211	0.221	0.276	1.05	1.31	5.626
8	3 279 361	6 553 600	0.106	0.111	0.138	1.05	1.31	5.626

$H^1(\mathcal{T}_n)$ norm for the exponential and polynomial a posteriori estimators, respectively, with $AEE = \{\mathrm{Exp}, \mathrm{Pol}\}$. The quantities

$$p = 2\ln\left(\|e_n\| \cdot \|e_{n+1}\|^{-1}\right) / \ln\left(\mathrm{Card}_{n+1}\mathrm{Card}_n^{-1}\right),$$

$p^{AEE} = 2\ln\left(\|\varepsilon_n^{AEE}\| \cdot \|\varepsilon_{n+1}^{AEE}\|^{-1}\right) / \ln\left(\mathrm{Card}_{n+1}\mathrm{Card}_n^{-1}\right)$ represent the convergence rates of the a posteriori estimators and of the true error, computed using the current and the previous meshes; $k^{\mathrm{Exp}} = \|\varepsilon_n^{\mathrm{Exp}}\|\|e_n\|^{-1}$ is the efficiency index of the estimator; $\|u_n\|^2 = \int_\Omega \left(u_n^2 + |\nabla u_n|^2\right) dx$ is the approximation norm computed in the space $H^1(\mathcal{T}_n)$.

As can be seen from Tabl. 1 and 2, both error estimators provide an upper bound for the true error while achieving the theoretically expected convergence rates.

However, the efficiency index of the exponential a posteriori estimator deviates less from unity, which indicates a more accurate reproduction of the true error and, consequently, higher efficiency when used in h-adaptive schemes.

Example 2. *The data in problem (25) are taken as* $\mu = 1$, $\beta_1(x) = -75x_2$, $\beta_2(x) = -75x_1$, $f(x) = 16 \times 10^4 x_1^2 x_2^2$, $u = 0$ *on* Γ, *and* $\Omega = (0,1)^2$. *This problem is also singularly perturbed, since the Peclet similarity criterion is* $Pe \approx 105$. *Fig. 3 shows the piecewise linear approximation obtained at the second step of uniform mesh refinement with* 1600 *finite elements, and Fig. 4 presents the magnitude of its gradient. In Tabl. 3, we report* *the convergence characteristics of the approximations computed on uniformly refined triangulations with accuracy $TOL = 3\%$.*

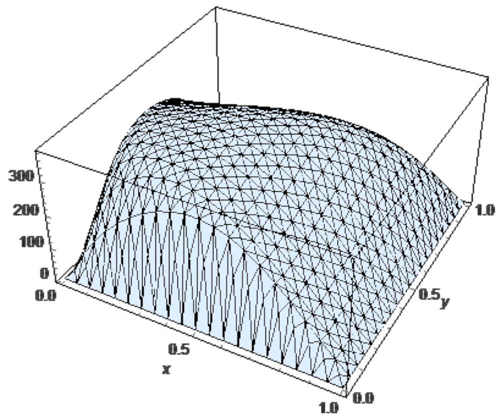

Fig. 3. Graph of the approximation $u_h|_K \in P_1(K)$, $n = 2$

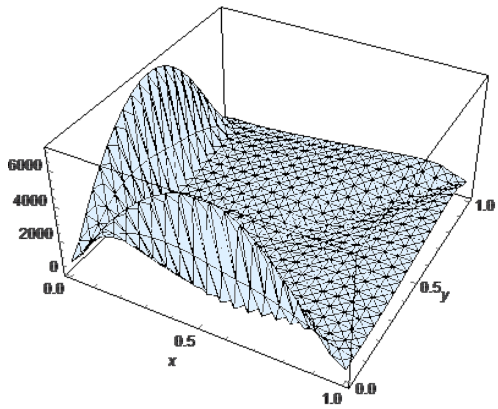

Fig. 4. Graph of the gradient magnitude $|\nabla u_h|$, $n = 2$

Table 3

Convergence of piecewise linear FEM approximations on uniformly refined meshes

i	$N_h(\mathcal{T}_n)$	$\mathrm{Card}(\mathcal{T}_n)$	$\delta_n^{\mathrm{Exp}}, \%$	$\delta_n^{\mathrm{Pol}}, \%$	p^{Exp}	p^{Pol}	$\Vert\varepsilon_n^{\mathrm{Exp}}\Vert$	$\Vert\varepsilon_n^{\mathrm{Pol}}\Vert$	$\Vert u_n\Vert$
1	221	400	71.4	94.3			1181.51	1560.88	1654.87
2	841	1 600	43.0	50.0	0.7	0.9	717.43	833.22	1667.22
3	3 281	6 400	23.2	25.6	0.9	1.0	387.44	427.10	1671.57
4	12 961	25 600	11.8	12.9	1.0	1.0	198.13	215.07	1672.83
5	51 521	102 400	6.0	6.4	1.0	1.0	99.65	107.73	1673.16
6	205 441	409 600	3.0	3.2	1.0	1.0	49.90	53.89	1673.24
7	820 481	1 638 400	1.5	1.6	1.0	1.0	24.96	26.95	1673.27

We observe that the theoretically expected convergence rates are achieved. The values of the exponential error estimator are smaller than those of the residual estimator, as in the previous example.

8. Conclusions

In this paper, an exponential a posteriori error estimator for linear finite element approximations on triangular elements has been constructed. Based on the conducted numerical experiments, the possibility of obtaining an upper bound for the true error of the convection-diffusion boundary value problem has been demonstrated.

The upper error bounds computed by the exponential a posteriori estimator in model problems with known exact solutions demonstrate better efficiency indices on uniformly refined meshes compared with the residual-type estimator constructed using polynomial basis functions.

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ЕКСПОНЕНЦІАЛЬНИЙ АПОСТЕРІОРНИЙ ОЦІНЮВАЧ, ЗДАТНИЙ ОБЧИСЛЮВАТИ ВЕРХНЮ МЕЖУ ПОХИБКИ МСЕ ДЛЯ ЗАДАЧ АДВЕКЦІЇ-ДИФУЗІЇ

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Ця стаття присвячена розробці апостеріорного оцінювача похибки (АОП), що базується на експоненціальних базисних функціях, для кусково-лінійних апроксимацій методу скінченних елементів на трикутних елементах. Запропонований АОП надійно обчислює верхню межу норми похибки апроксимації для крайових задач адвекції-дифузії. Індикаторна функція АОП кожного скінченного елемента є аналітичним розв'язком допоміжної крайової задачі з усередненими на ньому значеннями даних вихідного диференціального рівняння та крайових умовах Діріхле, визначених вузловими значеннями обчисленої апроксимації. Завдяки поелементному характеру обчислення АОП можуть бути ефективно розпаралелені та адаптовані для обчислень на GPGPU. Верхні оцінки похибки, отримані запропонованим АОП у модельних задачах з відомими точними розв'язками на рівномірно згущених сітках, демонструють кращі індекси ефективності порівняно з оцінювачем залишкового типу, побудованим на поліноміальних базисних функціях.

Ключові слова: апостеріорна оцінка похибки, експоненціальний апостеріорний оцінювач похибки, метод скінченних елементів, варіаційна постановка, білінійна форма, трикутні скінченні елементи, лінійні базисні функції, барицентричні координати, дифузія-адвекція, індекс ефективності оцінювача похибки.