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OPTIMALITY OF PENULTIMATE AND FINAL TIME MOMENTS IN 1-BULLET SILENT DUELS WITH GENERALIZED EXPONENTIALLY-CONVEX REWARDS

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Silent duels are a class of timing games that model competitive interaction among a group of participants through a given time span under informational uncertainty and reward limitation. The circumstances of the interaction are such that the participant does not learn about actions of the other participants until the duel end, and the participant benefits from acting as late as possible but only by acting first. The finite 1-bullet silent duel is considered, in which the duel time span is equidistantly quantized, and each of the two duelists has a generalized exponentially-convex reward function. In this paper, optimality of penultimate and final time moments is investigated, which are of particular interest in modeling systems, where actions (decisions) are forced to be made as late as possible. The duel is a symmetric matrix game whose optimal value is 0, and each of the duelists has the same optimal behavior, whether it is in pure or mixed strategies. It is proved that the final time moment is single optimal in 3×3 duels, whichever the factor of reward steepness is. It is also ascertained that in bigger duels the final time moment is single optimal if the factor of reward steepness is higher than the unique root of an exponential equation. If the factor is the root, both penultimate and final time moments become optimal.

Key words: 1-bullet silent duel, exponentially-convex reward, matrix game, optimal time moment, penultimate moment, final moment.

1. SILENT DUELS WITH EXPONENTIALLY-CONVEX REWARD

Silent duels are a class of timing games that model competitive interaction among a group of participants through a given time span under informational uncertainty and reward limitation [1, 5, 12, 13]. The circumstances of the interaction are such that the participant does not learn about actions of the other participants until the duel end, and the participant benefits from acting as late as possible but only by acting first [8, 10, 11, 20]. The duel time span is standardized being commonly quantized into a finite set [6, 9, 14]

$$T_N = \{t_q\}_{q=1}^{N} = \left\{ \frac{q-1}{N-1} \right\}_{q=1}^{N} \subset [0;\ 1] \ \text{ for } \ N \in \mathbb{N} \backslash \{1,\ 2\} \tag{1}$$

of N successive time moments of possible acting [7, 15, 16, 18]. The latter is also metaphorically spoken as shooting bullets, where having the bullet is a metaphor of the possibility to make a definite decision or an action. A finite 1-bullet silent duel is a special timing game

$$\langle X_N,\ Y_N,\ \mathbf{U}_N \rangle = \left\langle \{x_i\}_{i=1}^{N},\ \{y_j\}_{j=1}^{N},\ \mathbf{U}_N \right\rangle \tag{2}$$

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that models one-decision-making competition between two identical participants (duelists) through time span (1) whose pure strategy sets are

$$X_N = \{x_i\}_{i=1}^{N} = \left\{ \frac{i-1}{N-1} \right\}_{i=1}^{N} = T_N, \;\; Y_N = \{y_j\}_{j=1}^{N} = \left\{ \frac{j-1}{N-1} \right\}_{j=1}^{N} = T_N \tag{3}$$

by a skew-symmetric payoff (reward) matrix

$$\mathbf{U}_N = [u_{ij}]_{N \times N} = [-u_{ji}]_{N \times N} = -\mathbf{U}_N^{\mathrm{T}}. \tag{4}$$

Game (2) by (3), (1), (4) is a symmetric matrix game whose optimal game value is 0, and each of the duelists has the same optimal behavior [7, 13]. The structure of the optimal behavior depends on how reward matrix (4) is built [13, 14, 16, 19, 21]. Generally speaking, the matrix entry

$$u_{ij} = g(x_i) - g(y_j) + g(x_i)\, g(y_j) \operatorname{sign}(y_j - x_i) \;\; \text{for} \;\; i = \overline{1,\ N} \;\; \text{and} \;\; j = \overline{1,\ N} \tag{5}$$

by some discrete increasing accuracy functions $g(x_i)$ and $g(y_j)$ of the first and second duelists, respectively, where

$$g(x_1) = g(y_1) = g(0) = 0 \;\; \text{and} \;\; g(x_N) = g(y_N) = g(1) = 1. \tag{6}$$

Accuracy functions $g(x_i)$ and $g(y_j)$ can also be called reward functions [3, 17, 23]. If the duelist's reward is considered apart from the other duelist's reward, then it must exponentially increase with time. So, instead of (5), entry u_{ij} of reward matrix (4) is calculated as

$$u_{ij} = g\left(e^{\mu x_i}\right) - g\left(e^{\mu y_j}\right) + g\left(e^{\mu x_i}\right) g\left(e^{\mu y_j}\right) \operatorname{sign}(y_j - x_i)$$
$$\text{for} \;\; i = \overline{1,\ N} \;\; \text{and} \;\; j = \overline{1,\ N} \tag{7}$$

with some factor of steepness $\mu > 0$ and by still obeying requirements similar to (6):

$$g\left(e^{\mu x_1}\right) = g\left(e^{\mu y_1}\right) = g\left(e^{0}\right) = g(1) = 0 \;\; \text{and} \;\; g\left(e^{\mu x_N}\right) = g\left(e^{\mu y_N}\right) = g\left(e^{\mu}\right) = 1. \tag{8}$$

The primary general objective is to determine a subset $T_N^*(\mu) \subset T_N$ of optimal time moments in duel (2) by (3), (1), (4) for (7) and any given μ. Obviously, it is quite plausible that for some integers N and some positives μ this subset can turn out to be empty:

$$T_N^*(\mu) = \emptyset \;\; \text{for} \;\; N \in \mathrm{N}_{\emptyset} \subset \mathbb{N} \backslash \{1,\ 2\} \;\; \text{and} \;\; \mu \in \mathrm{M} \subset \mathbb{R}_{>0}, \tag{9}$$

where

$$\mathbb{R}_{>0} = \{\mu \in \mathbb{R} : \mu > 0\}.$$

So, the secondary general objective is to determine subsets $\mathrm{N}_{\emptyset}$ and M such that (9) holds. In this paper, we will investigate optimality of penultimate and final time moments, which are of particular interest in modeling systems, where actions (decisions) are forced to be made as late as possible [3, 6, 12, 17, 23].

2. Explicit reward function

In accordance with the abovementioned, an exponentially-increasing reward function of the duelist is

$$g\left(e^{\mu z}\right) = ae^{\mu z} + b \ \text{ for } \ z \in T_N \tag{10}$$

and some real numbers a and b, where, obviously, $a \neq 0$. As function (10) of variable z must obey requirements (8), then

$$g\left(e^{0}\right) = g(1) = a + b = 0,$$

$$g\left(e^{\mu}\right) = ae^{\mu} + b = 1,$$

whence

$$b = -a = 1 - ae^{\mu},$$

$$a\left(e^{\mu} - 1\right) = 1 \ \text{ or } \ e^{\mu} = \frac{1+a}{a},$$

and

$$a = \frac{1}{e^{\mu} - 1} > 0, \ \ b = \frac{1}{1 - e^{\mu}} < 0. \tag{11}$$

Hence, function (10) is

$$g\left(e^{\mu z}\right) = \frac{e^{\mu z}}{e^{\mu} - 1} - \frac{1}{e^{\mu} - 1} = \frac{e^{\mu z} - 1}{e^{\mu} - 1}. \tag{12}$$

Then, upon plugging (12) into (7), entry u_{ij} of reward matrix (4) is calculated as

$$\begin{gathered} u_{ij} = \frac{e^{\mu x_i} - 1}{e^{\mu} - 1} - \frac{e^{\mu y_j} - 1}{e^{\mu} - 1} + \frac{e^{\mu x_i} - 1}{e^{\mu} - 1} \cdot \frac{e^{\mu y_j} - 1}{e^{\mu} - 1} \cdot \operatorname{sign}\left(y_j - x_i\right) = \\ = \frac{e^{\mu x_i} - e^{\mu y_j}}{e^{\mu} - 1} + \frac{\left(e^{\mu x_i} - 1\right)\left(e^{\mu y_j} - 1\right)}{\left(e^{\mu} - 1\right)^2} \cdot \operatorname{sign}\left(y_j - x_i\right) \\ \text{for } \ i = \overline{1,\ N} \ \text{ and } \ j = \overline{1,\ N}. \end{gathered} \tag{13}$$

Owing to $e^{\mu} > 1$ and $e^{\mu y_j} > 1 \ \forall j = \overline{2,\ N}$, it is easy to see that

$$u_{1j} = \frac{1 - e^{\mu y_j}}{e^{\mu} - 1} + \frac{(1-1) \cdot \left(e^{\mu y_j} - 1\right)}{\left(e^{\mu} - 1\right)^2} = \frac{1 - e^{\mu y_j}}{e^{\mu} - 1} < 0 \ \ \forall j = \overline{2,\ N}$$

and thus the starting moment $t_1 = 0$ is never optimal in duel (2) by (3), (1), (4) for (13) and any given $\mu > 0$, whichever the number of time moments is.

3. The fewest number of moments to shoot

Theorem 1. *In 1-bullet silent duel (2) by (3), (1), (4) for (13) with the fewest number of moments to shoot, which is*

$$\langle X_3,\ Y_3,\ \mathbf{U}_3 \rangle = \left\langle \left\{0,\ \frac{1}{2},\ 1\right\},\ \left\{0,\ \frac{1}{2},\ 1\right\},\ \mathbf{U}_3 \right\rangle, \tag{14}$$

the duelist has the single optimal time moment $t_3 = 1$.

Proof. Upon plugging elements of set T_3 into (13) for $N = 3$, the entries of the respective reward matrix (4) are:

$$u_{12} = \frac{e^0 - e^{\frac{\mu}{2}}}{e^{\mu} - 1} + \frac{\left(e^0 - 1\right)\left(e^{\frac{\mu}{2}} - 1\right)}{\left(e^{\mu} - 1\right)^2} = \frac{1 - e^{\frac{\mu}{2}}}{e^{\mu} - 1} = -u_{21} < 0, \tag{15}$$

$$u_{13} = \frac{e^0 - e^{\mu}}{e^{\mu} - 1} + \frac{\left(e^0 - 1\right)\left(e^{\mu} - 1\right)}{\left(e^{\mu} - 1\right)^2} = \frac{1 - e^{\mu}}{e^{\mu} - 1} = -1 = -u_{31}, \tag{16}$$

$$u_{23} = \frac{e^{\frac{\mu}{2}} - e^{\mu}}{e^{\mu} - 1} + \frac{\left(e^{\frac{\mu}{2}} - 1\right)\left(e^{\mu} - 1\right)}{\left(e^{\mu} - 1\right)^2} = \frac{2e^{\frac{\mu}{2}} - e^{\mu} - 1}{e^{\mu} - 1}. \tag{17}$$

Hence, with (15)–(17) reward matrix (4) here is

$$\mathbf{U}_3 = \left[u_{ij}\right]_{3\times 3} = \begin{bmatrix} 0 & \dfrac{1 - e^{\frac{\mu}{2}}}{e^{\mu} - 1} & -1 \\ \dfrac{1 - e^{\frac{\mu}{2}}}{1 - e^{\mu}} & 0 & \dfrac{2e^{\frac{\mu}{2}} - e^{\mu} - 1}{e^{\mu} - 1} \\ 1 & \dfrac{2e^{\frac{\mu}{2}} - e^{\mu} - 1}{1 - e^{\mu}} & 0 \end{bmatrix}. \tag{18}$$

If entry (17) is negative, then the third row of matrix (18) is positive except for entry $u_{33} = 0$, and the final time moment $t_3 = 1$ is single optimal. The sign of entry (17) is determined by the numerator of the fraction in (17). Denote $\beta = e^{\frac{\mu}{2}}$, where always $\beta > 1$ due to $\mu > 0$. Then the numerator of the fraction in (17) is

$$2\beta - \beta^2 - 1 = -\left(\beta - 1\right)^2 < 0,$$

and thus time moment $t_3 = 1$ is single optimal. □

4. OPTIMAL TIME MOMENT IN BIGGER DUELS

Lemma 2. *Entry u_{nj} by (13), considered as a discrete function of index $j = \overline{1,\ n-1}$ by $n \in \left\{\overline{2,\ N}\right\}$, strictly decreases as index j is increased.*

Proof. Plugging $i = n$ into (13) for $n \in \left\{\overline{2,\ N}\right\}$, entry

$$u_{nj} = \frac{e^{\mu x_n} - e^{\mu y_j}}{e^{\mu} - 1} - \frac{\left(e^{\mu x_n} - 1\right)\left(e^{\mu y_j} - 1\right)}{\left(e^{\mu} - 1\right)^2} =$$
$$= \frac{e^{\mu x_n} - 1}{e^{\mu} - 1} - \left(1 + \frac{e^{\mu x_n} - 1}{e^{\mu} - 1}\right) \cdot \frac{e^{\mu y_j} - 1}{e^{\mu} - 1}$$
$$\text{for } j = \overline{1,\ n-1} \text{ at } n \in \left\{\overline{2,\ N}\right\}. \tag{19}$$

Due to $e^{\mu x_n} > 1$ and $e^{\mu y_j} \geqslant 1$ by $x_n > 0$ and $y_j \geqslant 0$, respectively, entry (19) is a negatively-sloped line with respect to exponent $e^{\mu y_j}$. Therefore, entry (19) strictly decreases as index j is increased off 1 up to $n - 1$. □

Lemma 3. *Entry u_{nj} by (13), considered as a discrete function of index $j = \overline{n+1,\ N}$ by $n \in \left\{\overline{2,\ N-1}\right\}$, strictly decreases as index j is increased.*

Proof. Plugging $i = n$ into (13) for $n \in \left\{\overline{2,\ N-1}\right\}$, entry

$$u_{nj} = \frac{e^{\mu x_n} - e^{\mu y_j}}{e^{\mu} - 1} + \frac{\left(e^{\mu x_n} - 1\right)\left(e^{\mu y_j} - 1\right)}{\left(e^{\mu} - 1\right)^2} =$$
$$= \frac{e^{\mu x_n} - 1}{e^{\mu} - 1} - \left(1 - \frac{e^{\mu x_n} - 1}{e^{\mu} - 1}\right) \cdot \frac{e^{\mu y_j} - 1}{e^{\mu} - 1}$$
$$\text{for } \ j = \overline{n+1,\ N} \ \text{ at } \ n \in \left\{\overline{2,\ N-1}\right\}. \tag{20}$$

Inasmuch as

$$\mu > \mu x_n \ \ \forall \, \mu > 0 \ \text{ and } \ \forall \, x_n \in \left\{\frac{i-1}{N-1}\right\}_{i=2}^{N-1} \subset \left[\frac{1}{N-1};\ \frac{N-2}{N-1}\right],$$

then $e^{\mu x_n} < e^{\mu}$ and, consequently,

$$1 > \frac{e^{\mu x_n} - 1}{e^{\mu} - 1},$$

whence entry (20) is a negatively-sloped line with respect to exponent $e^{\mu y_j}$. Therefore, entry (20) strictly decreases as index j is increased off $n+1$ up to N. □

Theorem 4. *In 1-bullet silent duel (2) by (3), (1), (4) for (13) the final time moment* $t_N = 1$ *is single optimal by* $\mu > \mu_0$ *for* $N \in \mathbb{N} \backslash \{1,\ 2,\ 3\}$, *where* μ_0 *is the unique root of equation*

$$e^{\mu} - 2e^{\mu \cdot \frac{N-2}{N-1}} + 1 = 0 \tag{21}$$

and, in addition,

$$\mu_0 > (N-1)\ln\left(2 \cdot \frac{N-2}{N-1}\right). \tag{22}$$

Proof. The final moment in the duel is single optimal if only the last row of matrix (4) is positive except for entry $u_{NN} = 0$:

$$u_{Nj} = \frac{e^{\mu x_N} - e^{\mu y_j}}{e^{\mu} - 1} - \frac{\left(e^{\mu x_N} - 1\right)\left(e^{\mu y_j} - 1\right)}{\left(e^{\mu} - 1\right)^2} > 0 \ \ \forall \, y_j < x_N = 1 \ \text{ by } \ j = \overline{1,\ N-1}. \tag{23}$$

Inequality (23) is simplified to inequality

$$u_{Nj} = \frac{e^{\mu} - e^{\mu y_j}}{e^{\mu} - 1} - \frac{\left(e^{\mu} - 1\right)\left(e^{\mu y_j} - 1\right)}{\left(e^{\mu} - 1\right)^2} =$$
$$= \frac{e^{\mu} - 2e^{\mu y_j} + 1}{e^{\mu} - 1} > 0 \ \ \forall \, y_j < x_N = 1 \ \text{ by } \ j = \overline{1,\ N-1}. \tag{24}$$

Owing to Lemma 1, function u_{Nj} is decreasing with respect to index $j = \overline{1,\ N-1}$, and hence inequality (24) is equivalent to inequality

$$u_{N,N-1} = \frac{e^{\mu} - 2e^{\mu y_{N-1}} + 1}{e^{\mu} - 1} > 0,$$

whence

$$e^{\mu} - 2e^{\mu \cdot \frac{N-2}{N-1}} + 1 > 0. \tag{25}$$

To solve inequality (25), denote

$$f\left(\mu\right) = e^{\mu} - 2e^{\mu\cdot\frac{N-2}{N-1}} + 1. \tag{26}$$

The first derivative of function (26) is

$$\frac{df}{d\mu} = e^{\mu} - 2\cdot\frac{N-2}{N-1}\cdot e^{\mu\cdot\frac{N-2}{N-1}}. \tag{27}$$

First derivative (27) turns into zero if

$$e^{\mu} = 2\cdot\frac{N-2}{N-1}\cdot e^{\mu\cdot\frac{N-2}{N-1}},$$

whence

$$\mu = \ln\left(2\cdot\frac{N-2}{N-1}\cdot e^{\mu\cdot\frac{N-2}{N-1}}\right) =$$

$$= \ln\left(2\cdot\frac{N-2}{N-1}\right) + \ln\left(e^{\mu\cdot\frac{N-2}{N-1}}\right) = \ln\left(2\cdot\frac{N-2}{N-1}\right) + \mu\cdot\frac{N-2}{N-1},$$

$$\mu\cdot\frac{1}{N-1} = \ln\left(2\cdot\frac{N-2}{N-1}\right),$$

and thus

$$\mu = \mu_{*} = (N-1)\ln\left(2\cdot\frac{N-2}{N-1}\right) \tag{28}$$

is the unique critical point of function (26). The second derivative of function (26) is

$$\frac{d^2 f}{d\mu^2} = e^{\mu} - 2\cdot\left(\frac{N-2}{N-1}\right)^2\cdot e^{\mu\cdot\frac{N-2}{N-1}}. \tag{29}$$

Second derivative (29) at critical point (28) is

$$\left.\frac{d^2 f}{d\mu^2}\right|_{\mu=\mu_{*}} = e^{(N-1)\ln\left(2\cdot\frac{N-2}{N-1}\right)} - 2\cdot\left(\frac{N-2}{N-1}\right)^2\cdot e^{(N-1)\ln\left(2\cdot\frac{N-2}{N-1}\right)\cdot\frac{N-2}{N-1}} =$$

$$= e^{\ln\left(2\cdot\frac{N-2}{N-1}\right)^{N-1}} - 2\cdot\left(\frac{N-2}{N-1}\right)^2\cdot e^{\ln\left(2\cdot\frac{N-2}{N-1}\right)^{N-2}} =$$

$$= \left(2\cdot\frac{N-2}{N-1}\right)^{N-1} - 2^{N-1}\cdot\left(\frac{N-2}{N-1}\right)^{N} =$$

$$= 2^{N-1}\cdot\left(\frac{N-2}{N-1}\right)^{N-1}\left(1-\frac{N-2}{N-1}\right) =$$

$$= 2^{N-1}\cdot\frac{\left(N-2\right)^{N-1}}{\left(N-1\right)^{N}} > 0,$$

and therefore critical point (28) is the unique minimum point of function (26). This function at minimum point (28) drops to value

$$f\left(\mu_*\right) = e^{(N-1)\ln\left(2\cdot\frac{N-2}{N-1}\right)} - 2e^{(N-1)\ln\left(2\cdot\frac{N-2}{N-1}\right)\cdot\frac{N-2}{N-1}} + 1 =$$

$$= e^{\ln\left(2\cdot\frac{N-2}{N-1}\right)^{N-1}} - 2e^{\ln\left(2\cdot\frac{N-2}{N-1}\right)^{N-2}} + 1 =$$

$$= \left(2\cdot\frac{N-2}{N-1}\right)^{N-1} - 2\cdot\left(2\cdot\frac{N-2}{N-1}\right)^{N-2} + 1 =$$

$$= 2^{N-1}\cdot\left(\frac{N-2}{N-1}\right)^{N-1} - 2^{N-1}\cdot\left(\frac{N-2}{N-1}\right)^{N-2} + 1 =$$

$$= 2^{N-1}\cdot\left(\frac{N-2}{N-1}\right)^{N-2}\cdot\left(\frac{N-2}{N-1} - 1\right) + 1 =$$

$$= 2^{N-1}\cdot\left(\frac{N-2}{N-1}\right)^{N-2}\cdot\left(-\frac{1}{N-1}\right) + 1 =$$

$$= 1 - 2^{N-1}\cdot\frac{\left(N-2\right)^{N-2}}{\left(N-1\right)^{N-1}}. \tag{30}$$

Value (30) is a function of N, so denote it by

$$\rho\left(N\right) = 1 - 2^{N-1}\cdot\frac{\left(N-2\right)^{N-2}}{\left(N-1\right)^{N-1}}. \tag{31}$$

Note that

$$\rho\left(3\right) = 1 - 2^2\cdot\frac{1^1}{2^2} = 0. \tag{32}$$

The first derivative of function (31) is

$$\frac{d\rho}{dN} = \frac{2^{N-1}\left(\ln 2\right)\cdot\left(\frac{N-2}{N-1}\right)^{N-2}}{1-N} +$$

$$+\frac{2^{N-1}\cdot\left(\frac{N-2}{N-1}\right)^{N-2}}{1-N}\cdot\left(\ln\frac{N-2}{N-1} + \left(\frac{1}{N-1} - \frac{N-2}{\left(N-1\right)^2}\right)\left(N-1\right)\right) +$$

$$+\frac{2^{N-1}\cdot\left(\frac{N-2}{N-1}\right)^{N-2}}{\left(1-N\right)^2} =$$

$$= \frac{2^{N-1}\cdot\left(\frac{N-2}{N-1}\right)^{N-2}}{1-N}\cdot\left(\ln 2 + \ln\frac{N-2}{N-1} + \frac{1}{N-1} + \frac{1}{1-N}\right) =$$

$$= \frac{2^{N-1}\cdot\left(\frac{N-2}{N-1}\right)^{N-2}}{1-N}\cdot\ln\left(2\cdot\frac{N-2}{N-1}\right) < 0 \tag{33}$$

due to

$$\frac{N-2}{N-1} > \frac{1}{2},$$

$$2 \cdot \frac{N-2}{N-1} > 1,$$

and thus

$$\ln\left(2 \cdot \frac{N-2}{N-1}\right) > 0.$$

Inequality (33) means that function (31) decreases as N increases. Furthermore, due to (32), this function is negative beyond $N = 3$:

$$\rho(N) < 0 \;\text{ by }\; N > 3.$$

Therefore, $f(\mu_*) < 0$. Meanwhile, $f(0) = 0$ and

$$\lim_{\mu\to\infty} f(\mu) = \lim_{\mu\to\infty}\left(e^{\mu} - 2e^{\mu\cdot\frac{N-2}{N-1}} + 1\right) = +\infty. \tag{34}$$

Equality (34) is true because, for instance, inequality

$$e^{\mu} > 2e^{\mu\cdot\frac{N-2}{N-1}} \tag{35}$$

holds for a sufficiently great μ: indeed, it follows from (35) that

$$\mu > \ln\left(2e^{\mu\cdot\frac{N-2}{N-1}}\right),$$

$$\mu > \ln 2 + \mu \cdot \frac{N-2}{N-1},$$

whence

$$\mu > (N-1)\ln 2. \tag{36}$$

Whichever N is, μ can always be taken great enough to satisfy inequality (36), and this confirms inequality (35) for a sufficiently great μ. Hence, function (26) being negative at its unique minimum point (28) and positive at sufficiently great values of μ, must have the unique zero point μ_0 beyond minimum point (28), i. e. equation (21) has a unique root μ_0 that satisfies inequality (22). ☐

Theorem 5. *An estimation for the unique root of equation (21) is*

$$\mu_0 \in (\mu_*;\ \mu_1) = \left((N-1)\ln\left(2\cdot\frac{N-2}{N-1}\right);\ (N-1)\ln 2\right). \tag{37}$$

Proof. The left endpoint in membership (37) is taken straightforwardly from (22) in Theorem 2. The right endpoint in membership (37) means that

$$\mu_0 < (N-1)\ln 2. \tag{38}$$

If some $\mu = \mu_1$ is taken such that

$$e^{\mu_1} = 2e^{\mu_1\cdot\frac{N-2}{N-1}},$$

then

$$\mu_1 = (N-1)\ln 2 \tag{39}$$

and inequality (25) still holds, which means that inequality $\mu_0 < \mu_1$ holds. This confirms (38) and estimation (37) overall. ☐

It is noteworthy that at the upper bound (39) function (26) is constant not depending on N:

$$f(\mu_1) = e^{(N-1)\ln 2} - 2e^{((N-1)\ln 2)\cdot\frac{N-2}{N-1}} + 1 = 2^{N-1} - 2\cdot 2^{N-2} + 1 = 1. \tag{40}$$

Owing to $f(\mu_*) < 0$ and (40), the unique root (37) of equation (21) can be approximated with any desired accuracy by the bisection method [22, 24] applied to function (26) with endpoints (28) and (39).

Theorem 6. *In 1-bullet silent duel (2) by (3), (1), (4) for (13) both the penultimate and final time moments* $t_{N-1} = \dfrac{N-2}{N-1}$ *and* $t_N = 1$ *are only optimal by* $\mu = \mu_0$ *for* $N \in \mathbb{N}\backslash\{1,\ 2,\ 3\}$*, where* μ_0 *is the unique root of equation (21).*

Proof. It follows from Theorem 2 that at $\mu = \mu_0$, where μ_0 is the unique root of equation (21),

$$u_{N,N-1} = u_{N-1,N} = u_{N-1,N-1} = u_{NN} = 0. \tag{41}$$

Owing to (41) and to Lemma 1, according to which function u_{Nj} is decreasing with respect to index $j = \overline{1,\ N-1}$, entry $u_{N,N-2} > 0$ and thus the last row of matrix (4) is positive except for entries $u_{N,N-1} = 0$ and $u_{NN} = 0$. This means that the final moment $t_N = 1$ in the duel is optimal. Due to Lemma 1, function $u_{N-1,j}$ is decreasing as well with respect to index $j = \overline{1,\ N-2}$, so the penultimate moment $t_{N-1} = \dfrac{N-2}{N-1}$ here is optimal along with the final moment $t_N = 1$ if inequality

$$u_{N-1,N-2} = \frac{e^{\mu_0 x_{N-1}} - e^{\mu_0 y_{N-2}}}{e^{\mu_0} - 1} - \frac{\left(e^{\mu_0 x_{N-1}} - 1\right)\left(e^{\mu_0 y_{N-2}} - 1\right)}{\left(e^{\mu_0} - 1\right)^2} > 0 \tag{42}$$

holds, by which the penultimate row of matrix (4) is positive except for entries $u_{N-1,N-1} = 0$ and $u_{N-1,N} = 0$. Inequality (42) is rewritten as

$$u_{N-1,N-2} = \frac{e^{\mu_0 x_{N-1}} - e^{\mu_0 y_{N-2}}}{e^{\mu_0} - 1} - \frac{\left(e^{\mu_0 x_{N-1}} - 1\right)\left(e^{\mu_0 y_{N-2}} - 1\right)}{\left(e^{\mu_0} - 1\right)^2} =$$

$$= \frac{e^{\mu_0 x_{N-1}} e^{\mu_0} - e^{\mu_0 x_{N-1}} - e^{\mu_0 y_{N-2}} e^{\mu_0} + e^{\mu_0 y_{N-2}} - e^{\mu_0 x_{N-1}} e^{\mu_0 y_{N-2}} + e^{\mu_0 y_{N-2}} + e^{\mu_0 x_{N-1}} - 1}{\left(e^{\mu_0} - 1\right)^2} =$$

$$= \frac{e^{\mu_0 x_{N-1}} e^{\mu_0} - e^{\mu_0 y_{N-2}} e^{\mu_0} + 2e^{\mu_0 y_{N-2}} - e^{\mu_0 x_{N-1}} e^{\mu_0 y_{N-2}} - 1}{\left(e^{\mu_0} - 1\right)^2} > 0. \tag{43}$$

Inequality (43) is simplified to inequality

$$e^{\mu_0 x_{N-1}} e^{\mu_0} - e^{\mu_0 y_{N-2}} e^{\mu_0} + 2e^{\mu_0 y_{N-2}} - e^{\mu_0 x_{N-1}} e^{\mu_0 y_{N-2}} - 1 > 0 \tag{44}$$

written with respect to integer variable N. To solve inequality (44), denote

$$h(\mu,\ N) = e^{\mu x_{N-1}} e^{\mu} - e^{\mu y_{N-2}} e^{\mu} + 2e^{\mu y_{N-2}} - e^{\mu x_{N-1}} e^{\mu y_{N-2}} - 1 =$$

$$= e^{\mu\cdot\frac{N-2}{N-1}} e^{\mu} - e^{\mu\cdot\frac{N-3}{N-1}} e^{\mu} + 2e^{\mu\cdot\frac{N-3}{N-1}} - e^{\mu\cdot\frac{N-2}{N-1}} e^{\mu\cdot\frac{N-3}{N-1}} - 1 =$$

$$= e^{\mu\cdot\frac{2N-3}{N-1}} - e^{\mu\cdot\frac{2N-4}{N-1}} + 2e^{\mu\cdot\frac{N-3}{N-1}} - e^{\mu\cdot\frac{2N-5}{N-1}} - 1. \tag{45}$$

With using the bisection method for approximating value μ_0, inequality (44) is easily confirmed for $N=4$ and $N=5$:

$$0.21 < h(\mu_0,\ 4) < 0.22 \ \text{ by } \ 1.4436354 < \mu_0 < 1.4436355, \tag{46}$$

$$0.15 < h(\mu_0,\ 5) < 0.16 \ \text{ by } \ 2.4375114 < \mu_0 < 2.4375115, \tag{47}$$

i. e. in the duels with four and five shooting moments by $\mu = \mu_0$ the penultimate and final time moments are only optimal. Next,

$$h(0,\ N) = e^0 e^0 - e^0 e^0 + 2e^0 - e^0 e^0 - 1 = 1 - 1 + 2 - 1 - 1 = 0 \tag{48}$$

and

$$\begin{gathered} h(N-1,\ N) = e^{(N-1)\cdot\frac{N-2}{N-1}} e^{N-1} - e^{(N-1)\cdot\frac{N-3}{N-1}} e^{N-1} + \\ +2e^{(N-1)\cdot\frac{N-3}{N-1}} - e^{(N-1)\cdot\frac{N-2}{N-1}} e^{(N-1)\cdot\frac{N-3}{N-1}} - 1 = \\ = e^{2N-3} - e^{2N-4} + 2e^{N-3} - e^{2N-5} - 1 = \\ = e^{N-3}\cdot\left(e^N - e^{N-1} + 2 - e^{N-2}\right) - 1 = \\ = e^{N-3}\cdot\left(e^{N-2}\cdot\left(e^2 - e - 1\right) + 2\right) - 1 > 0, \end{gathered} \tag{49}$$

and will show that

$$h(1,\ N) < 0 \ \ \forall\, N \geqslant 6. \tag{50}$$

Consider function

$$\begin{gathered} h(1,\ N) = e^{x_{N-1}} e - e^{y_{N-2}} e + 2e^{y_{N-2}} - e^{x_{N-1}} e^{y_{N-2}} - 1 = \\ = e^{\frac{2N-3}{N-1}} - e^{\frac{2N-4}{N-1}} + 2e^{\frac{N-3}{N-1}} - e^{\frac{2N-5}{N-1}} - 1. \end{gathered} \tag{51}$$

The first derivative of function (51) is

$$\begin{gathered} \frac{dh(1,\ N)}{dN} = \left(\frac{2}{N-1} - \frac{2N-3}{(N-1)^2}\right) e^{\frac{2N-3}{N-1}} - \left(\frac{2}{N-1} - \frac{2N-4}{(N-1)^2}\right) e^{\frac{2N-4}{N-1}} + \\ +2\cdot\left(\frac{1}{N-1} - \frac{N-3}{(N-1)^2}\right) e^{\frac{N-3}{N-1}} - \left(\frac{2}{N-1} - \frac{2N-5}{(N-1)^2}\right) e^{\frac{2N-5}{N-1}} = \\ = \frac{1}{(N-1)^2} e^{\frac{2N-3}{N-1}} - \frac{2}{(N-1)^2} e^{\frac{2N-4}{N-1}} + \frac{4}{(N-1)^2} e^{\frac{N-3}{N-1}} - \frac{3}{(N-1)^2} e^{\frac{2N-5}{N-1}} = \\ = \frac{1}{(N-1)^2}\left(e^{\frac{2N-3}{N-1}} - 2e^{\frac{2N-4}{N-1}} + 4e^{\frac{N-3}{N-1}} - 3e^{\frac{2N-5}{N-1}}\right). \end{gathered} \tag{52}$$

In the numerator of (52), denote functions

$$\varphi_1(N) = e^{\frac{2N-3}{N-1}} - 2e^{\frac{2N-4}{N-1}} \tag{53}$$

and

$$\varphi_2(N) = 4e^{\frac{N-3}{N-1}} - 3e^{\frac{2N-5}{N-1}}. \tag{54}$$

First, $\varphi_1(N) = 0$ when

$$e^{\frac{2N-3}{N-1}} = 2e^{\frac{2N-4}{N-1}},$$

whence

$$\frac{2N-3}{N-1} = \ln 2 + \frac{2N-4}{N-1},$$

$$\frac{1}{N-1} = \ln 2,$$

$$N = \frac{\ln 2 + 1}{\ln 2}, \tag{55}$$

where

$$2 < \frac{\ln 2 + 1}{\ln 2} < 3. \tag{56}$$

Besides,

$$\varphi_1(2) = e - 2 > 0 \tag{57}$$

and

$$\varphi_1(3) = e^{\frac{3}{2}} - 2e = e\left(\sqrt{e} - 2\right) < 0. \tag{58}$$

Therefore, inasmuch as function (53) has only one zero at (55) by (56) and it changes its sign from positive to negative due to (57) and (58),

$$\varphi_1(N) < 0 \ \ \forall\, N \geqslant 3. \tag{59}$$

Second, $\varphi_2(N) = 0$ when

$$4e^{\frac{N-3}{N-1}} = 3e^{\frac{2N-5}{N-1}},$$

whence

$$\ln 4 + \frac{N-3}{N-1} = \ln 3 + \frac{2N-5}{N-1},$$

$$\ln 4 - \ln 3 = \frac{N-2}{N-1},$$

$$N = \frac{2 - \ln 4 + \ln 3}{1 - \ln 4 + \ln 3}, \tag{60}$$

where

$$2 < \frac{2 - \ln 4 + \ln 3}{1 - \ln 4 + \ln 3} < 3. \tag{61}$$

Besides,

$$\varphi_2(2) = 4e^{-1} - 3e^{-1} = e^{-1} > 0 \tag{62}$$

and

$$\varphi_2(3) = 4 - 3\sqrt{e} < 0. \tag{63}$$

Therefore, inasmuch as function (54) has only one zero at (60) by (61) and it changes its sign from positive to negative due to (62) and (63),

$$\varphi_2(N) < 0 \ \ \forall\, N \geqslant 3. \tag{64}$$

Inequalities (59) and (64) imply that first derivative (52) of function (51) is negative for any integer N starting off 3. So, function (51) decreases $\forall\, N \geqslant 3$. Meanwhile,

$$h(1,\ 6) = e^{\frac{9}{5}} - e^{\frac{8}{5}} + 2e^{\frac{3}{5}} - e^{\frac{7}{5}} - 1 < 0,$$

i. e. inequality (50) is true. Hence, (48), (50), and (49) mean that function (45), considered for integer N starting off 6, must have at least a local minimum between $\mu = 0$ and $\mu = N - 1$. Generally speaking, function (45), considered as a function of μ at a fixed integer N, is a sum of four exponential functions and, therefore, according to the theorem of summing exponential functions [2, 4], it can have at most three extrema. One of those, being a local minimum, has just been widely localized. For estimating other possible extrema, the first partial derivative of function (45) with respect to variable μ is:

$$\frac{\partial h}{\partial \mu} = \frac{2N-3}{N-1} e^{\mu \cdot \frac{2N-3}{N-1}} - \frac{2N-4}{N-1} e^{\mu \cdot \frac{2N-4}{N-1}} + \frac{2N-6}{N-1} e^{\mu \cdot \frac{N-3}{N-1}} - \frac{2N-5}{N-1} e^{\mu \cdot \frac{2N-5}{N-1}}. \tag{65}$$

First partial derivative (65) at point $\mu = 0$ is

$$\left.\frac{\partial h}{\partial \mu}\right|_{\mu=0} = \frac{2N-3}{N-1} - \frac{2N-4}{N-1} + \frac{2N-6}{N-1} - \frac{2N-5}{N-1} = 0,$$

and thus $\mu = 0$ is a critical point of function (45) as a function of variable μ. The second partial derivative of function (45) with respect to variable μ is:

$$\frac{\partial^2 h}{\partial \mu^2} = \left(\frac{2N-3}{N-1}\right)^2 e^{\mu \cdot \frac{2N-3}{N-1}} - \left(\frac{2N-4}{N-1}\right)^2 e^{\mu \cdot \frac{2N-4}{N-1}} +$$

$$+2 \cdot \left(\frac{N-3}{N-1}\right)^2 e^{\mu \cdot \frac{N-3}{N-1}} - \left(\frac{2N-5}{N-1}\right)^2 e^{\mu \cdot \frac{2N-5}{N-1}}. \tag{66}$$

Second partial derivative (66) at point $\mu = 0$ is

$$\left.\frac{\partial^2 h}{\partial \mu^2}\right|_{\mu=0} = \left(\frac{2N-3}{N-1}\right)^2 - \left(\frac{2N-4}{N-1}\right)^2 + \frac{2 \cdot (N-3)^2}{(N-1)^2} - \left(\frac{2N-5}{N-1}\right)^2 =$$

$$= \frac{4N^2 - 12N + 9 - 4N^2 + 16N - 16 + 2N^2 - 12N + 18 - 4N^2 + 20N - 25}{(N-1)^2} =$$

$$= \frac{-2N^2 + 12N - 14}{(N-1)^2} = -\frac{2 \cdot \left(N - 3 - \sqrt{2}\right)\left(N - 3 + \sqrt{2}\right)}{(N-1)^2} < 0, \quad \forall\, N \geqslant 5,$$

and thus $\mu = 0$ is a maximum point of function (45) as a function of variable μ.

At the lower bound of the unique root of equation (21) function (45) is:

$$h\left(\mu_*,\ N\right) = h\left((N-1)\ln\left(2 \cdot \frac{N-2}{N-1}\right),\ N\right) =$$

$$= e^{\ln\left(2 \cdot \frac{N-2}{N-1}\right) \cdot (2N-3)} - e^{\ln\left(2 \cdot \frac{N-2}{N-1}\right) \cdot (2N-4)} +$$

$$+2e^{\ln\left(2 \cdot \frac{N-2}{N-1}\right) \cdot (N-3)} - e^{\ln\left(2 \cdot \frac{N-2}{N-1}\right) \cdot (2N-5)} - 1 =$$

$$= \left(2 \cdot \frac{N-2}{N-1}\right)^{2N-3} - \left(2 \cdot \frac{N-2}{N-1}\right)^{2N-4} +$$

$$+2 \cdot \left(2 \cdot \frac{N-2}{N-1}\right)^{N-3} - \left(2 \cdot \frac{N-2}{N-1}\right)^{2N-5} - 1 =$$

$$\begin{aligned}
&= \left(2\cdot\frac{N-2}{N-1}\right)^{N-3}\left[\left(2\cdot\frac{N-2}{N-1}\right)^{N}-\left(2\cdot\frac{N-2}{N-1}\right)^{N-1}+2-\left(2\cdot\frac{N-2}{N-1}\right)^{N-2}\right]-1=\\
&= \left(2\cdot\frac{N-2}{N-1}\right)^{N-3}\left[\left(2\cdot\frac{N-2}{N-1}\right)^{N-2}\left(\left(2\cdot\frac{N-2}{N-1}\right)^{2}-2\cdot\frac{N-2}{N-1}-1\right)+2\right]-1=\\
&= \left(2\cdot\frac{N-2}{N-1}\right)^{N-3}\left[\left(2\cdot\frac{N-2}{N-1}\right)^{N-2}\cdot\frac{2\cdot(N-2)(N-3)-(N-1)^{2}}{(N-1)^{2}}+2\right]-1=\\
&= \left(2\cdot\frac{N-2}{N-1}\right)^{N-3}\left[\left(2\cdot\frac{N-2}{N-1}\right)^{N-2}\cdot\frac{N^{2}-8N+11}{(N-1)^{2}}+2\right]-1.
\end{aligned}\tag{67}$$

As

$$N^2-8N+11\geqslant 0 \text{ by } N\in\left(-\infty;\ 4-\sqrt{5}\right]\cup\left[4+\sqrt{5};\ \infty\right)$$

and

$$6<4+\sqrt{5}<7,$$

where

$$2\cdot\frac{N-2}{N-1}>1\ \ \forall\, N\geqslant 4,$$

function (67) of variable N is positive $\forall\, N\geqslant 7$. In addition,

$$\begin{aligned}
h\left(\mu_*,\ 6\right) &= \left(2\cdot\frac{4}{5}\right)^{3}\left[\left(2\cdot\frac{4}{5}\right)^{4}\cdot\frac{36-48+11}{5^2}+2\right]-1=\\
&= \frac{2^9}{5^3}\cdot\left(-\frac{2^{12}}{5^6}+2\right)-1=\frac{11949723}{1953125}.
\end{aligned}$$

So,

$$h\left(\mu_*,\ N\right)>0\ \ \forall\, N\geqslant 6. \tag{68}$$

Hence, (48), (50), and (68) mean that function (45), considered for integer N starting off 6, must have at least a local minimum μ_{**} between points $\mu=0$ and (28). But suppose that this function has another local extremum $\mu^{(*)}$ by $\mu>0$. If $\mu^{(*)}\in(0;\ \mu_{**})$ then $\mu^{(*)}$ must be a local minimum, and there must be another local extremum (a local maximum) between points $\mu^{(*)}$ and μ_{**}. However, the case with such four local extrema is impossible due to function (45) of variable μ for $N\geqslant 4$ can have at most three extrema. Then suppose that this function, apart from local maximum $\mu=0$ and local minimum μ_{**}, has another local extremum $\mu^{(*)}$ by $\mu^{(*)}>\mu_{**}$. Then it must be a local maximum. But

$$\lim_{\mu\to\infty}h\left(\mu,\ N\right)=\lim_{\mu\to\infty}\left(e^{\mu\cdot\frac{2N-3}{N-1}}-e^{\mu\cdot\frac{2N-4}{N-1}}+2e^{\mu\cdot\frac{N-3}{N-1}}-e^{\mu\cdot\frac{2N-5}{N-1}}-1\right)=+\infty,$$

and hence local maximum $\mu^{(*)}$ must be followed by a local minimum, which is again impossible. Therefore, function (45) of variable μ for $N\geqslant 4$ has only two local extrema – the unique local maximum $\mu=0$ with local maximum value (48) and the unique local minimum

$$\mu_{**}\in\left(0;\ \mu_*\right)=\left(0;\ (N-1)\ln\left(2\cdot\frac{N-2}{N-1}\right)\right),$$

at which $h(\mu_{**},\ N) < 0$. Owing to this and to (68), function (45) of variable $\mu > 0$ for $N \geqslant 6$ has the single zero point between μ_{**} and μ_{*}. This, in its turn, implies that

$$h(\mu_0,\ N) > 0 \ \ \forall\, N \geqslant 6,$$

where the latter along with (46) and (47) implies that inequality (44) holds for integer N starting off 4. Therefore, inequality (42) holds for $N \in \mathbb{N} \backslash \{1,\ 2,\ 3\}$. $\square$

5. Conclusion

In 1-bullet silent duel (2) by (3), (1), (4) for (13), the starting moment is never optimal, and a subset of optimal time moments is:

$$T_3^*(\mu) = \{1\} \ \ \forall\, \mu > 0 \ \ \text{(Theorem 1)},$$

$$T_N^*(\mu) = \{1\} \ \ \text{for} \ \ N \in \mathbb{N} \backslash \{1,\ 2\} \ \ \text{and} \ \ \forall\, \mu > \mu_0 \ \ \text{(Theorem 2)},$$

$$T_N^*(\mu_0) = \left\{\frac{N-2}{N-1},\ 1\right\} \ \ \text{for} \ \ N \in \mathbb{N} \backslash \{1,\ 2,\ 3\} \ \ \text{(Theorem 4)},$$

where optimality or non-optimality of other time moments is yet to be determined in subsequent investigations. Optimality of penultimate and final time moments in 1-bullet silent duels with generalized exponentially-convex rewards is a result of steeper reward curve (12), which becomes more curvilinear as the number of shooting moments is increased. This ensues from estimation (37), according to which a greater N implies a higher factor of reward steepness, by which the final time moment is optimal. Compared to the linear-reward 1-bullet silent duel [14], which is solved in pure strategies only when $N \in \{3,\ 4,\ 5,\ 7\}$ and the duelist's best decision is to shoot at the duel span middle $\frac{1}{2}$ by the only exception of the 4×4 duel and its optimal third moment $t_3 = \frac{2}{3}$, 1-bullet silent duels with generalized exponentially-convex rewards can always have an optimal time moment by just setting a sufficiently high factor of reward steepness. At least, this is true for the final time moment and, by exactly hitting value μ_0 as the unique root of equation (21), for the penultimate time moment.

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ОПТИМАЛЬНІСТЬ ПЕРЕДОСТАННЬОГО ТА КІНЦЕВОГО МОМЕНТІВ ЧАСУ В ОДНОПОСТРІЛОВІЙ БЕЗШУМНІЙ ДУЕЛІ З УЗАГАЛЬНЕНИМИ ЕКСПОНЕНЦІАЛЬНО-ОПУКЛИМИ ВИНАГОРОДАМИ

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Безшумні дуелі – це клас ігор на вибір моменту дії, які моделюють конкурентну взаємодію між учасниками протягом заданого інтервалу часу за умова інформаційної невизначеності та обмеженості винагороди. Обставини взаємодії такі, що учасник не дізнається про дії інших до завершення дуелі та отримує вигоду від якнайпізнішої дії, але лише за умови, що діє першим. Розглядається скінченна однопострілова безшумна дуель, у якій часовий інтервал рівновіддалено квантизований, а кожен із двох дуелянтів має узагальнену експоненціально-опуклу функцію винагороди. У статті досліджується оптимальність передостаннього та кінцевого моментів часу, що є особливо важливими для моделювання систем, у яких дії (рішення) змушено приймаються якомога пізніше. Дуель є симетричною матричною грою з нульовим значенням, причому обидва дуелянти мають однакову оптимальну поведінку як у чистих, так і в змішаних стратегіях. Доведено, що в 3 × 3-дуелях кінцевий момент часу є єдиним оптимальним незалежно від значення параметра крутизни винагороди. Також встановлено, що в дуелях більшої розмірності кінцевий момент часу є єдиним оптимальним, якщо параметр крутизни винагороди перевищує єдиний корінь певного експоненціального рівняння. Якщо ж параметр дорівнює цьому кореню, оптимальними стають як передостанній, так і кінцевий моменти часу.

Ключові слова: однопострілова безшумна дуель, експоненціально-опукла винагорода, матрична гра, оптимальний момент часу, передостанній момент, кінцевий момент.