UDC 519.6 http://dx.doi.org/10.30970/vam.2024.34.00000

A COMPARISON BETWEEN THREE METHODS OF ORDER SIX USING SIMILAR INFORMATION

I. Argyros, S. Shakhno, M. Havdiak, M. Argyros

[1]*Cameron University, Lawton, 73505, OK, USA,*
e-mail: iargyros@cameron.edu,
[2]*Ivan Franko National University of Lviv,*
1, Universytetska str., 79000, Lviv, Ukraine,
e-mail: stepan.shakhno@lnu.edu.ua, mykhailo.havdiak@lnu.edu.ua,
[2]*College of Computing and Engineering, Nova Southeastern University,*
Florida, 33328, USA

This study provides a comparative analysis of three sixth-order iterative methods for solving nonlinear equations. Unlike many existing approaches that rely on higher-order derivatives, this work develops an optimized local convergence analysis based on the operator F and its derivative F'. More significantly, introduce a semi-local convergence analysis for these methods using majorizing sequences, which provides a practical framework for establishing convergence based on the conditions at the initial guess. Both local and semi-local convergence analysis are performed within the context of Banach spaces. This approach enhances the theoretical robustness and practical applicability of the methods. To validate the theoretical findings, a series of numerical experiments is conducted on various standard benchmark problems, including small- to large-scale nonlinear systems. The performance of the methods is compared against the classical Newton's method. The findings confirm that the sixth-order methods consistently outperform Newton's method in terms of the total number of iterations. We analyze the Computational Order of Convergence (COC) and Approximate Computational Order of Convergence (ACOC), which empirically confirm the high convergence order of the proposed methods. Furthermore, we investigate the numerical stability and performance under high-precision requirements, utilizing arbitrary-precision arithmetic to solve problems where standard double precision fails. These results underscore the practical advantages and theoretical robustness of the methods. The methodology presented can be applied to other similar iterative methods.

Key words: iterative methods, Banach space, Fréchet derivative, local convergence, semi-local convergence, nonlinear equations.

1. Introduction

A wide range of problems arising in mathematics, computational science and various applied fields can be effectively reformulated through mathematical modeling as a nonlinear equation of the form

$$F(x) = 0, \tag{1}$$

where $F : D \subset X \longrightarrow Y$ is a Fréchet-differentiable operator. Here, X, Y represent Banach spaces and the domain D is assumed to be an open, convex and nonempty subset of X. The goal is to find a solution $x^* \in D$ that satisfies the equation. In general, obtaining an analytical solution is highly nontrivial and possible only in exceptional cases. As a result, considerable effort is dedicated to the design and analysis of iterative methods that approximate the solution under suitable assumptions on the operator and the initial guess, ensuring convergence to the desired solution [1–7]. This problem provides a unified framework for modeling and solving many real-world problems.

There are differences among certain types of convergence. In local convergence, the conditions are imposed on the solution and a convergence ball is identified around it. This reflects the difficulty in choosing a suitable initial approximation, especially since the solution is usually unknown. On the other hand, in semi-local convergence, the conditions are imposed on the initial approximation itself and the solution is then shown to exist within a ball centered at x_0 [16].

One of the most fundamental and widely studied iterative methods for solving nonlinear equations of the form $F(x) = 0$ is Newton's method. Given an initial guess x_0, Newton's method generates a sequence defined by

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n),\ n = 0, 1, 2, ... \quad , \tag{2}$$

under the assumption that F' is invertible near the solution. Newton's method is known for its quadratic convergence when the initial approximation is sufficiently close to the exact solution and F is sufficiently smooth. However, it may fail to converge in certain conditions, which motivates the development of higher-order or more robust variants [8–13]. Several such methods have been proposed in [15, 17, 24, 25].

We now consider the following iterative methods, which generalize Newton's approach. These are defined for an initial approximation $x_0 \in D$ and for each iteration index $n = 0, 1, 2, ...$, the sequences $\{x_n\}$ are generated as follows:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ z_n &= x_n - \frac{1}{2}(F'(x_n)^{-1} + F'(y_n)^{-1})F(x_n), \\ x_{n+1} &= z_n - \frac{1}{2}(F'(x_n)^{-1} + F'(y_n)^{-1}F'(x_n)F'(y_n)^{-1})F(z_n), \end{aligned} \tag{3}$$

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \quad A_n = F'(x_n) + F'(y_n), \\ z_n &= x_n - 2A_n^{-1}F(x_n), \\ x_{n+1} &= z_n - \frac{1}{2}(7I - 8F'(x_n)^{-1}F'(y_n) + 3(F'(x_n)^{-1}F'(y_n))^2)F'(x_n)^{-1}F(z_n), \end{aligned} \tag{4}$$

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \quad E_n = F'(x_n) - 3F'(y_n), \\ z_n &= y_n + \frac{1}{3}(F'(x_n)^{-1} + 2E_n^{-1})F(x_n), \\ x_{n+1} &= z_n + \frac{1}{3}(-F'(x_n)^{-1} + 4E_n^{-1})F(z_n). \end{aligned} \tag{5}$$

High-order iterative methods were proposed in [14], [17], [23]. Despite the significant progress in developing high-order iterative methods for solving nonlinear equations, several theoretical and practical limitations remain:

(P_1) Existing convergence results rely on Taylor series expansions and require higher-order derivatives, in this case $F^{(6)}$, limiting their applicability to problems where such derivatives are unavailable or costly to compute. In contrast, the methods analyzed in this paper require only the operator F and its first derivative F'.

(P_2) Computable a priori estimates for the error bounds $\|x_n - x^*\|$ are not provided, making it difficult to predict the number of iterations needed to achieve a desired level of accuracy.

(P_3) Traditional local convergence theory assumes knowledge of the exact solution x^*, which is typically unknown in practice. Moreover, the more important and challenging semi-local convergence, which has not been studied previously, is developed using majorizing sequences. Both analyses are presented in a more general setting of a Banach space and also rely on the concept of generalized continuity used to control the derivative F' and sharpen the error bounds $\|x^* - x_n\|$ and $\|x_{n+1} - x_n\|$. The same technique can be used to extend the applicability of other methods using the same schema.

In this paper, we present a comparative analysis of three sixth-order iterative methods for solving nonlinear equations. Section 2 provides a local convergence analysis for each method. In Section 3, we extend the analysis to the semi-local convergence framework. Section 4 presents a series of numerical experiments that evaluate the performance of the proposed methods in comparison with Newton's method. Finally, Section 5 concludes the study with a summary of findings and remarks.

2. Local Convergence

In this section, we analyze the local convergence behavior of the three sixth-order iterative methods. For each method, we derive a sequence of error bounds using a majorizing function approach, which allows us to establish convergence without requiring higher-order derivatives. The convergence analysis is carried out in the setting of Banach spaces.

2.1. Analysis of method (3)

We begin by defining the majorant function $g_1(t)$, which provides an upper bound on the error at the first intermediate step y_n

$$g_1(t) = \frac{\int_0^1 w((1-\theta)t)d\theta}{1 - w_0(t)},$$

where w_0, w are continuous and nondecreasing functions. Using this, define $\overline{w}$ that will be used in subsequent steps

$$\overline{w}(t) = \begin{cases} w((1 + g_1(t))t) \\ \quad \text{or} \\ w_0(t) + w_0(g_1(t)t). \end{cases}$$

Next, we derive the second majorant function $g_2(t)$, which captures the error behavior at the second intermediate step z_n

$$g_2(t) = \frac{\int_0^1 w((1-\theta)t)d\theta}{1 - w_0(t)} + \frac{\overline{w}(t)(1 + \int_0^1 w_0(\theta t)d\theta)}{2(1 - w_0(t))(1 - w_0(g_1(t)t))}.$$

The third majorant function $g_3(t)$ is then constructed to describe the error at the final step x_{n+1}

$$\overline{\overline{w}}(t) = \begin{cases} w((1+g_2(t))t) \\ \quad \text{or} \\ w_0(t) + w_0(g_2(t)t), \end{cases}$$

$$\overline{\overline{\overline{w}}}(t) = \begin{cases} w((g_1(t)+g_2(t))t) \\ \quad \text{or} \\ w_0(g_1(t)t) + w_0(g_2(t)t), \end{cases}$$

$$g_3(t) = \left[\frac{\int_0^1 w((1-\theta)g_2(t)t)d\theta}{1-w_0(g_2(t)t)} + \frac{\overline{\overline{w}}(t)(1+\int_0^1 w_0(\theta g_2(t)t)d\theta)}{2(1-w_0(t))(1-w_0(g_2(t)t))} \right.$$
$$\left. + \frac{\overline{\overline{\overline{w}}}(t)(1+\int_0^1 w_0(\theta g_2(t)t)d\theta)}{2(1-w_0(g_1(t)t))(1-w_0(g_2(t)t)))} \quad + \frac{\overline{w}(t)(1+\int_0^1 w_0(\theta g_2(t)t)d\theta)}{2(1-w_0(t))(1-w_0(g_1(t)t))} \right] g_2(t).$$

The justification for these majorant functions is provided by the following step-by-step error analysis. The error at each substep of the iterative method is bounded by a majorant function of the error from the previous step. This process confirms that the method systematically reduces the distance to the solution at each iteration. The analysis uses properties of the Fréchet derivative, controlled by the functions w_0 and w, along with a lemma on invertible operators to establish invertibility and bound operator norms.
We obtain the error for the first intermediate step, $y_n - x^*$, as:

$$y_n - x^* = x_n - x^* - F'(x_n)^{-1}F(x_n),$$

The norm of this error is bounded by the majorant function g_1

$$\|y_n - x^*\| \le \frac{\int_0^1 w((1-\theta)\|x_n - x^*\|)d\theta \|x_n - x^*\|}{1 - w_0(\|x_n - x^*\|)}$$
$$\le g_1(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| < r.$$

At the next step, we get

$$z_n - x^* = x_n - x^* - F'(x_n)^{-1}F(x_n) - \frac{1}{2}(F'(y_n)^{-1} - F'(x_n)^{-1})F(x_n)$$
$$= x_n - x^* - F'(x_n)^{-1}F(x_n) - \frac{1}{2}F'(y_n)^{-1}(F'(x_n) - F'(y_n))F'(x_n)^{-1}F(x_n),$$
$$\|z_n - x^*\| \le \left[\frac{\int_0^1 w((1-\theta)\|x_n - x^*\|)\, d\theta}{1 - w_0(\|x_n - x^*\|)} \right.$$
$$\left. + \frac{\overline{w}_n \left(1 + \int_0^1 w_0(\theta\|x_n - x^*\|)\, d\theta\right)}{2(1 - w_0(\|x_n - x^*\|))(1 - w_0(\|y_n - x^*\|))} \right] \|x_n - x^*\|$$
$$\le g_2(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\|.$$

Finally, we analyze the error at the final step of the iteration, $x_{n+1} - x^*$

$$\begin{aligned} x_{n+1} - x^* &= z_n - x^* - F'(z_n)^{-1}F(z_n) + F'(z_n)^{-1}F(z_n) \\ &-\frac{1}{2}F'(x_n)^{-1}F(z_n) - \frac{1}{2}F'(y_n)^{-1}(F'(x_n) - F'(y_n) + F'(y_n))F'(y_n)^{-1}F(z_n) \\ &= z_n - x^* - F'(z_n)^{-1}F(z_n) + \frac{1}{2}(F'(z_n)^{-1} - F'(x_n)^{-1})F(z_n) \\ &+\frac{1}{2}(F'(z_n)^{-1} - F'(y_n)^{-1})F(z_n) \\ &-\frac{1}{2}F'(y_n)^{-1}(F'(x_n) - F'(y_n))F'(y_n)^{-1}F(z_n). \end{aligned}$$

The norm of this final error is bounded by the majorant function g_3

$$\begin{aligned} \|x_{n+1} - x^*\| \leq &\left[\frac{\int_0^1 w((1-\theta)\|z_n - x^*\|)d\theta}{1 - w_0(\|z_n - x^*\|)}\right. \\ &+\frac{\overline{\overline{w}}_n(1 + \int_0^1 w_0(\theta\|z_n - x^*\|)d\theta)}{2(1 - w_0(\|x_n - x^*\|))(1 - w_0(\|z_n - x^*\|))} \\ &+\frac{\overline{\overline{\overline{w}}}_n(1 + \int_0^1 w_0(\theta\|z_n - x^*\|)d\theta)}{(1 - w_0(\|y_n - x^*\|))(1 - w_0(\|z_n - x^*))} \\ &\left.+\frac{\overline{w}_n(1 + \int_0^1 w_0(\theta\|z_n - x^*\|)d\theta)}{2(1 - w_0(\|x_n - x^*\|))(1 - w_0(\|y_n - x^*\|))}\right]\|z_n - x^*\| \\ &\leq g_3(\|x_n - x^*\|)\|x_n - x^*\|. \end{aligned}$$

These bounds demonstrate that the method reduces the distance to the solution at each iteration.

2.2. Analysis of method (4)

The first majorant function $g_1(t)$ remains the same as defined in method (3). It characterizes the behavior of the error at the intermediate step y_n.
To proceed, we define an additional function

$$p(t) = \frac{1}{2}(w_0(t) + w_0(g_1(t)t)).$$

Using $p(t)$, we construct the second majorant function $g_2(t)$

$$g_2(t) = \frac{\int_0^1 w((1-\theta)t)d\theta}{1 - w_0(t)} + \frac{\overline{w}(t)(1 + \int_0^1 w_0(\theta t)d\theta)}{2(1 - w_0(t))(1 - p(t))}.$$

The third majorant function $g_3(t)$ estimates the error at the updated step x_{n+1}

$$\begin{aligned} g_3(t) = &\left[\frac{\int_0^1 w((1-\theta)g_2(t)t)d\theta}{1 - w_0(g_2(t)t)} + \frac{\overline{\overline{w}}(t)(1 + \int_0^1 w_0(\theta g_2(t)t)d\theta)}{(1 - w_0(t))(1 - w_0(g_2(t)t))}\right. \\ &\left.+\frac{1}{2}\frac{\overline{w}(t)}{1 - w_0(t)}\left(3\frac{\overline{w}(t)}{1 - w_0(t)} + 2\right)\frac{(1 + \int_0^1 w_0(\theta t)d\theta)}{1 - w_0(t)}\right]g_2(t). \end{aligned}$$

Analysis proceeds by estimating the distance between the current approximation x_n and the exact solution x^* through the intermediate points y_n and z_n. For example, the difference $z_n - x^*$ is expressed in terms of known x_n, y_n and the derivative operator F'

$$z_n - x^* = x_n - x^* - F'(x_n)^{-1}F(x_n) + (F'(x_n)^{-1} - 2A_n^{-1})F(x_n)$$

$$= x_n - x^* - F'(x_n)^{-1}F(x_n) - A_n^{-1}(F'(x_n) - F'(y_n))F'(x_n)^{-1}F(x_n).$$

The motivational calculations for functions g_2 and g_3 are:

$$||z_n - x^*|| \leq [\frac{\int_0^1 w((1-\theta)||x_n - x^*||)d\theta}{1 - w_0(||x_n - x^*||)} + \frac{\overline{w}_n(1 + \int_0^1 w_0(\theta||x_n - x^*||)d\theta)}{2(1 - w_0(||x_n - x^*||))(1 - p_n)}]\|x_n - x^*\|$$

$$\leq g_2(||x_n - x^*||) \cdot ||x_n - x^*|| \leq ||x_n - x^*||,$$

where we also used the estimate

$$||(2L)^{-1}(A_n - 2L)|| \leq \frac{1}{2}(||L^{-1}(F'(x_n) - L)|| + ||L^{-1}(F'(y_n) - L)||)$$

$$\leq \frac{1}{2}(w_0(||x_n - x^*||) + w_0(||y_n - x^*||)) = p_n < 1.$$

Therefore, according to Banach lemma [20]

$$||A_n^{-1}L|| \leq \frac{1}{2(1 - p_n)}.$$

This leads to the following bound for the next iterate

$$x_{n+1} - x^* = z_n - x^* - F'(z_n)^{-1}F(z_n) + (F'(z_n)^{-1} - F'(x_n)^{-1})F(z_n)$$

$$-\frac{1}{2}(5I - 8F'(x_n)^{-1}F'(y_n) + 3(F'(x_n)^{-1}F'(y_n))^2)F'(x_n)^{-1}F(z_n)$$

$$= z_n - x^* - F'(z_n)^{-1}F(z_n) + (F'(z_n)^{-1} - F'(x_n)^{-1})F(z_n)$$

$$-\frac{1}{2}(F'(x_n)^{-1}F'(y_n) - I)(3(F'(x_n)^{-1}F'(y_n) - I) - 2I)F'(x_n)^{-1}F(z_n),$$

$$||x_{n+1} - x^*|| \leq \Bigg[\frac{\int_0^1 w((1-\theta)||z_n - x^*||)d\theta}{1 - w_0(||z_n - x^*||)} + \frac{\overline{\overline{w}}_n(1 + \int_0^1 w_0(\theta||z_n - x^*||)d\theta)}{(1 - w_0(||x_n - x^*||))(1 - w_0(||z_n - x^*||))}$$

$$+\frac{1}{2}\frac{\overline{w}_n}{1 - w_0(||x_n - x^*||)}\left(\frac{3\overline{w}_n}{1 - w_0(||x_n - x^*||)} + 2\right)$$

$$\times \frac{(1 + \int_0^1 w_0(\theta||z_n - x^*||)d\theta)}{1 - w_0(||z_n - x^*||)}\Bigg] \cdot ||z_n - x^*||$$

$$\leq g_3(||x_n - x^*||) \cdot ||x_n - x^*|| \leq ||x_n - x^*||.$$

2.3. Analysis of method (5)

The function $c(t)$ is defined and then used to construct the second and third majorant functions $g_2(t)$, $g_3(t)$

$$c(t) = \frac{1}{2}(3w_0(g_1(t)t) + w_0(t)),$$
$$g_2(t) = g_1(t) + \frac{1}{3}\left(\frac{1}{1-w_0(t)} + \frac{2}{1-c(t)}\right)\left(1 + \int_0^1 w_0(\theta t)d\theta\right),$$
$$g_3(t) = \left[g_2(t) + \frac{1}{3}\left(\frac{1}{1-w_0(t)} + \frac{4}{1-c(t)}\right)\left(1 + \int_0^1 w_0(\theta g_2(t)t)d\theta\right)\right] g_2(t).$$

The underlying calculations are:

$$z_n - x^* = y_n - x^* + \frac{1}{3}(F'(x_n)^{-1} + 2E_n^{-1})F(x_n),$$
$$\|z_n - x^*\| \le \left[g_1(\|x_n - x^*\|) + \frac{1}{3}\left(\frac{1}{1 - w_0(\|x_n - x^*\|)}\right.\right.$$
$$\left.\left. + \frac{1}{1-c_n}\right)\left(1 + \int_0^1 w_0(\theta\|x_n - x^*\|)\, d\theta\right)\right] \|x_n - x^*\|$$
$$\le g_2(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\|.$$

Next, the error at the final step, $x_{n+1} - x^*$, is expressed as

$$x_{n+1} - x^* = z_n - x^* + \frac{1}{3}(-F(x_n) + 4E_n^{-1})F(z_n),$$
$$\|x_{n+1} - x^*\| \le \left[g_2(\|x_n - x^*\|) + \frac{1}{3}\left(\frac{1}{1 - w_0(\|x_n - x^*\|)}\right.\right.$$
$$\left.\left. + \frac{2}{1-c_n}\right)\left(1 + \int_0^1 w_0(\theta\|z_n - x^*\|)\, d\theta\right)\right] \|z_n - x^*\|$$
$$\le g_3(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\|,$$

where we also used the estimates [20]:

$$\|(2L)^{-1}(E_n - 2L)\| \le \frac{1}{3}(3w_0(\|y_n - x^*\|) + w_0(\|x_n - x^*\|)) = c_n < 1,$$
$$\|E_n^{-1}L\| \le \frac{1}{2(1-c_n)}.$$

3. Semi-local convergence

In this section, we analyze the semi-local convergence of the methods where conditions are now imposed on the initial approximation. Consequently x^*, w_0, w are replaced by x_0, v_0, v respectively.

3.1. Analysis of method (3)

For this method, the majorant sequence $\{\alpha_n\}$ is defined for

$$\alpha_0 = 0, b_0 \ge \|F'(x_0)^{-1}F(x_0)\|$$

as follows for each $n = 0, 1, 2, \dots$. First, we define a sequence $\overline{v}_n$ and an intermediate sequence γ_n as follows:

$$\overline{v}_n = \begin{cases} v(b_n - \alpha_n) \\ \quad \text{or} \\ v_0(\alpha_n) + v_0(b_n), \end{cases}$$
$$\gamma_n = b_n + \frac{\overline{v}_n(b_n - \alpha_n)}{2(1 - v_0(b_n))}.$$

Next, we define λ_n, which will serve as a bound for the norm of $L^{-1}F(z_n)$

$$\lambda_n = \int_0^1 v((1-\theta)(\gamma_n - \alpha_n))d\theta(\gamma_n - \alpha_n) + (1 + v_0(\alpha_n))(\gamma_n - b_n),$$
$$\alpha_{n+1} = \gamma_n + \frac{1}{2}\left(\frac{1}{1 - v_0(\alpha_n)} + \frac{1}{1 - v_0(b_n)} + \frac{\overline{v}_n}{(1 - v_0(b_n))^2}\right)\lambda_n,$$
$$\mu_{n+1} = \int_0^1 v((1-\theta)(\alpha_{n+1} - \alpha_n))d\theta(\alpha_{n+1} - \alpha_n) + (1 + v_0(\alpha_n))(\alpha_{n+1} - b_n),$$
$$b_{n+1} = \alpha_{n+1} + \frac{\mu_{n+1}}{1 - v_0(\alpha_{n+1})}.$$

The motivational calculations are

$$z_n - y_n = \left(F'(x_n)^{-1} - \frac{1}{2}(F'(x_n)^{-1} + F'(y_n)^{-1})\right)F(x_n)$$
$$= -\frac{1}{2}F'(y_n)^{-1}(F'(x_n) - F'(y_n))F'(x_n)^{-1}F(x_n),$$
$$||z_n - y_n|| \le \frac{\overline{v}_n(b_n - \alpha_n)}{2(1 - v_0(b_n))} = \gamma_n - b_n.$$

Next, we analyze the function value at z_n

$$F(z_n) = F(z_n) - F(x_n) - F'(x_n)(y_n - x_n)$$
$$= F(z_n) - F(x_n) - F'(x_n)(z_n - x_n) + F'(x_n)(z_n - y_n).$$

The norm of $L^{-1}F(z_n)$ is bounded by λ_n

$$||L^{-1}F(z_n)|| \le \int_0^1 v((1-\theta)(\gamma_n - \alpha_n))d\theta(\gamma_n - \alpha_n) + (1 + v_0(\alpha_n))(\gamma_n - b_n) = \lambda_n,$$

$$x_{n+1} - z_n = -\frac{1}{2}F'(x_n)^{-1}F(z_n) - \frac{1}{2}F'(y_n)^{-1}(F'(x_n) - F'(y_n))F'(y_n)^{-1}F(z_n)$$
$$-\frac{1}{2}F'(y_n)^{-1}F(z_n),$$

$$||x_{n+1} - z_n|| \le \frac{1}{2}\left(\frac{1}{1 - v_0(\alpha_n)} + \frac{1}{1 - v_0(\beta_n)} + \frac{\overline{v}_n}{(1 - v_0(\beta_n))^2}\right)\lambda_n$$
$$= \alpha_{n+1} - \gamma_n.$$

Following this, we analyze the function value at the new iterate x_{n+1}

$$\begin{aligned} F(x_{n+1}) &= F(x_{n+1}) - F(x_n) - F'(x_n)(y_n - x_n) \\ &= F(x_{n+1}) - F(x_n) - F'(x_n)(x_{n+1} - x_n) + F'(x_n)(x_{n+1} - y_n), \\ ||L^{-1}F(x_{n+1})|| &\le \int_0^1 v((1-\theta)(\alpha_{n+1} - \alpha_n))d\theta(\alpha_{n+1} - \alpha_n) \\ &+ (1 + v_0(\alpha_n))(\alpha_{n+1} - b_n) = \mu_{n+1}, \\ ||y_{n+1} - x_{n+1}|| &\le ||F'(x_{n+1})^{-1}L|| \cdot ||L^{-1}F(x_{n+1})|| \\ &\le \frac{\mu_{n+1}}{1 - v_0(\alpha_{n+1})} = b_{n+1} - \alpha_{n+1}. \end{aligned}$$

3.2. Analysis of method (4)

The majorant sequence $\{\alpha_n\}$ is defined for

$$\alpha_0 = 0, b_0 \ge ||F'(x_0)^{-1}F(x_0)||$$

and each $n = 0, 1, 2, ...$ by

$$\begin{aligned} q_n &= \frac{1}{2}(v_0(\alpha_n) + v_0(b_n)), \\ \gamma_n &= b_n + \frac{\overline{v}_n(b_n - \alpha_n)}{2(1 - q_n)}, \end{aligned}$$

$$\begin{aligned} \alpha_{n+1} &= \gamma_n + \left(1 + \frac{1}{2}\frac{\overline{v}_n}{1 - v_0(\alpha_n)}\left(3\frac{\overline{v}_n}{1 - v_0(\alpha_n)} + 2\right)\right)\frac{\lambda_n}{1 - v_0(\alpha_n)}, \\ b_{n+1} &= \alpha_{n+1} + \frac{\mu_{n+1}}{1 - v_0(\alpha_{n+1})}. \end{aligned}$$

The motivational calculations supporting these definitions are as follows:

$$\begin{aligned} z_n - y_n &= -A_n^{-1}(F'(x_n) - F'(y_n))F'(x_n)^{-1}F(x_n), \\ \|z_n - y_n\| &\le \frac{\overline{v}_n(b_n - \alpha_n)}{2(1 - q_n)} = \gamma_n - b_n, \\ x_{n+1} - z_n &= -F'(x_n)^{-1}F(z_n) \\ &- \frac{1}{2}(F'(x_n)^{-1}F'(y_n) - I)[3(F'(x_n)^{-1}F'(y_n) - I) - 2I]F'(x_n)^{-1}F(z_n), \\ \|x_{n+1} - z_n\| &\le \left(1 + \frac{1}{2}\frac{\overline{v}_n}{1 - v_0(\alpha_n)}\left(3\frac{\overline{v}_n}{1 - v_0(\alpha_n)} + 2\right)\right)\frac{\lambda_n}{1 - v_0(\alpha_n)} \\ &= \alpha_{n+1} - \gamma_n. \end{aligned}$$

The computation for b_{n+1} is the same as in method (3).

3.3. Analysis of method (5)

We define the majorant sequence $\{\alpha_n\}$ with initial value $\alpha_0 = 0$ and

$$b_0 \geq ||F'(x_0)^{-1}F(x_0)||,$$

such that for each $n = 0, 1, 2, ...$, the sequence satisfies:

$$\begin{aligned}
d_n &= \frac{1}{2}(3v_0(b_n) + v_0(\alpha_n)),\\
\gamma_n &= b_n + \left(\frac{1 + v_0(\alpha_n)}{3(1 - d_n)} + \frac{1}{3}\right)(b_n - \alpha_n),\\
\alpha_{n+1} &= \gamma_n + \frac{1}{3}\left(\frac{1}{1 - v_0(\alpha_n)} + \frac{2}{1 - d_n}\right)\lambda_n,\\
b_{n+1} &= \alpha_{n+1} + \frac{\mu_{n+1}}{1 - v_0(\alpha_{n+1})}.
\end{aligned}$$

The motivational calculations are provided to justify the sequence:

$$\begin{aligned}
z_n - y_n &= \frac{2}{3}E_n^{-1}F(x_n) + \frac{1}{3}F'(x_n)^{-1}F(x_n),\\
\|z_n - y_n\| &\leq \left(\frac{1 + v_0(\alpha_n)}{3(1 - d_n)} + \frac{1}{3}\right)(b_n - \alpha_n) = \gamma_n - b_n,\\
x_{n+1} - z_n &= \frac{1}{3}(-F'(x_n)^{-1} + 4E_n^{-1})F(z_n),\\
\|x_{n+1} - z_n\| &\leq \frac{1}{3}\left(\frac{1}{1 - v_0(\alpha_n)} + \frac{2}{1 - d_n}\right)\lambda_n = \alpha_{n+1} - \gamma_n,
\end{aligned}$$

where we also used the estimates

$$\begin{aligned}
F(x_n) &= F'(x_n)(y_n - x_n),\\
\|L^{-1}F(x_n)\| &\leq \|L^{-1}((F'(x_n) - L) + L)\|\|y_n - x_n\| \leq (1 + v_0(\alpha_n))(b_n - \alpha_n).
\end{aligned}$$

The iteration b_{n+1} and the rest of the convergence analysis are the same as in the previous chapters.

4. Numerical Examples

In this section, we present numerical experiments to demonstrate the performance and effectiveness of the three sixth-order methods analyzed previously: Methods (3), (4) and (5).

We apply the methods to several nonlinear systems of equations, commonly used as benchmarks in the literature. The stopping criterion is set based on a fixed tolerance for the residual norm or a maximum number of iterations. For all numerical experiments, the iteration process was terminated when $\|F(x_n)\| < \varepsilon$, with ε varying between 10^{-9} to 10^{-20}. The number of iterations was limited to 100. All computations were performed using Python 3.11.0 with the mpmath library for arbitrary-precision arithmetic. The experiments ran on a machine with a 2.4 GHz 8-core Intel Core i9 processor.

The following problems are used to evaluate the performance of the proposed methods [14, 21]:

- System of 3 equations

$$\begin{cases} 10v_1 + \sin(v_1 + v_2) - 1 = 0, \\ 8v_2 - \cos^2(v_3 - v_2) - 1 = 0, \\ 12v_3 + \sin(v_3) - 1 = 0, \end{cases} \tag{6}$$

where initial approximation $v_0 = (v_1, v_2, v_3)^T = (0, 1, 0)^T$. Solution is in the point $v^* = (0.069..., 0.246..., 0.0769...)^T$.

- System of 200 nonlinear equations

$$f_i(v) = \sum_{j=1, j\neq i}^{200} v_j - e^{-v_i} = 0, \quad 1 \le i \le 200, \tag{7}$$

with initial value $(\frac{3}{2}, \frac{3}{2}, \ldots, \frac{3}{2})^T$ and solution $v^* = (0.0050..., ..., 0.0050...)^T$.

- System of 10 equations

$$f_i(v) = \tan^{-1}(v_i) + 1 - 2 \sum_{j=1, j\neq i}^{10} v_j^2, \quad 1 \le i \le 10. \tag{8}$$

Solution is $v^* = (0.2644, ..., 0.2644)^T$ and initial approximation $(0.7, ..., .0.7)^T$.

- A large-scale system with 500 nonlinear equations

$$\begin{cases} v_i^2 v_{i+1} - 1 = 0, \quad 1 \le i \le 499, \\ v_{500}^2 v_1 - 1 = 0, \quad i = 500. \end{cases} \tag{9}$$

Chosen initial value is $v_0 = (1.8, ..., 1.8)^T$ and solution is $v^* = (1, ..., 1)^T$.

- Broyden tridiagonal function

$$\begin{gathered} f_i(v) = (3 - 2v_i)v_i - v_{i-1} - 2v_{i+1} + 1, \\ v_0 = v_{n+1} = 0, \\ n = 20 \end{gathered} \tag{10}$$

with initial approximation $v_0 = (-10, ..., -10)^T$.

- A large-scale exponential function:

$$\begin{cases} f_i(v) = 3v_i^3 + 2v_{i+1} - 5 + \sin^2(v_i) - \sin^2(v_{i+1}), \quad i = \overline{0, n-2}, \\ f_{n-1}(v) = 3v_{n-1}^3 + 2v_0 - 5 + \sin^2(v_{n-1}) - \sin^2(v_0). \end{cases} \tag{11}$$

- Powell singular function:

$$\begin{gathered} f_1(v) = v_1 + 10v_2, \\ f_2(v) = 5^{1/2}(v_3 - v_4), \\ f_3(v) = (v_2 - 2v_3)^2, \\ f_4(v) = 10^{1/2}(v_1 - v_4)^2. \end{gathered} \tag{12}$$

Initial approximation is $v_0 = (2, 2, 2, 1)^T$ and solution $v^* = (0, 0, 0, 0)^T$.

- Brown almost linear function:

$$f_i(v) = v_i + \sum_{j=0}^{n-1} v_j - (n+1), \quad i = \overline{0, n-2}, \tag{13}$$

$$f_{n-1}(v) = \left(\prod_{j=0}^{n-1} v_j\right) - 1, \quad v^* = (1, ..., 1)^T.$$

- Chandrasekhar equation:

$$f_i(v) = v_i - \left(1 - \frac{c}{2n}\sum_{j=0}^{n-1} \frac{\mu_j v_j}{\mu_i + \mu_j}\right)^{-1}, \quad c = 0.9. \tag{14}$$

Example 1. *Performance of methods with $\varepsilon = 10^{-9}$ and $\varepsilon = 10^{-15}$.*

Each method was applied to the list of problems and the corresponding iteration counts are presented in Table 1 and Table 2.

Table 1

Iteration counts for various problems with tolerance $\varepsilon = 10^{-9}$

Problem	x_0	**n**	**Iterations**			
			(3)	**(4)**	**(5)**	**Newton (2)**
System of 3 equation (6)	$(0.0, 1.0, 0.0)$	3	2	2	2	3
System of 10 equations (8)	$(0.7, \dots, 0.7)$	10	2	3	3	5
System of 200 equations (7)	$(1.5, \dots, 1.5)$	200	2	2	2	3
System of 500 equations (9)	$(1.8, \dots, 1.8)$	500	3	3	3	6
Broyden tridiagonal (10)	$(-10, \dots, -10)$	20	3	4	4	8
Large-scale exponential (11)	$(-10, \dots, -10)$	20	8	15	16	13
Large-scale exponential (11)	$(15, \dots, 15)$	200	5	5	3	11
Powell singular (12)	$(2, 2, 2, 1)$	4	7	8	1	17
Brown almost linear (13)	$(10, \dots, 10)$	5	12	46	10	22
Chandrasekhar equation (14)	$(5, \dots, 5)$	10	7	6	7	9

Based on the results presented in Table 1, method (5) shows strong, consistent performance across a variety of nonlinear systems, achieving the lowest iteration count in most cases and demonstrating better scalability. Method (3) is competitive with (5). Newton's method often converges successfully and demonstrates solid performance, particularly on smaller or well-behaved systems, though it typically requires a higher number of iterations compared to the newer methods.

Table 2 presents results that align closely with those observed in Table 1, reinforcing the earlier conclusions about the efficiency of the methods. Nonetheless, in a few cases, some methods failed to converge within the prescribed maximum of 100 iterations.

Example 2. *Revisiting convergence using arbitrary-precision arithmetic.*

In this example, higher tolerances of 10^{-20} and 10^{-40} were used, which led to an increased number of cases where the methods failed to converge. To determine whether the issue comes from rounding errors or the limitations of double-precision floating-point arithmetic, the algorithm was reimplemented using arbitrary-precision arithmetic.

Table 2

Iteration counts for various problems with tolerance $\varepsilon = 10^{-15}$

Problem	x_0	**n**	**Iterations**			
			(3)	**(4)**	**(5)**	**Newton (2)**
System of 3 equations (6)	$(0.0, 1.0, 0.0)$	3	2	2	2	4
System of 10 equations (8)	$(0.7, \ldots, 0.7)$	10	3	3	3	6
System of 200 equations (7)	$(1.5, \ldots, 1.5)$	200	2	2	2	8
System of 500 equations (9)	$(1.8, \ldots, 1.8)$	500	3	3	3	6
Broyden tridiagonal (10)	$(-10, \ldots, -10)$	20	-	-	-	-
Large-scale exponential (11)	$(-10, \ldots, -10)$	20	9	15	16	14
Large-scale exponential (11)	$(15, \ldots, 15)$	200	5	5	4	11
Powell singular (12)	$(2, 2, 2, 1)$	4	11	13	1	27
Brown almost linear (13)	$(10, \ldots, 10)$	5	13	47	10	-
Chandrasekhar equation (14)	$(5, \ldots, 5)$	10	7	-	-	10

Computations were performed with precision levels of 25 and 50 decimal places, and the corresponding results are shown in Table 3.

Table 3

Comparison of the number of iterations required by different methods under high-precision conditions. Results for double precision are omitted, as convergence was not achieved

(a) Number of iterations for precision level $p = 25$, tolerance $\varepsilon = 10^{-20}$

Problem	**Method (3)**	**Method (4)**	**Method (5)**	**Newton (2)**
System of 10 equations	3	3	3	7
System of 200 equations	2	2	2	4
Broyden tridiagonal	4	4	4	9
Brown almost linear	13	–	10	24
Chandrasekhar equation	8	6	7	11

(b) Number of iterations for precision level $p = 50$, tolerance $\varepsilon = 10^{-40}$

Problem	**Method (3)**	**Method (4)**	**Method (5)**	**Newton (2)**
System of 10 equations	3	3	3	8
System of 200 equations	2	2	2	5
Broyden tridiagonal	4	5	5	10
Brown almost linear	13	47	11	25
Chandrasekhar equation	8	7	7	12

In all the problems presented in the Table 3, double precision fails to achieve convergence, while higher arithmetic precision significantly improves convergence. Newton's method typically exhibits around twice the iteration count compared to the other methods. Methods (3), (4) and (5) generally perform better across both small and large-scale problems. An exception is method (4) applied to the Brown almost linear problem, where convergence was achieved only with the highest precision setting after 47 iterations. These observations underscore the importance of selecting an appropriate arithmetic precision for reliable convergence. Moreover, increasing the number of decimal places in arbitrary-precision arithmetic should be done with caution, as it can substantially increase the cost of each iteration. Therefore, it is crucial to estimate the minimal level of precision required to achieve convergence efficiently. In practice, techniques such as ExBLAS or ReproBLAS can also be employed to enhance numerical accuracy and performance [19, 22].

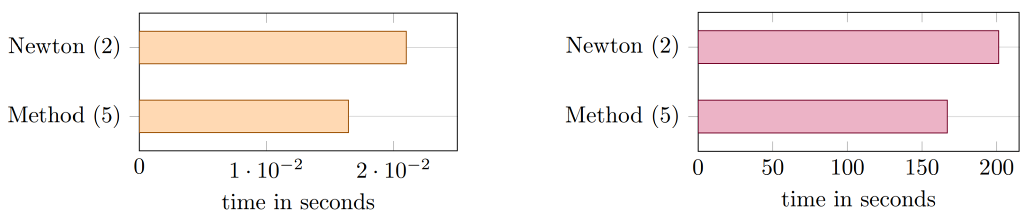
Example 3. *Computational order of convergence and approximate computational order of convergence.* The computational order of convergence (COC) [18] was calculated for the Brown almost linear problem (13). Methods (3), (4) and (5) show convergence orders of 4.99, 5.32 and 5.59, respectively and Newton's method showed an order of 1.99. All computations were performed with a tolerance of $\varepsilon = 10^{-20}$.

Additionally, the approximate computational order of convergence (ACOC) [18] was used to compare method (3) and Newton's method (2) across several problems: (8), (10), (13) and (14). Method (3) demonstrated ACOC values: 5.78, 5.81, 5.06 and 6.95. Newton's method shows a consistent value of 2.00 in all cases.

Example 4. *Comparison of execution times.*

In this example, we compare the computational efficiency of method (5) with the classical Newton method (2) when solving a system of 200 nonlinear equations (7). Starting point is $x^{(0)} = (\frac{3}{2}, \frac{3}{2}, \ldots, \frac{3}{2})$. *The comparison is conducted under two numerical precision settings: standard double precision and extended-precision arithmetic.*

The total execution times required for each method to converge under these conditions are illustrated in Figure 1.



(a) Double precision, $\varepsilon = 10^{-15}$

(b) Extended precision ($p = 50$), $\varepsilon = 10^{-20}$

Fig. 1. Comparison of execution times for method (5) and method (2) in double and extended precision

As shown in Figure 1, method (5) outperforms method (2) in both precision settings. These results confirm that method (5) provides a measurable computational benefit, particularly for high-precision applications.

5. Conclusion

This work presented a comparative analysis of sixth-order iterative methods for solving nonlinear equations in Banach spaces. By focusing on the use of only the operator F and its first derivative F', we provided a refined local and semi-local convergence analysis, moving beyond reliance on higher-order derivatives. Through the development of majorizing sequences, we established semi-local convergence results for each method, extending their theoretical robustness. Numerical experiments were conducted across a range of nonlinear systems, including both small and large-scale problems. These experiments demonstrated that all three methods outperform the classical Newton's method in terms of convergence speed and number of iterations. The computational order of convergence was computed for both methods, showing close to sixth-order behavior for the proposed methods and second-order behavior for Newton's method. Additionally, we ran experiments using arbitrary-precision arithmetic for cases where double precision was insufficient. This study shows that the proposed sixth-order methods are theoretically and practically more effective than Newton's method. The developed techniques also provide a framework for analyzing other high-order methods for nonlinear equations.

References

1. Sharma J. An improved Newton-Traub composition for solving systems of nonlinear equations / J.R. Sharma, R. Sharma, A. Bahl // Applied Mathematics and Computation. – 2016. – Vol. 290. – P. 98–110.
2. Sharma J. A novel family of composite Newton-Traub methods for solving systems of nonlinear equations / J.R. Sharma, R. Sharma, N. Kalra // Applied Mathematics and Computation. – 2015. – Vol. 269. – P. 520–535.
3. Wang X. Semilocal convergence of a sixth-order Jarrat method in Banach spaces / X. Wang, J. Kou, C. Gu // Numerical Algorithms. – 2011. – Vol. 57. – P. 441–456.
4. Argyros I. Extended local convergence for the combined Newton-Kurchatov method under the generalized Lipschitz conditions / I. Argyros, S. Shakhno // Mathematics. – 2019. – Vol. 7. – № 207. – https://doi.org/10.3390/math7020207.
5. Argyros I. Extended two-step-Kurchatov method for solving Banach space valued nondifferentiable equations / I. Argyros, S. Shakhno // International Journal of Applied and Computational Mathematics. – 2020. – Vol. 6. – № 3. – https://doi.org/10.1007/s40819-020-0784-y.
6. Argyros I. Two-step solver for nonlinear equations / I. Argyros, S. Shakhno, H. Yarmola // Symmetry. – 2019. – Vol. 11. – № 128. – https://doi.org/10.3390/sym11020128.
7. Grau-Sanchez M. On the approximation of derivatives using divided difference operators preserving the local convergence order of iterative methods / M. Grau-Sánchez, M. Noguera, S. Amat // Journal of Computational and Applied Mathematics. – 2013. – Vol. 237. – P. 363–372.
8. Homeier H.H.H. A modified Newton method with cubic convergence: The multivariable case / H.H.H. Homeier // Journal of Computational and Applied Mathematics. – 2004. – Vol. 169. – P. 161–169.
9. Behl R. Efficient family of sixth-order methods for nonlinear models with its dynamics / R. Behl, P. Maroju, S.S. Motsa // International Journal of Computer Methods. – 2018. – Vol. 15. – P. 1–26.
10. Cordero A. Variants of Newton's method using fifth-order quadrature formulas / A. Cordero, J.R. Torregrosa // Applied Mathematics and Computation. – 2007. – Vol. 190. – P. 686–698.
11. Frontini M. Third-order methods from quadrature formulae for solving systems of nonlinear equations / M. Frontini, E. Sormani // Applied Mathematics and Computation. – 2004. – Vol. 149. – P. 771–782.
12. Waseem H. Efficient method for solving a system of nonlinear equations / H. Waseem, M. A. Noor, K.I. Noor // Applied Mathematics and Computation. – 2016. – Vol. 275. – P. 134–146.
13. Khan W.A. A new fourth order Newton-type method for solution of system of nonlinear equations / W.A. Khan, K.I. Noor, K. Bhatti, F.A. Ansari // Applied Mathematics and Computation. – 2015. – Vol. 270. – P. 724–730.

14. Sharma R. A modified Newton-Ozban composition for solving nonlinear systems / R. Sharma, J.R. Sharma, N. Kalra // International Journal of Computational Methods. – 2020. – Vol. 17 (08). – https://doi.org/10.1142/S0219876219500476.
15. Narang M. New two-parameter Chebyshev-Halley-like family of fourth and sixth-order methods for systems of nonlinear equations / M. Narang, S. Bhatia, V. Kanwar // Applied Mathematics and Computation. – 2016. – Vol. 275. – P. 394–403.
16. Argyros I.K. Computational Theory of Iterative Methods / I.K. Argyros New-York: Elsevier, 2007.
17. Lotfi T. Some new efficient multipoint iterative methods for solving nonlinear systems of equations / T. Lotfi, P. Bakhtiari, A. Cordero, K. Mahdiani, J.R. Torregrosa // International Journal of Computer Mathematics. – 2014. – Vol. 92. – P. 1921–1934.
18. Grau-Sanchez M. On some computational orders of convergence / M. Grau-Sanchez, M. Noguera, J.M. Gutierrez // Applied Mathematics Letters. – 2010. – Vol. 23 (4). – P. 472–478. – https://doi.org/10.1016/j.aml.2009.12.006.
19. Iakymchuk R. ExBLAS: Reproducible and Accurate BLAS Library / R. Iakymchuk, C. Collange, D. Defour, S. Graillat // Proceedings of the Numerical Reproducibility at Exascale (NRE2015) Workshop, held as part of the Supercomputing Conference (SC15). – Austin, TX, United States. – 2015. – https://hal.science/hal-01202396.
20. Kantorovich L.V. Functional Analysis. – 2nd ed. / Kantorovich L.V., Akilov G.P. // Pergamon. – 1982.
21. More J.J. Testing unconstrained optimization software / J.J. Moré, B.S. Garbow, K.E. Hillstrom // ACM Transactions on Mathematical Software. – 1981. – Vol. 7 (1). – P. 17–41.
22. Demmel J. Efficient reproducible floating point summation and BLAS / J. Demmel, W. Ahrens, H.D. Nguyen // Technical Report UCB/EECS-2016-121. – 2016. – http://www2.eecs.berkeley.edu/Pubs/TechRpts/2016/EECS-2016-121.html.
23. Esmaeili H. An efficient three-step method to solve system of nonlinear equations / H. Esmaeili, M. Ahmadi // Applied Mathematics and Computation. – 2015. – Vol. 266. – P. 1093–1101.
24. Xiao X. A new class of methods with higher-order of convergence for solving systems of nonlinear equations / X. Xiao, H. Yin // Applied Mathematics and Computation. – 2015. – Vol. 264. – P. 300–309.
25. Ozban A.Y. Some new variants of Newton's Method / A.Y. Ozban // Applied Mathematics Letters. – 2004. – Vol. 17. – P. 677–682.

Article: received 17.09.2025
revised 08.10.2025
printing adoption 15.10.2025

ПОРІВНЯННЯ ТРЬОХ МЕТОДІВ ШОСТОГО ПОРЯДКУ З ВИКОРИСТАННЯМ ПОДІБНОЇ ІНФОРМАЦІЇ

І. Аргирос[1], С. Шахно[2], М. Гавдяк[2], М. Аргирос[3]

[1] *Університет Камерона, Лоутон, 73505, Оклахома, США,*
e-mail: iargyros@cameron.edu,
[2] *Львівський національний університет імені Івана Франка,*
вул. Університетська 1, Львів, 79000, Україна,
e-mail: stepan.shakhno@lnu.edu.ua, mykhailo.havdiak@lnu.edu.ua,
[3] *Коледж комп'ютерної інженерії, університет Nova Southeastern,*
Флорида, 33328, США

У цій статті проведено порівняльний аналіз трьох ітераційних методів шостого порядку для розв'язування нелінійних рівнянь. На відміну від підходів, що ґрунтуються на використанні похідних вищих порядків, у цій роботі аналіз локальної збіжності використовує лише інформацію про оператор F та його похідну F'. Також проведено напівлокальний аналіз збіжності для цих методів із використанням мажоруючих послідовностей, що забезпечує практичну основу для встановлення збіжності в умовах початкового наближення. Як локальний, так і напівлокальний аналіз збіжності виконано в банахових просторах. Цей підхід покращує теоретичну стійкість та практичну застосовність методів. Для підтвердження теоретичних результатів проведено низку чисельних експериментів на стандартних тестових задачах як малої, так і великої розмірності. Ефективність запропонованих методів порівнюється з класичним методом Ньютона. Результати показують, що методи шостого порядку завжди перевершують метод Ньютона за кількістю необхідних ітерацій. Було проаналізовано обчислювальний порядок збіжності (COC) та апроксимований обчислювальний порядок збіжності (ACOC), які емпірично підтвердили високий порядок збіжності запропонованих методів. Крім того, досліджено чисельну стійкість та ефективність методів у задачах із підвищеними вимогами до точності. Для розв'язання задач, у яких стандартна подвійна точність є недостатньою, використовувалася арифметика довільної точності. Отримані результати підтверджують практичні переваги та теоретичну стійкість методів. Розглянута методологія може бути застосована й до інших подібних ітераційних методів.

Ключові слова: ітераційні методи, банахові простори, похідна Фреше, локальна збіжність, напівлокальна збіжність.