СИСТЕМНИЙ АНАЛІЗ

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PURE STRATEGY SOLUTIONS OF THE SILENT DUEL ON THE UNIFORM LATTICE WITH IDENTICAL LINEAR ACCURACY FUNCTIONS

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A finite zero-sum game defined on a uniform lattice of the unit square is solved as a game of timing. The game is a discrete silent duel, in which the kernel is skew-symmetric, and the players, referred to as duelists, have identical linear accuracy functions featured with an accuracy proportionality factor. Due to the skew-symmetry, both the duelists have the same optimal strategies and the game optimal value is 0. If the accuracy factor is not less than the number of possible shooting moments decreased by 2 then the duelist's optimal behavior is to shoot at the middle of the duel time span. A boundary case of the accuracy factor is determined, where the factor is equal to the reciprocal of the number of possible shooting moments decreased by 2. In this case, the duel has four pure strategy solutions issued by the two last moments of possible shooting. Otherwise, the duel has a single optimal pure strategy situation by when the accuracy factor belongs to the definite nonempty interval. A 4×4 or bigger duel is not solved in pure strategies if this membership is false or the interval in the membership is empty. The trivial duels, with just two or three moments of possible shooting, are always solved in pure strategies.

Key words: game of timing, silent duel, linear accuracy, uniform lattice, matrix game, pure strategy saddle point.

1. Introduction to duels on the uniform lattice

Games of timing model competitive interactions involving two or more intelligent participants. Such interactions are typical for most of economic processes, sports, social and ecological processes, jurisprudence [1, 4, 9, 10, 17]. A distinct feature of the competition is the moment of decision making or acting. Another feature is a time span during which the participant (player) must make a decision of acting [2, 3, 7, 14]. Usually the span is standardized to a unit interval.

The most common games of timing involve two players, where the player does not learn about the action of the other player until the time span elapses [4, 11, 15, 20]. Such two-person games are referred to as silent duels, and the players are often called duelists [2, 5, 11, 16, 18]. The duelist possesses a number of bullets, where the bullet is an abstraction implying an implementation of the decision of acting during the duel time span. Silent duels model delays typical for time-lagged systems like those in economics, ecosystems, jurisprudence [6, 8, 10, 16, 20].

In general, the silent duel is a continuous-time game of timing

$$\langle X,\ Y,\ K(x,\ y)\rangle \tag{1}$$

being an infinite zero-sum game with a kernel $K(x,\ y)$ defined on unit square

$$X \times Y = [0;\ 1] \times [0;\ 1] \tag{2}$$

being the Cartesian product of the duel time spans (i. e., the product is the square of the span), where $x \in X$, $y \in Y$. This is a zero-sum game due to the following reasons. Given a single bullet, the duelist is featured with an accuracy function which, generally speaking, is a nondecreasing function of time [3, 4, 10, 16]. In silent duel (1), it is unknown to the duelist whether the bullet was fired by the other duelist or not until the end of the duel time span [2, 11, 16, 18]. The duelist may obtain a greater payoff by firing (shooting) as late as possible, but then the loss likelihood increases due to the other duelist may shoot first. If both the duelists shoot simultaneously, the payoff of each of them is 0 [6, 7, 10, 19].

When the accuracy functions $p_X(x)$, $p_Y(y)$ of the duelists are linear and identical, the kernel is [4, 11]

$$K(x,\ y) = ax - ay + a^2 xy \operatorname{sign}(y - x) \quad \text{by} \quad a > 0, \tag{3}$$

where $p_X(x) = ax$, $p_Y(y) = ay$, and a is an accuracy proportionality factor. Kernel (3) is skew-symmetric, i. e.

$$K(y,\ x) = ay - ax + a^2 yx \operatorname{sign}(x - y) = -K(x,\ y), \tag{4}$$

and so both the duelists have the same optimal strategies, and the game optimal value is 0 [7, 8, 10, 16].

The standardized case of $a = 1$ simplifies kernel (3) to

$$K(x,\ y) = x - y + xy \operatorname{sign}(y - x). \tag{5}$$

The duel with kernel (5) was considered in [10], and its solution is the duelist's optimal strategy

$$\rho(u) = \begin{cases} 0, & u \in \left[0;\ \dfrac{1}{3}\right) \\ 0.25 \cdot u^{-3}, & u \in \left[\dfrac{1}{3};\ 1\right] \end{cases} \tag{6}$$

being a non-continuous probability distribution as a mixed strategy with an uncountably infinite support whose measure is less than the duel time span length [4, 5, 16, 17]. Therefore, an infinite duel solution cannot be completely implemented in practice due to any sequence of real-world actions is naturally limited [4, 6, 7, 12, 13]. This is why discrete silent duels are considered instead.

While the duelist is allowed to shoot at any moment during the infinite duel time span, in a discrete silent duel the duelist can shoot only at specified time moments. These moments constitute the duelist's set of pure strategies. The set is finite by default (however, the duelist still can possess a countable infinite set of pure strategies and so the respective discrete silent duel will be infinite in this case). Therefore, the kernel of the discrete silent duel is defined on a finite set of the pairs of pure strategies of the duelists. The moments of the duel start $x = y = 0$ and duel end $x = y = 1$ are included [2, 10, 16, 20].

Moments of possible shooting can be specifically defined by uniformly breaking the time span. Then the identical sets of pure strategies of the duelists are [4, 10, 11]

$$X = \{x_i\}_{i=1}^{N} = \left\{\frac{i-1}{N-1}\right\}_{i=1}^{N} \subset [0;\ 1] \quad \text{for} \quad N \in \mathbb{N} \backslash \{1\} \tag{7}$$

and

$$Y = \{y_j\}_{j=1}^{N} = \left\{\frac{j-1}{N-1}\right\}_{j=1}^{N} \subset [0;\ 1] \quad \text{for} \quad N \in \mathbb{N} \backslash \{1\}. \tag{8}$$

Then game (1) by kernel (3) defined on a finite lattice

$$X \times Y = \left\{\frac{i-1}{N-1}\right\}_{i=1}^{N} \times \left\{\frac{j-1}{N-1}\right\}_{j=1}^{N} \subset [0;\ 1] \times [0;\ 1] \tag{9}$$

is a discrete silent duel with identical linear accuracy functions of the duelists scaled with factor a. The discrete silent duel is an $N \times N$ matrix game whose payoff matrix is skew-symmetric. Any solution of this matrix game is of finite supports only [7, 8, 10, 12, 16]. Owing to that, any solution of the discrete silent duel is computed far easier than that in the case of infinite game (1).

It is obvious that this duel solution depends on N and a. The goal is to study pure strategy solutions of this silent duel. In terms of the matrix game, its saddle points are to be found. To achieve the goal, the discrete silent duel is first formalized in Section 2. The most trivial case, when the duelist is allowed to shoot at either the very start or end of the duel, is considered in Section 3. Section 4 ascertains whether the duel can be solved in pure strategies at the very start. Pure strategy solutions are generally noted in Section 5. Then pure strategy saddle points in the case of the duelist's three possible actions (the start, middle, and end of the duel time span) are deduced in Section 6. The existence of pure strategy solutions is proved in Section 7. The singleness of the pure strategy solution is ascertained in Section 8, whereupon the study is discussed, recapitulated, and concluded in Section 9.

2. Discrete silent duel

In fact, the discrete silent duel with identical linear accuracy functions is a matrix game

$$\left\langle \{x_i\}_{i=1}^{N},\ \{y_j\}_{j=1}^{N},\ \mathbf{K}_N \right\rangle \tag{10}$$

by (7), (8), and payoff matrix

$$\mathbf{K}_N = [k_{ij}]_{N \times N} \quad \text{by}$$
$$k_{ij} = K(x_i,\ y_j) = ax_i - ay_j + a^2 x_i y_j \operatorname{sign}(y_j - x_i) \quad \text{by} \quad a > 0. \tag{11}$$

Followed by (4), matrix (11) is skew-symmetric. Therefore, owing to the skew-symmetry of matrix (11) and identical sets of pure strategies (7), (8) of the duelists, the sets of optimal strategies of the duelists are identical, and the game optimal value is 0.

If $a = 1$ then the accuracy is exactly equal to the time moment at which the bullet is fired. Although by $a \neq 1$ the linear accuracy is just scaled by $a > 0$, the scaling does influence the solution, and this is yet to be shown below.

3. The duel with only two shooting moments

Whereas the case of $N = 1$ is obviously excluded, the case of $N = 2$ is the most trivial, where the duelist is allowed to shoot at either the very start or end of the duel. As the shooting is allowed only at moments $t_1 = 0$ and $t_2 = 1$, the respective payoff matrix (11) is

$$\mathbf{K}_2 = [k_{ij}]_{2\times 2} = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} \tag{12}$$

whence situation

$$\{x_2,\ y_2\} = \{1,\ 1\} \tag{13}$$

is the single saddle point. Therefore, the silent duels with only two shooting moments $t_1 = 0$ and $t_2 = 1$ are solved identically: the best action of the duelist is to shoot at the end of the duel.

4. No pure strategy solutions at the very start

In general, the first row of matrix (11) contains a negative entry:

$$K(x_1,\ y_N) = K(0,\ 1) = -a = -K(1,\ 0) = -K(x_N,\ y_1)\,. \tag{14}$$

Therefore, the minimum of the first row does not exceed $-a < 0$ and thus the game optimal value cannot be reached in this row, whichever number N is. So, the first row of matrix (11) does not contain saddle points. Due to the skew-symmetry of matrix (11), the stated inference is immediately followed by that the first column does not contain saddle points either. Hence, there are no pure strategy solutions at the very start of the duel, whichever the number of possible shooting moments is.

5. Pure strategy solutions in general

Inasmuch as the game optimal value is 0, it is quite obvious that only a zero entry of skew-symmetric matrix (11) can be a saddle point (i. e., a pure strategy solution). Obviously, if a row contains a negative entry, this row does not contain saddle points. Due to the skew-symmetry of matrix (11), the stated inference is immediately followed by that the respective column sharing only the main diagonal zero entry with the abovementioned row does not contain saddle points either. Henceforward, mentioning a row without saddle points immediately implies that there are no saddle points in the respective column (of the same number).

If a row contains only nonnegative entries, this row contains at least one saddle point located on the main diagonal. Besides, if a row of a skew-symmetric matrix contains only nonnegative entries, and a zero entry located off the main diagonal in this row is a saddle point, the matrix game has at least four saddle points [8, 10, 11]. If there are two saddle points on the main diagonal of a skew-symmetric matrix, the matrix game has at least four saddle points also [2, 7, 8, 10, 16].

6. Pure strategy solutions for three shooting moments

Another trivial case is when the shooting, apart from the very start and end moments $t_1 = 0$, $t_3 = 1$, is also allowed at moment $t_2 = \frac{1}{2}$. This is the case of $N = 3$, i. e. the duelist has three possible actions.

Theorem 1. *In a discrete silent duel (10) by (7), (8), (11) for $N = 3$, situation*

$$\{x_2,\ y_2\} = \left\{\frac{1}{2},\ \frac{1}{2}\right\} \tag{15}$$

is optimal only by $a \geqslant 1$ but it is never optimal by $0 < a < 1$. Besides, saddle point (15) is single by $a > 1$. Any pure strategy situation in the 3×3 duel, not containing the duel start moment, is optimal by $a = 1$, whereas situation

$$\{x_3,\ y_3\} = \{1,\ 1\} \tag{16}$$

is the single saddle point by $0 < a < 1$.

Proof. Due to (14), situation

$$\{x_1,\ y_1\} = \{0,\ 0\}$$

is never optimal in the duel. The respective payoff matrix is

$$\mathbf{K}_3 = [k_{ij}]_{3\times3} = \begin{bmatrix} 0 & -\dfrac{a}{2} & -a \\ \dfrac{a}{2} & 0 & \dfrac{a}{2}(a-1) \\ a & -\dfrac{a}{2}(a-1) & 0 \end{bmatrix}. \tag{17}$$

If $a - 1 > 0$, i. e. $a > 1$, then the second row of matrix (17) is nonnegative and the third row contains a negative entry. The only zero entry in the second row is k_{22}. So, situation (15) is optimal and it is the single saddle point for (17) by $a > 1$. If $a = 1$ then the respective payoff matrix is

$$\mathbf{K}_3 = [k_{ij}]_{3\times3} = \begin{bmatrix} 0 & -\dfrac{1}{2} & -1 \\ \dfrac{1}{2} & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{18}$$

This matrix game has four saddle points: situation (15), situation (16), and non-symmetric situations

$$\{x_2,\ y_3\} = \left\{\frac{1}{2},\ 1\right\} \tag{19}$$

and

$$\{x_3,\ y_2\} = \left\{1,\ \frac{1}{2}\right\}. \tag{20}$$

If $0 < a < 1$ then the second row of matrix (17) contains a negative entry and the third row is nonnegative. The only zero entry in the third row is k_{33}. So, situation (16) is the single saddle point for (17) by $0 < a < 1$. □

7. Existence of pure strategy solutions

Now, the general case for $N \in \mathbb{N} \backslash \{1,\ 2,\ 3\}$ is to be considered in order to find where pure strategy solutions exist. As the first row of matrix (11) never has a saddle point, the consideration is started with the second row of matrix (11).

Theorem 2. *In a progressive discrete silent duel (10) by (7), (8), (11) for* $N \in \mathbb{N}\backslash\{1,\ 2,\ 3\}$*, situation*

$$\{x_2,\ y_2\} = \left\{\frac{1}{N-1},\ \frac{1}{N-1}\right\} \tag{21}$$

is optimal only if

$$a \geqslant N - 2. \tag{22}$$

Proof. If situation (21) is optimal, then inequalities

$$K(x_2,\ y_j) = ax_2 - ay_j - a^2x_2y_j = \frac{a}{N-1} - ay_j - \frac{a^2y_j}{N-1} \geqslant 0 \ \forall\, y_j < x_2 \tag{23}$$

and

$$K(x_2,\ y_j) = ax_2 - ay_j + a^2x_2y_j = \frac{a}{N-1} - ay_j + \frac{a^2y_j}{N-1} \geqslant 0 \ \forall\, y_j > x_2 \tag{24}$$

must hold. Inequality (23) is simplified to inequality

$$K(x_2,\ y_1) = K(x_2,\ 0) = \frac{a}{N-1} \geqslant 0 \tag{25}$$

that always holds. From inequality (24) it follows that

$$\frac{1}{N-1} - y_j + \frac{ay_j}{N-1} \geqslant 0,$$

whence

$$a \geqslant N - 1 - \frac{1}{y_j} \ \ \forall\, y_j > x_2. \tag{26}$$

As

$$1 \geqslant y_j \geqslant \frac{2}{N-1} > \frac{1}{N-1} = x_2$$

then

$$a \geqslant N - 1 - 1 = N - 2$$

that is (22). So, situation (21) is a saddle point if just inequality (22) holds. □

In a way, Theorem 2 extends Theorem 1: situation (21) is optimal only if (22) is true for $N \in \mathbb{N}\backslash\{1,\ 2\}$. The optimality of other pure strategy situations is to be ascertained yet.

Theorem 3. *In a progressive discrete silent duel (10) by (7), (8), (11) for* $N \in \mathbb{N}\backslash\{1,\ 2,\ 3\}$*, situation*

$$\{x_n,\ y_n\} = \left\{\frac{n-1}{N-1},\ \frac{n-1}{N-1}\right\} \tag{27}$$

is optimal only if

$$a \in \left[\frac{N-n}{n-1};\ \frac{N-1}{(n-2)\cdot(n-1)}\right] \subset \left(0;\ \frac{N-1}{(n-2)\cdot(n-1)}\right] \tag{28}$$

is true for $n \in \left\{\overline{3,\ N}\right\}$.

Proof. If situation (27) is optimal, then inequalities

$$K(x_n,\ y_j) = ax_n - ay_j - a^2x_ny_j \geqslant 0 \ \ \forall\, y_j < x_n \tag{29}$$

and

$$K(x_n,\ y_j) = ax_n - ay_j + a^2 x_n y_j \geqslant 0 \ \ \forall\, y_j > x_n \tag{30}$$

must hold. From inequality (29) it follows that

$$x_n - y_j - ax_n y_j \geqslant 0,$$

whence

$$\frac{x_n}{1 + ax_n} \geqslant y_j \ \ \forall\, y_j < x_n. \tag{31}$$

As

$$y_j \leqslant \frac{n-2}{N-1} < \frac{n-1}{N-1} = x_n$$

then

$$\frac{n-1}{N-1} \cdot \frac{1}{1 + a \cdot \dfrac{n-1}{N-1}} \geqslant \frac{n-2}{N-1},$$

$$\frac{n-1}{N - 1 + a \cdot (n-1)} \geqslant \frac{n-2}{N-1},$$

$$(n-1) \cdot (N-1) \geqslant (n-2) \cdot (N-1) + a \cdot (n-2) \cdot (n-1)\,,$$

$$N - 1 \geqslant a \cdot (n-2) \cdot (n-1)\,,$$

whence

$$\frac{N-1}{(n-2) \cdot (n-1)} \geqslant a. \tag{32}$$

From inequality (30) it follows that

$$x_n - y_j + ax_n y_j \geqslant 0,$$

whence

$$a \geqslant \frac{y_j - x_n}{x_n y_j} = \frac{1}{x_n} - \frac{1}{y_j} \ \ \forall\, y_j > x_n. \tag{33}$$

As

$$1 \geqslant y_j \geqslant \frac{n}{N-1} > \frac{n-1}{N-1} = x_n \ \text{ for } \ n \in \left\{\overline{2,\ N-1}\right\}$$

then

$$a \geqslant \frac{N-1}{n-1} - 1 = \frac{N-n}{n-1}. \tag{34}$$

So, according to simultaneous inequalities (32) and (34), if

$$a \in \left[\frac{N-n}{n-1};\ \frac{N-1}{(n-2) \cdot (n-1)}\right] \tag{35}$$

for some $N \in \mathbb{N} \backslash \{1,\ 2,\ 3\}$ and $n \in \left\{\overline{3,\ N-1}\right\}$ then situation (27) is a saddle point. If $n = N$ then only inequality (29) must hold so that

$$\{x_N,\ y_N\} = \{1,\ 1\} \tag{36}$$

be a saddle point. This is true by only (32), i. e. by

$$\frac{1}{N-2} \geqslant a,$$

or, being more precise,

$$a \in \left(0;\ \frac{1}{N-2}\right]. \tag{37}$$

As a generalization, if (28) is true for some $N \in \mathbb{N}\backslash\{1,\ 2,\ 3\}$ and $n \in \left\{\overline{3,\ N}\right\}$ then situation (27) is a saddle point. □

As of this place, it has not been ascertained yet whether the main diagonal of matrix (11) can have more than just a one saddle point. Nevertheless, there is an easily noticeable case in which the duel has four saddle points.

Theorem 4. *In a progressive discrete silent duel (10) by (7), (8), (11) for* $N \in \mathbb{N}\backslash\{1,\ 2,\ 3\}$, *situations*

$$\{x_{N-1},\ y_{N-1}\} = \left\{\frac{N-2}{N-1},\ \frac{N-2}{N-1}\right\}, \tag{38}$$

$$\{x_{N},\ y_{N-1}\} = \left\{1,\ \frac{N-2}{N-1}\right\}, \tag{39}$$

$$\{x_{N-1},\ y_{N}\} = \left\{\frac{N-2}{N-1},\ 1\right\}, \tag{40}$$

and (36) are optimal only if $a = \frac{1}{N-2}$.

Proof. Owing to Theorem 3, if $a = \frac{1}{N-2}$ then (36) is a saddle point. For $n = N-1$ situation (38) is a saddle point if, in accordance with (28),

$$a = \frac{1}{N-2} \in \left[\frac{N-n}{n-1};\ \frac{N-1}{(n-2)\cdot(n-1)}\right],$$

that is

$$a = \frac{1}{N-2} \in \left[\frac{1}{N-2};\ \frac{N-1}{(N-3)\cdot(N-2)}\right]. \tag{41}$$

As

$$\frac{N-1}{(N-3)\cdot(N-2)} - \frac{1}{N-2} = \frac{N-1-N+3}{(N-3)\cdot(N-2)} = \frac{2}{(N-3)\cdot(N-2)} > 0$$

then the membership in (41) is indeed true, and $a = \frac{1}{N-2}$ satisfies condition (28) issuing saddle point (38). Moreover, due to the skew-symmetry of matrix (11) having saddle points (38) and (36) in its main diagonal, situations (39) and (40) are saddle points also. □

It is easy to get convinced that the membership in (28) is not true for any N and n. After all, Theorem 3 does not ensure that the interval in the membership in (28) is nonempty ever.

Theorem 5. *A progressive discrete silent duel (10) by (7), (8), (11) for* $N \in \{4,\ 5,\ 6,\ 7\}$ *has an optimal situation (27) by (28), wherein the interval in the membership is nonempty by the respective* $n \in \left\{\overline{3,\ N}\right\}$.

Proof. According to Theorem 3, situation (27) is optimal if (28) is true, where the interval in the membership should be nonempty. Consider the difference between the endpoints of the interval:

$$\frac{N-1}{(n-2)\cdot(n-1)} - \frac{N-n}{n-1} = \\ = \frac{n^2 - n(N+2) + 3N - 1}{(n-2)\cdot(n-1)}. \tag{42}$$

A quadratic equation

$$n^2 - n(N+2) + 3N - 1 = 0 \tag{43}$$

from the numerator of the last term in (42) has the disciminant

$$D = (N+2)^2 - 4\cdot(3N-1) = N^2 - 8N + 8, \tag{44}$$

where

$$N^2 - 8N + 8 \leqslant 0$$

if

$$1 < 4 - 2\sqrt{2} < 2 \leqslant N \leqslant 6 < 4 + 2\sqrt{2} < 7.$$

The latter means that if $N \in \{2,\ 3,\ 4,\ 5,\ 6\}$ then discriminant (44) of equation (43) is nonpositive. This implies the numerator of the last term in (42) is nonnegative for any $n \in \left\{\overline{2,\ N}\right\}$ and so the interval in the membership of (28) is nonempty. The latter means that an optimal situation (27) always exists by (28) for $N \in \{4,\ 5,\ 6\}$.

If $N = 7$ then $D = 1$ and so the numerator of the last term in (42) can be negative. However, it is negative for

$$n \in \left(\frac{N+2-\sqrt{D}}{2};\ \frac{N+2+\sqrt{D}}{2}\right) = (4;\ 5)$$

which is impossible for integer n. So, the numerator of the last term in (42) is nonnegative for $N = 7$ as well. Then the interval in the membership of (28) is nonempty by $N = 7$ also, and an optimal situation (27) always exists by (28) and $N = 7$. □

Now, consider bigger duels for $N \in \mathbb{N}\backslash\left\{\overline{1,\ 7}\right\}$. It turns out that the bigger duels already lose optimality of some rows of matrix (11) due to the empty interval in the membership of (28).

Theorem 6. *In a progressive discrete silent duel (10) by (7), (8), (11)* $\exists\, n \in \left\{\overline{3,\ N}\right\}$ *for every* $N \in \mathbb{N}\backslash\left\{\overline{1,\ 7}\right\}$ *such that situation (27) is non-optimal if*

$$n \in \left(\frac{N+2-\sqrt{N^2-8N+8}}{2};\ \frac{N+2+\sqrt{N^2-8N+8}}{2}\right), \tag{45}$$

wherein the interval in the membership of (28) is empty.

Proof. Discriminant (44) of equation (43) is positive by

$$N \geqslant 7 > 4 + 2\sqrt{2},$$

i. e. it is always positive by $N \in \mathbb{N} \backslash \left\{ \overline{1,\ 7} \right\}$. Because of its positivity property, the numerator of the last term in (42) is negative if (45) is true. To study the difference between the endpoints of the interval in (45), denote them by

$$n_* (N) = \frac{N + 2 - \sqrt{N^2 - 8N + 8}}{2} \tag{46}$$

and

$$n^* (N) = \frac{N + 2 + \sqrt{N^2 - 8N + 8}}{2}. \tag{47}$$

Thus,

$$\delta (N) = n^* (N) - n_* (N) = \\ = \frac{N + 2 + \sqrt{N^2 - 8N + 8}}{2} - \frac{N + 2 - \sqrt{N^2 - 8N + 8}}{2} = \\ = \sqrt{N^2 - 8N + 8} \tag{48}$$

and it is obvious that difference (48) increases as N increases. In particular, $\delta (8) = 2\sqrt{2}$ that formally would mean that at least two $n \in \left\{ \overline{3,\ 8} \right\}$ exist such that situation (27) is non-optimal. In reality, situation (27) is non-optimal by $n \in \{4,\ 5,\ 6\}$. As N increases, the number of such $n \in \left\{ \overline{3,\ 8} \right\}$ at which situation (27) is non-optimal cannot decrease. □

The non-optimality consideration is not exhausted by Theorem 6. The following assertion ascertains which rows of matrix (11) will not contain saddle points.

Theorem 7. *In a progressive discrete silent duel (10) by (7), (8), (11) for* $N \in \mathbb{N} \backslash \left\{ \overline{1,\ 7} \right\}$ *situation (27) is non-optimal for every* $n = \overline{4,\ N - 2}$.

Proof. According to Theorem 6, situation (27) is non-optimal for some $n \in \left\{ \overline{3,\ N} \right\}$ if (45) is true. The left endpoint (46) being a function of N has the derivative:

$$\frac{dn_* (N)}{dN} = \frac{1}{2} \cdot \left(1 - \frac{2N - 8}{2\sqrt{N^2 - 8N + 8}} \right) = \frac{\sqrt{N^2 - 8N + 8} - N + 4}{2\sqrt{N^2 - 8N + 8}}. \tag{49}$$

Ratio (49) is simplified:

$$\frac{\sqrt{N^2 - 8N + 8} - N + 4}{2\sqrt{N^2 - 8N + 8}} = \\ = \frac{\sqrt{N^2 - 8N + 8} - (N - 4)}{2\sqrt{N^2 - 8N + 8}} \cdot \frac{\sqrt{N^2 - 8N + 8} + (N - 4)}{\sqrt{N^2 - 8N + 8} + (N - 4)} = \\ = \frac{N^2 - 8N + 8 - N^2 + 8N - 16}{2\sqrt{N^2 - 8N + 8} \cdot \left(\sqrt{N^2 - 8N + 8} + (N - 4) \right)} = \\ = \frac{-8}{2\sqrt{N^2 - 8N + 8} \cdot \left(\sqrt{N^2 - 8N + 8} + (N - 4) \right)} < 0.$$

Therefore, function $n_* (N)$ is decreasing. Its maximal value

$$\max_{N \geqslant 7} n_* (N) = n_* (7) = 4.$$

The minimal value of function (46) is

$$\min_{N \geqslant 7} n_* (N) = \lim_{N \to \infty} n_* (N) =$$

$$= \lim_{N\to\infty} \frac{N+2-\sqrt{N^2-8N+8}}{2} = 1+\frac{1}{2}\cdot\lim_{N\to\infty}\left(N-\sqrt{N^2-8N+8}\right) =$$

$$= 1+\frac{1}{2}\cdot\lim_{N\to\infty}\frac{\left(N-\sqrt{N^2-8N+8}\right)\cdot\left(N+\sqrt{N^2-8N+8}\right)}{N+\sqrt{N^2-8N+8}} =$$

$$= 1+\frac{1}{2}\cdot\lim_{N\to\infty}\frac{8N-8}{N+\sqrt{N^2-8N+8}} =$$

$$= 1+\frac{1}{2}\cdot\lim_{N\to\infty}\left(\frac{8N}{N+N\sqrt{1-\dfrac{8}{N}+\dfrac{8}{N^2}}}-\frac{8}{N+\sqrt{N^2-8N+8}}\right) =$$

$$= 1+\frac{1}{2}\cdot\lim_{N\to\infty}\left(\frac{8}{1+\sqrt{1-\dfrac{8}{N}+\dfrac{8}{N^2}}}-\frac{8}{N+\sqrt{N^2-8N+8}}\right) =$$

$$= 1+\frac{1}{2}\cdot\left(\frac{8}{1+\sqrt{1-0}}-0\right) = 1+\frac{1}{2}\cdot 4 = 3.$$

Therefore, $n_*(N) \in (3;\ 4]$. Inasmuch as

$$4 > n_*(N) > 3 \ \ \forall\, N \in \mathbb{N}\backslash\left\{\overline{1,\ 7}\right\},$$

the situation

$$\{x_3,\ y_3\} = \left\{\frac{2}{N-1},\ \frac{2}{N-1}\right\} \tag{50}$$

is optimal. Then the task is to ascertain how many integer values of n starting from 4 are less than the right endpoint (47). According to the assertion in this theorem, an equality

$$N-2 < n^*(N) < N-1 \tag{51}$$

must hold. So, it must be

$$n^*(N)-(N-2) = \frac{N+2+\sqrt{N^2-8N+8}}{2} - N + 2 =$$
$$= 3-\frac{N}{2}+\frac{\sqrt{N^2-8N+8}}{2} > 0. \tag{52}$$

As $N^2-8N+8>0$, inequality (52) is equivalent to the following:

$$\frac{\sqrt{N^2-8N+8}}{2} > \frac{N}{2}-3,$$

$$\frac{N^2-8N+8}{4} > \left(\frac{N}{2}-3\right)^2 = \frac{N^2}{4}-3N+9,$$

$$-2N+2 > -3N+9,$$

$$N > 7,$$

which is true. On the other side of inequality (51), it must be

$$n^*(N) - (N-1) = \frac{N+2+\sqrt{N^2-8N+8}}{2} - N + 1 =$$
$$= 2 - \frac{N}{2} + \frac{\sqrt{N^2-8N+8}}{2} < 0, \tag{53}$$

where inequality (53) is equivalent to the following:

$$\sqrt{N^2-8N+8} < N-4,$$

$$N^2 - 8N + 8 < (N-4)^2 = N^2 - 8N + 16,$$

$$8 < 16,$$

which is true as well. So, inequality (51) holds for the increasing function (47) that, according to Theorem 6, implies the non-optimality of situation (27) for every $n \in \left\{\overline{4,\ N-2}\right\}$. □

By the way, the optimality of situation (50) can be checked in a simpler way. Plugging $n = 3$ into (35) gives

$$a \in \left[\frac{N-3}{2};\ \frac{N-1}{2}\right]. \tag{54}$$

The interval in the membership of (54) is obviously nonempty, which, according to Theorem 3, implies the optimality of situation (50). In the same way to be shown, situation

$$\{x_4,\ y_4\} = \left\{\frac{3}{N-1},\ \frac{3}{N-1}\right\} \tag{55}$$

is never optimal in duels with 8 and greater pure strategies at the duelist: plugging $n = 4$ into (35) gives

$$a \in \left[\frac{N-4}{3};\ \frac{N-1}{6}\right],$$

where

$$\frac{N-1}{6} - \frac{N-4}{3} = \frac{N-1-2N+8}{6} = \frac{7-N}{6} < 0 \ \ \forall\, N \in \mathbb{N} \backslash \left\{\overline{1,\ 7}\right\}$$

and so situation (55) violating condition (28) in Theorem 3 by the empty interval in the membership of (28) cannot be optimal.

8. Singleness

Strictly speaking, Theorem 3 does not claim that pure strategy solution (27) by the respective condition (28) with the nonempty interval in the membership exists only for one $n \in \left\{\overline{3,\ N}\right\}$. Nor claims Theorem 5 the singleness. Theorem 2 does not claim it, too. The following assertions clarify the cases of when saddle point (27) is single.

Theorem 8. *A progressive discrete silent duel (10) by (7), (8), (11) for* $N \in \mathbb{N} \backslash \left\{\overline{1,\ 7}\right\}$ *has a single optimal situation (27) by the respective* $n \in \{2,\ 3,\ N-1,\ N\}$ *and* $a \neq \dfrac{1}{N-2}$.

Proof. According to Theorem 7, situation (27) is non-optimal for every $n = \overline{4,\ N-2}$. According to Theorem 2, situation (21) is optimal (here $n = 2$) if (22) is true, i. e.

$$a \in [N-2;\ \infty)\,. \tag{56}$$

Situation (50) is optimal (here $n = 3$) if (54) is true. Situation (38) is optimal (here $n = N-1$) by $a \neq \dfrac{1}{N-2}$ if

$$a \in \left(\frac{1}{N-2};\ \frac{N-1}{(N-3)\cdot(N-2)}\right]. \tag{57}$$

Situation (36) is optimal (here $n = N$) by $a \neq \dfrac{1}{N-2}$ if

$$a \in \left(0;\ \frac{1}{N-2}\right). \tag{58}$$

As

$$\frac{1}{N-2} < \frac{N-1}{(N-3)\cdot(N-2)} < \frac{N-3}{2} < \frac{N-1}{2} < N-2,$$

the intervals in (56), (54), (57), (58) are pairwise non-overlapping. This means that for each $N \in \mathbb{N}\backslash\left\{\overline{1,\ 7}\right\}$ exists only one $n \in \{2,\ 3,\ N-1,\ N\}$ such that situation (27) is optimal by just if (28) is true. □

Clearly, if $a = \dfrac{1}{N-2}$ then it is the condition for Theorem 4, which also extends Theorem 1: situations (38) – (40), (36) are optimal only by $a = \dfrac{1}{N-2}$ for $N \in \mathbb{N}\backslash\{1,\ 2\}$. The condition for the single saddle point for $N \in \{4,\ 5,\ 6,\ 7\}$ is the same.

Theorem 9. *A progressive discrete silent duel (10) by (7), (8), (11) for* $N \in \{4,\ 5,\ 6,\ 7\}$ *has a single optimal situation (27) by the respective* $n \in \left\{\overline{2,\ N}\right\}$ *and* $a \neq \dfrac{1}{N-2}$.

Proof. According to Theorem 2, situation (21) is optimal (here $n = 2$) if (56) is true. According to Theorem 3, situation (27) is optimal if: (54) is true for $n = 3$,

$$a \in \left[\frac{N-4}{3};\ \frac{N-1}{6}\right] \tag{59}$$

is true for $n = 4$,

$$a \in \left[\frac{N-5}{4};\ \frac{N-1}{12}\right] \tag{60}$$

is true for $n = 5$,

$$a \in \left[\frac{N-6}{5};\ \frac{N-1}{20}\right] \tag{61}$$

is true for $n = 6$,

$$a \in \left[\frac{N-7}{6};\ \frac{N-1}{30}\right] \tag{62}$$

is true for $n = 7$ (surely, by keeping $a > 0$). According to Theorem 5, each of the intervals in (54), (59) – (62) is nonempty. Intervals in (56) and (54) do not overlap. Intervals in

(54) and (59) overlap at point $\frac{1}{2} = \frac{1}{N-2}$ (where $N = 4$ for their respective left and right endpoints), but

$$\frac{N-1}{6} < \frac{N-3}{2}$$

owing to

$$\frac{N-1-3N+9}{6} = \frac{8-2N}{6} < 0 \text{ for } N > 4.$$

Intervals in (59) and (60) overlap at point $\frac{1}{3} = \frac{1}{N-2}$ (where $N = 5$ for their respective left and right endpoints), but

$$\frac{N-1}{12} < \frac{N-4}{3}$$

owing to

$$\frac{N-1-4N+16}{12} = \frac{15-3N}{12} < 0 \text{ for } N > 5.$$

Intervals in (60) and (61) overlap at point $\frac{1}{4} = \frac{1}{N-2}$ (where $N = 6$ for their respective left and right endpoints), but

$$\frac{N-1}{20} < \frac{N-5}{4}$$

owing to

$$\frac{N-1-5N+25}{20} = \frac{24-4N}{20} < 0 \text{ for } N > 6.$$

Intervals in (61) and (62) overlap at point $\frac{1}{5} = \frac{1}{N-2}$ (where $N = 7$ for their respective left and right endpoints), but

$$\frac{N-1}{30} < \frac{N-6}{5}$$

owing to

$$\frac{N-1-6N+36}{30} = \frac{35-5N}{30} < 0 \text{ for } N > 7.$$

Hence, the sequence of inequalities

$$\begin{gathered}\frac{N-7}{6} < \frac{N-1}{30} < \\ < \frac{N-6}{5} < \frac{N-1}{20} < \\ < \frac{N-5}{4} < \frac{N-1}{12} < \\ < \frac{N-4}{3} < \frac{N-1}{6} < \\ < \frac{N-3}{2} < \frac{N-1}{2} < N-2\end{gathered}$$

holds by $a \neq \frac{1}{N-2}$ and it means the intervals in (56), (54), (59) – (62) are pairwise non-overlapping. This means that for each $N \in \{4, 5, 6, 7\}$ exists only one $n \in \left\{\overline{2, N}\right\}$ such that situation (27) is optimal by just if (28) is true. □

And what about the duel end moment and its respective situation (36) being optimal by (37)? It is the easiest to see when this optimal situation is single.

Theorem 10. *A progressive discrete silent duel (10) by (7), (8), (11) for* $N \in \mathbb{N} \backslash \{1, 2, 3\}$ *by*

$$a \in \left(0; \frac{1}{N-2}\right) \tag{63}$$

has the single saddle point (36).

Proof. Owing to Theorem 3, situation (36) is a saddle point by (63). Considering the N-th row of matrix (11), there is an inequality

$$y_j \leqslant \frac{N-2}{N-1} < 1 = x_N. \tag{64}$$

From (63) it follows that

$$a \cdot (N-2) < 1,$$

$$a \cdot (N-2) + N - 2 < N - 1,$$

$$\frac{N-2}{N-1} < \frac{1}{1+a},$$

whence

$$\frac{1}{1+a} > y_j \;\; \forall\, y_j < x_N = 1 \tag{65}$$

by using inequality (64). Inequality (65) is followed by inequality

$$1 - y_j - a y_j > 0$$

implying that inequality

$$K(x_N, y_j) = K(1, y_j) = a - a y_j - a^2 y_j > 0 \;\; \forall\, y_j < x_N = 1 \tag{66}$$

holds in the N-th row of matrix (11). Inequality (66) implies that situation (36) is the single saddle point in the N-th row, and entries $k_{iN} < 0 \;\forall\; i = \overline{1, N-1}$ in the N-th column. The latter implies that the N-th row is the single nonnegative row by (63) and saddle point (36) is the single one also. □

So, according to Theorems 8–10, if $a \neq \frac{1}{N-2}$ and the duel has a pure strategy solution, this solution is a single situation. If $a = \frac{1}{N-2}$ then, according to Theorem 4, the duel has four pure strategy solutions issued by the two last moments $t_{N-1} = \frac{N-2}{N-1}$, $t_N = 1$ of possible shooting. Apart from those solutions, there are no other saddle points in the duel. Indeed, according to Theorem 7, for each $N \in \mathbb{N} \backslash \left\{\overline{1, 7}\right\}$ situation (27) can be optimal only for $n \in \{2, 3, N-1, N\}$. For $a = \frac{1}{N-2}$ situation (21) is non-optimal due to (22) does not hold (Theorem 2). Situation (50) is also non-optimal due to (54) does not hold by $a = \frac{1}{N-2}$. For each $N \in \{4, 5, 6, 7\}$ only every two intervals in (54) and (59), (59) and (60), (60) and (61), (61) and (62) overlap at point $a = \frac{1}{N-2}$, which implies the existence of just two saddle points on the main diagonal of matrix (11).

9. Discussion and conclusion

Theorem 2 extending Theorem 1 claims that if the accuracy factor is not less than the number of possible shooting moments decreased by 2 then the duelist's optimal behavior is to shoot at the middle of the duel time span. Theorem 3 ascertains when the situation corresponding to an entry on the main diagonal is optimal. Theorem 4 clarifies it for the accuracy factor being equal to the reciprocal of the number of possible shooting moments decreased by 2, where the duelist's optimal behavior is to shoot at one of the two last moments $t_{N-1} = \frac{N-2}{N-1}$, $t_N = 1$ of possible shooting. Therefore, Theorem 4 is supplemented by Theorem 1 in the part of when the accuracy factor is 1 in the 3×3 duel. Theorem 5 claims that the conditions of optimality in Theorem 3 are always true and a saddle point on the main diagonal exists at the respective values of the accuracy factor when the number of possible shooting moments varies between 4 and 7. Theorem 6 clarifies when the conditions of optimality in Theorem 3 are violated for 8 and more moments of possible shooting. For this case Theorem 7 additionally ascertains that the main diagonal can contain saddle points only on second, third, and the last two rows of matrix (11). Theorem 8 extending Theorem 7 further ascertains that the saddle point is single in those rows if the accuracy factor is not equal to the reciprocal of the number of possible shooting moments decreased by 2. The solution singleness for such accuracy factors is proved in Theorem 9 for the duels in which the number of possible shooting moments varies between 4 and 7. In a way, Theorem 9 also extends Theorem 1 for bigger duels up to 7×7 duels, but Theorem 9 claiming the solution singleness does not specify the solution (for the specification, the reference to Theorem 3 is implied). Finally, Theorem 10 ascertains that the duel end moment is the only optimal strategy of the duelist if the accuracy factor is less than the reciprocal of the number of possible shooting moments decreased by 2.

Value $a = \frac{1}{N-2}$ is a boundary case of the accuracy factor. This is the only case where the duel has multiple pure strategy solutions (having no fewer than three moments of possible shooting). These four optimal situations are constituted by two last moments $t_{N-1} = \frac{N-2}{N-1}$, $t_N = 1$ of possible shooting. Otherwise, excluding the boundary case, the duel has either single optimal pure strategy situation (21) by

$$a > N - 2 \ \text{ for } \ N \in \mathbb{N} \backslash \{1,\ 2\}$$

or single optimal pure strategy situation (27) by

$$a \in \left(\frac{N-n}{n-1};\ \frac{N-1}{(n-2)\cdot(n-1)} \right] \ \text{ for } \ N \in \mathbb{N} \backslash \{1,\ 2,\ 3\} \ \text{ and } \ n \in \left\{ \overline{3,\ N} \right\}.$$

A 4×4 or bigger duel is not solved in pure strategies if the membership in (28) is false or the interval in the membership is empty. Then, however, a mixed strategy solution always exists owing to that this is a matrix game [10, 16]. The trivial duels, with just two or three moments of possible shooting, are always solved in pure strategies.

Specifying locations of moments of possible shooting as (7), (8) defines an approximation of the initial continuous-time game of timing. The approximation is uniform in this case. Nevertheless, the locations can be jittered to some extent by adding random values to the values in sets (7), (8). And this is an open question of whether the proved assertions remain valid (by the respective modifications) after the jittering. Another open question is a nonlinearity in the accuracy functions and its influence on the duel solution.

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РОЗВ'ЯЗКИ У ЧИСТИХ СТРАТЕГІЯХ БЕЗШУМНОЇ ДУЕЛІ НА РІВНОМІРНІЙ РЕШІТЦІ З ІДЕНТИЧНИМИ ФУНКЦІЯМИ ЛІНІЙНОЇ ВЛУЧНОСТІ

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Розв'язується скінченна гра з нульовою сумою, яка визначена на рівномірній решітці одиничного квадрату, як гра розрахунку часу. Ця гра є дискретною безшумною дуеллю, у якій ядро є кососиметричним, і гравці, до яких посилаються як до дуелянтів, мають ідентичні функції лінійної влучності, змінювані коефіцієнтом пропорційності влучності. Внаслідок кососиметричності обидва дуелянти мають ті самі оптимальні стратегії, а оптимальне значення гри дорівнює 0. Якщо коефіцієнт пропорційності влучності є не меншим, ніж кількість моментів можливого пострілу, зменшена на 2, то оптимальною поведінка дуелянта є постріл у середині інтервалу часу тривання дуелі. Визначено граничне значення коефіцієнта пропорційності влучності, яке дорівнює оберненому значенню кількості моментів пострілу, зменшеній на 2. У цьому випадку дуель має чотири розв'язки у чистих стратегіях, в яких фігурують два останні моменти можливого пострілу. У протилежному випадку дуель має єдину оптимальну ситуацію у чистих стратегіях, коли коефіцієнт пропорційності влучності належить певному непорожньому інтервалу 4×4 або більша дуель не розв'язується у чистих стратегіях, якщо такої належності не існує або інтервал належності є порожнім. Тривіальні дуелі, з лише двома або трьома моментами можливого пострілу, завжди розв'язуються у чистих стратегіях.

Ключові слова: гра розрахунку часу, безшумна дуель, лінійна влучність, рівномірна решітка, матрична гра, сідлова точка у чистих стратегіях.