ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2018. Випуск 85. С. 82–90
Visnyk of the Lviv Univ. Series Mech. Math. 2018. Issue 85. P. 82–90
http://publications.lnu.edu.ua/bulletins/index.php/mmf
doi: http://dx.doi.org/10.30970/vmm.2018.85.082-090

УДК 517.53 + 519.213.2

ON THE BELONGING OF ANALYTIC IN UNIT DISK CHARACTERISTIC FUNCTIONS OF PROBABILITY LAWS TO GENERALIZED CONVERGENCE CLASS

Oksana Mulyava[1], **Myroslav Sheremeta**[2]

[1]*Kyiv National University of Food Technologies,*
Volodymyrska Str., 68, 01004, Kyiv, Ukraine
e-mail: info@nuft.edu.ua
[2]*Ivan Franko Lviv National University,*
Universitetska Str., 1, 79000, Lviv, Ukraine
e-mail: m.m.sheremeta@gmail.com

For continuous on $[x_0, +\infty)$ functions α and β increasing to $+\infty$ we say that an analytic in $\mathbb{D} = \{z : |z| < 1\}$ characteristic function φ of a probability law F belongs to the generalized convergence $\alpha\beta$-class if $\int\limits_{r_0}^{1} \frac{\alpha(\ln M(r,\varphi))}{(1-r)^2\beta(\frac{1}{1-r})}dr < +\infty$, where $M(r,\varphi) = \max\{|\varphi(z) : |z| = r\}$. Conditions on α, β and F are found under which the function φ belongs to the generalized convergence $\alpha\beta$-class if and only if $\int\limits_{x_0}^{\infty} \alpha'(x)\beta_1\left(\frac{x}{\ln(W_F(x)e^x)}\right)dx < +\infty$, where $\beta_1(x) = \int\limits_{x}^{\infty}\frac{dt}{\beta(t)}$ and $W_F(x) = 1 - F(x) + F(-x)$.

Key words: analytic function, probability law, characteristic function, generalized convergence class.

1. Introduction

A continuous on the left on $(-\infty, +\infty)$ non-decreasing function F is said [1, p. 10] to be a *probability law* if $\lim\limits_{x\to+\infty} F(x) = 1$ and $\lim\limits_{x\to-\infty} F(x) = 0$, and the function $\varphi(z) = \int\limits_{-\infty}^{\infty} e^{izx}dF(x)$ defined for real z is called [1, p. 12] a characteristic function of this law. If φ has an analytic continuation on the disk $\mathbb{D} = \{z : |z| < 1\}$ then we call φ an analytic in

2010 Mathematics Subject Classification: 30B50, 60E10

$\mathbb{D}$ characteristic function of the law F. Further we always assume that $\mathbb{D}$ is the maximal disk of the analicity of φ. It is known [1, p. 37–38] that φ is an analytic in $\mathbb{D}$ characteristic function of the law F if and only if for every $r \in [0,\, 1)$

(1) $$W_F(x) =: 1 - F(x) + F(-x) = O(e^{-rx}), \quad x \to +\infty.$$

Hence it follows that

(2) $$\lim_{x\to+\infty} \frac{1}{x} \ln \frac{1}{W_F(x)} = 1.$$

For $0 \le r < 1$ we put $M(r,\varphi) = \max\{|\varphi(z)| : |z| = r\}$, and if φ has the order

$$\varrho = \overline{\lim_{r\uparrow 1}} \frac{\ln \ln M(r,\varphi)}{-\ln(1-r)} > 0$$

a convergence class is defined [2] by the condition

(3) $$\int_{r_0}^{1} (1-r)^{\varrho-1} \ln M(r,\varphi) dr < +\infty.$$

For $\varrho = 2$ this condition is sufficient [3, p. 50] in order that φ belong to the class of Mac-Lane.

For an analytic in $\mathbb{D}$ characteristic function φ of the order $\varrho > 0$ in [4] it is proved that in order that φ belong to convergence class it is necessary and in the case when the function $v(x) = \ln \dfrac{1}{W_F(x)}$ is continuously differentiable and v' increases it is sufficient that

(4) $$\int_{x_0}^{\infty} \left\{ \left(1 + \frac{1}{x} \ln W_F(x)\right)^{+} \right\}^{\varrho+1} dx < +\infty.$$

Generalizing this result in [5] the concept of the convergence Φ-class is introduced as follows.

Let $\Omega(1)$ be the class of positive unbounded on $(0,1)$ functions Φ such that the derivative Φ' is positive continuously differentiable and increasing to $+\infty$ on $(0,1)$.

As in [5], we say that φ belongs to a convergence Φ-class if

(5) $$\int_{r_0}^{1} \frac{\Phi'(r) \ln M(r,\varphi)}{\Phi^2(r)} dr < +\infty,$$

and by $V(1)$ we denote the class of positive continuously differentiable on $(0,+\infty)$ functions v such that $v'(x) \uparrow 1$ as $x \to +\infty$.

The following theorem was proved in [5].

Theorem 1. *Let* $\Phi \in \Omega(1)$, $\dfrac{\Phi'(r)}{\Phi(r)}$ *be a function, nondecreasing on* $[r_0,\, 1)$, $\Phi'(r) > \dfrac{1}{1-r}$ $\Phi'\left(r + \dfrac{1}{\Phi'(r)}\right) \le H_1 \Phi'(r)$ *and* $\dfrac{\Phi''(r)\Phi(r)}{(\Phi'(r))^2} \le H_2$ *for all* $r \in [r_0,\, 1)$, *where* $H_j = \text{const} >$

0, and $\int\limits_{r_0}^{1}\frac{\Phi'(r)\ln\Phi'(r)}{\Phi^2(r)}dr<+\infty$*. Suppose that* φ *is an analytic in* $\mathbb{D}$ *characteristic function on a probability law* F *such that* $\overline{\lim\limits_{x\to+\infty}}W_F(x)e^x=+\infty$.

Then in order that φ *belong to a convergence* Φ*-class it is necessary and, in the case when* $\ln\frac{1}{W_F(x)}=v(x)\in V(1)$*, it is sufficient that*

$$\int\limits_{x_0}^{\infty}\frac{dx}{\Phi'\left(\frac{1}{x}\ln\frac{1}{W_F(x)}\right)}<+\infty. \tag{6}$$

Corollary 1. *Let* $0<\varrho<+\infty$ *and* φ *be an analytic in* $\mathbb{D}$ *characteristic function of a probability law* F *such that* $\overline{\lim\limits_{x\to+\infty}}W_F(x)e^x=+\infty$*. Then in order that* (3) *holds it is necessary and, in the case when* $\ln\frac{1}{W_F(x)}=v(x)\in V(1)$*, it is sufficient that*

$$\int\limits_{x_0}^{\infty}\left(\frac{\ln(W_F(x)e^x)}{x}\right)^{\varrho+1}dx<+\infty.$$

Let L be a class of continuous increasing functions α such that $\alpha(x)\ge 0$ for $x\ge x_0$, $\alpha(x)=\alpha(x_0)$ for $x\le x_0$ and on $[x_0,+\infty)$ the function α increases to $+\infty$. We say that $\alpha\in L^0$ if $\alpha\in L$ and $\alpha(x(1+o(1)))=(1+o(1))\alpha(x)$ as $x\to+\infty$.

Let $\alpha\in L$ and $\beta\in L$. We say that an analytic in $\mathbb{D}$ function φ belongs to the generalized convergence $\alpha\beta$-class, if

$$\int\limits_{r_0}^{1}\frac{\alpha(\ln M(r,\varphi))}{(1-r)^2\beta(\frac{1}{1-r})}dr<+\infty. \tag{7}$$

If $\alpha(x)\equiv x$ and $\beta\equiv x^{\varrho+1}$ for $x_0\le x<+\infty$ then (7) implies (3). Here we examine a problem of the belonging of the analytic characteristic function of probability law to the generalized convergence $\alpha\beta$-class.

2. Auxiliary results

Let $I(r,\varphi)=\int\limits_0^{\infty}W_F(x)e^{xr}dx$ and $\mu(r,\varphi)=\sup\{W_F(x)e^{xr}:\ x\ge 0\}$ be the maximum of integrand. Suppose that $M(r,\varphi)\uparrow+\infty$ as $r\uparrow 1$. Then [5]

$$\ln\mu(r,\varphi)\le(1+o(1))\ln M(r,\varphi)\le(1+o(1))\ln I(r,\varphi),\quad r\uparrow 1.$$

Hence it follows that if $\alpha\in L^0$ then

$$\int\limits_{r_0}^{1}\frac{\alpha(\ln\mu(r,\varphi))}{(1-r)^2\beta(\frac{1}{1-r})}dr\le\int\limits_{r_0}^{1}\frac{\alpha(\ln M(r,\varphi))}{(1-r)^2\beta(\frac{1}{1-r})}dr\le\int\limits_{r_0}^{1}\frac{\alpha(\ln I(r,\varphi))}{(1-r)^2\beta(\frac{1}{1-r})}dr. \tag{8}$$

On the other hand

$$I(r,\varphi)=\int_0^\infty W_F(x)e^{xr}dx=\int_0^\infty W_F(x)\exp\left\{\frac{r+1}{2}x\right\}\exp\left\{-\frac{1-r}{2}x\right\}dx\le$$

$$\le\mu\left(\frac{r+1}{2},\varphi\right)\frac{2}{1-r}. \tag{9}$$

In [6] it is proved that if $\alpha\in L^0$ then α is *RO*-varying and, thus [7, p. 86], $1\le\alpha(lx)/\alpha(x)\le M(l)<+\infty$ for each $l\in[1,+\infty)$ and all $x\ge x_0(l)$. Therefore, from (9) we obtain

$$\alpha(\ln I(r,\varphi))\le\alpha\left(2\max\left\{\ln\mu\left(\frac{r+1}{2},\varphi\right),\ \ln\frac{2}{1-r}\right\}\right)\le$$

$$\le M(2)\alpha\left(\max\left\{\ln\mu\left(\frac{r+1}{2},\varphi\right),\ \ln\frac{2}{1-r}\right\}\right)=$$

$$=M(2)\left(\max\left\{\alpha\left(\ln\mu\left(\frac{r+1}{2},\varphi\right)\right),\ \alpha\left(\ln\frac{2}{1-r}\right\}\right)\right)\le$$

$$\le M(2)\left(\alpha\left(\ln\mu\left(\frac{r+1}{2},\varphi\right)\right)+\alpha\left(\ln\frac{2}{1-r}\right)\right),$$

whence for $\beta\in L^0$ using the cite of result from [6] we obtain

$$\int_{r_0}^1\frac{\alpha(\ln I(r,\varphi))}{(1-r)^2\beta(\frac{1}{1-r})}dr\le M(2)\left(\int_{r_0}^1\frac{\alpha\left(\ln\mu\left(\frac{r+1}{2}\varphi\right)\right)}{(1-r)^2\beta(\frac{1}{1-r})}dr+\int_{r_0}^1\frac{\alpha\left(\ln\frac{2}{1-r}\right)}{(1-r)^2\beta(\frac{1}{1-r})}dr\right)=$$

$$=2M(2)\int_{r_0}^1\frac{\alpha\left(\ln\mu\left(\frac{r+1}{2},\varphi\right)\right)}{4(1-(r+1)/2)^2\beta(\frac{1}{2(1-(r+1)/2)})}d\frac{r+1}{2}+M(2)\int_{r_0}^1\frac{\alpha\left(\ln\frac{2}{1-r}\right)}{(1-r)^2\beta(\frac{1}{1-r})}dr\le$$

$$\le K_1\int_{t_0}^1\frac{\alpha(\ln\mu(t,\varphi))}{(1-t)^2\beta(\frac{1}{1-t})}dt+K_2\int_{x_0}^\infty\frac{\alpha(\ln x)}{\beta(x)}dx. \tag{10}$$

From (9) and (10) the following statement follows.

Proposition 1. *Let* $\alpha\in L^0$, $\beta\in L^0$ *and* $\int_{x_0}^\infty\frac{\alpha(\ln x)}{\beta(x)}dx<+\infty$. *Then* (7) *holds if and only if*

$$\int_{r_0}^1\frac{\alpha(\ln\mu(r,\varphi))}{(1-r)^2\beta(\frac{1}{1-r})}dr<+\infty. \tag{11}$$

The function $\ln\,\mu(r,\varphi)$ may be bounded. It is easy to show that $\mu(r,\varphi)\le K<+\infty$ for all $r\in[0,\,1)$ if and only if $W_F(x)e^x\le K<+\infty$ for all $x\ge 0$. Thus, $\mu(r,\varphi)\uparrow+\infty$ as $r\uparrow 1$ if and only if $\varlimsup\limits_{x\to+\infty} W_F(x)e^x=+\infty$. In [5] was proved that the function $\ln\,\mu(r,\varphi)$ is convex on $[0,\,1)$ and there exists a nondecreasing on $[0,\,R)$ function $\nu(r,\varphi)$ such that $(\ln\,\mu(r,\varphi))'=\nu(r,\varphi)$ for all $r\in(0,\,R)$ with the exception of an at most countable set, i.e.

$$\ln\,\mu(r,\varphi)=\ln\,\mu(r_0,\varphi)+\int\limits_{r_0}^{r}\nu(x,\varphi)dx,\quad 0\le r_0\le r<1. \tag{12}$$

Hence it follows that if $\mu(r,\varphi)\uparrow+\infty$ as $r\uparrow 1$ then $\nu(r,\varphi)\nearrow+\infty$ as $r\uparrow 1$.

If $\ln\,\dfrac{1}{W_F(x)}=v(x)\in V(1)$ then for every $r\in(0,\,1)$ the function $\ln\,W_F(x)+rx=$ $=-v(x)+rx$ has a unique point of the maximum $x=\nu(r,\varphi)$, which is a continuous on $(0,\,1)$ function increasing to $+\infty$, and

$$\ln\,\mu(r,\varphi)=\max\{\ln\,W_F(x)+rx:\,x\ge 0\}=\ln\,W_F(\nu(r,\varphi))+r\nu(r,\varphi),$$

whence

$$\frac{1}{\nu(r,,\varphi)}\ln\,\frac{1}{W_F(\nu(r,\varphi))}=r-\frac{\ln\,\mu(r,\varphi)}{\nu(r,\varphi)}\le r. \tag{13}$$

From (12) it follows that

$$\ln\,\mu(r,\varphi)=\ln\,\mu(r_0,\varphi)+\nu(r,\varphi)(r-r_0)\le\ln\,\mu(r_0,\varphi)+(1-r_0)\nu(r,\varphi),$$

and if $\alpha\in L^0$ then $\alpha(\ln\,\mu(r,\varphi))\le K_1\alpha(\nu(r,\varphi)$ for all $r\in[r_0,\,1)$.

On the other hand for $r\ge r_0$

$$\ln\,\mu\left(\frac{1+r}{2},\varphi\right)\ge\ln\,\mu(r_0\varphi)+\int\limits_{r}^{(1+r)/2}\nu(x,\varphi)dx\ge\ln\,\mu(r_0,\varphi)+\nu(r,\varphi)\frac{1-r}{2},$$

and if $\alpha(e^x)\in L^0$ then as above we obtain

$$\alpha(\nu(r,\varphi))\le\alpha\left(\exp\left\{\ln\,\frac{2}{1-r}+\ln\,\ln\,\mu\left(\frac{1+r}{2},\varphi\right)\right\}\right)\le$$
$$\le\alpha\left(\exp\left\{2\max\left\{\ln\,\frac{2}{1-r},\,\ln\,\ln\,\mu\left(\frac{1+r}{2},\varphi\right)\right\}\right\}\right)\le$$
$$\le K_2\left(\alpha\left(\ln\,\mu\left(\frac{1+r}{2},\varphi\right)\right)+\alpha\left(\ln\,\frac{2}{1-r}\right)\right).$$

Thus,

$$\int\limits_{r_0}^{1}\frac{\alpha(\ln\,\mu(r,\varphi))}{(1-r)^2\beta(\frac{1}{1-r})}dr\le K_1\int\limits_{r_0}^{1}\frac{\alpha(\ln\,\nu(r,\varphi))}{(1-r)^2\beta(\frac{1}{1-r})}dr\le$$
$$\le K_1K_2\int\limits_{r_0}^{1}\frac{\alpha(\ln\,\mu((r+1)/2,\varphi))}{(1-r)^2\beta(\frac{1}{1-r})}dr+K_1K_2\int\limits_{r_0}^{1}\frac{\alpha(\ln\,(2/(1-r)))}{(1-r)^2\beta(\frac{1}{1-r})}dr,$$

whence as above we obtain the following statement.

Proposition 2. *Let* $\alpha(e^x) \in L^0$, $\beta \in L^0$, $\int\limits_{x_0}^{\infty} \frac{\alpha(\ln x)}{\beta(x)} dx < +\infty$ *and* $\ln \frac{1}{W_F(x)} =$ $= v(x) \in V(R)$. *Then* (11) *holds if and only if*

$$\int\limits_{r_0}^{1} \frac{\alpha(\nu(r,\varphi))}{(1-r)^2\beta(\frac{1}{1-r})} dr < +\infty. \tag{14}$$

Thus, the problem of belonging of φ to the generalized convergence $\alpha\beta$-class is reduced to the problem of the fulfilment of (14).

3. Main result

Using Propositions 1 and 2 we may prove the following main theorem.

Theorem 2. *Let* $\alpha(e^x) \in L^0$, $\beta \in L^0$, $\int\limits_{x_0}^{\infty} \frac{\alpha(\ln x)}{\beta(x)} dx < +\infty$ *and* $\frac{x\beta'(x)}{\beta(x)} \geq 2 + h$ *for all* $x \geq x_0$. *Suppose that* φ *is an analytic in* $\mathbb{D}$ *characteristic function on probability law* F *such that* $W_F(0) = 1$, $\ln \frac{1}{W_F(x)} = v(x) \in V(1)$ *and* $\overline{\lim\limits_{x\to+\infty}} W_F(x)e^x = +\infty$.

Then in order that φ *belongs to a generalized convergence* $\alpha\beta$*-class it is necessary and sufficient that*

$$\int\limits_{x_0}^{\infty} \alpha(x)\beta_1\left(\frac{x}{\ln\left(W_F(x)e^x\right)}\right) dx < +\infty, \quad \beta_1(x) = \int\limits_{x}^{\infty} \frac{dt}{\beta(t)}. \tag{15}$$

Proof. Clearly,

$$\int\limits_{r_0}^{1} \frac{\alpha(\nu(r,\varphi))}{(1-r)^2\beta(\frac{1}{1-r})} dr = -\int\limits_{r_0}^{1} \alpha(\nu(r,\varphi)) d\beta_1\left(\frac{1}{1-r}\right) =$$

$$= -\alpha(\nu(r,\varphi))\beta_1\left(\frac{1}{1-r}\right)\Big|_{r_0}^{1} + \int\limits_{r_0}^{1} \alpha'(\nu(r,\varphi))\beta_1\left(\frac{1}{1-r}\right) d\nu(r,\varphi). \tag{16}$$

At first we suppose that (15) holds. Then, from (16) and (13), in view of the nonincreasing of β_1, we have

$$\int\limits_{r_0}^{1} \frac{\alpha(\nu(r,\varphi))}{(1-r)^2\beta(\frac{1}{1-r})} dr \leq K + \int\limits_{r_0}^{1} \alpha'(\nu(r,\varphi))\beta_1\left(\frac{1}{1 - \frac{1}{\nu(r,\varphi)} \ln \frac{1}{W_F(\nu(r,\varphi))}}\right) d\nu(r,\varphi) =$$

$$= K + \int\limits_{x_0}^{\infty} \alpha'(\nu(r,\varphi))\beta_1\left(\frac{\nu(r,\varphi)}{\ln\left(W_F(\nu(r,\varphi)x)e^{\nu(r,\varphi)}\right)}\right) d\nu(r,\varphi) < +\infty,$$

because the function $\nu(r,\varphi)$ is continuous. The sufficiency of (15) is proved.

Now we prove its necessity. From (14) for each $\varepsilon > 0$ and all $r \in [r_0(\varepsilon), 1)$ we have

$$\varepsilon > \int_r^1 \frac{\alpha(\nu(r,\varphi))}{(1-r)^2\beta(\frac{1}{1-r})}dr \geq \alpha(\nu(r,\varphi)) \int_r^1 \frac{dr}{(1-r)^2\beta(\frac{1}{1-r})} = \alpha(\nu(r,\varphi))\beta_1\left(\frac{1}{1-r}\right),$$

that is from (14) and (16) we obtain

$$\int_{r_0}^1 \alpha'(\nu(r,\varphi))\beta_1\left(\frac{1}{1-r}\right)d\nu(r,\varphi) < +\infty.$$

Since $\ln\dfrac{1}{W_F(x)} = v(x) \in V(1)$ and $x = \nu(r,\varphi)$ is a solution of the equation $-v'(x) + r = 0$, we have $r = v'(\nu(r,\varphi))$ and hence it follows that

$$\int_{r_0}^\infty \alpha'(\nu(r,\varphi))\beta_1\left(\frac{1}{1-v'(\nu(r,\varphi))}\right)d\nu(r,\varphi) < +\infty,$$

i.e.

$$\int_{x_0}^\infty \alpha'(x)\beta_1\left(\frac{1}{1-v'(x)}\right)dx < +\infty. \tag{17}$$

From a theorem proved in [8] it follows that if $a(x)$ and $\mu(x)$ are continuous functions on $(0, +\infty)$, $-\infty \leq A < a(x) < B \leq +\infty$, $\mu(x) \searrow \mu \geq 0$ as $x \to +\infty$, and for a positive function f on (A, B) the function $f^{1/p}$ with $p > 1$ is convex on (A, B), then

$$\int_0^y \mu(x)f\left(\frac{1}{x}\int_0^x a(t)dt\right)dx \leq \left(\frac{p}{p-1}\right)^p \int_0^y \mu(x)f(a(x))dx, \quad y \leq +\infty. \tag{18}$$

We choose $\mu(x) = \alpha'(x)$, $a(x) = v'(x)$, $f(x) = \beta_1\left(\dfrac{1}{1-x}\right)$ and show that the function $f^{1/p}$ is convex for some $p > 1$.

It is easy to see that $f^{1/p}$ is convex for $p > 1$ if $f(x)f''(x) - \dfrac{p-1}{p}(f'(x))^2 \geq 0$ that is if

$$\beta_1\left(\frac{1}{1-x}\right)\beta_1''\left(\frac{1}{1-x}\right) + 2(1-x)\beta_1\left(\frac{1}{1-x}\right)\beta_1'\left(\frac{1}{1-x}\right) \geq \frac{p-1}{p}\left(\beta_1'\left(\frac{1}{1-x}\right)\right)^2,$$

and thus, if

$$\beta_1(t)\beta_1''(t) + \frac{2}{t}\beta_1(t)\beta_1'(t) \geq \frac{p-1}{p}(\beta_1'(t))^2.$$

Since $\beta_1(t) = \displaystyle\int_t^\infty \frac{dx}{\beta(x)}$, the last inequality holds if

$$\left(\beta'(t) - \frac{2\beta(t)}{t}\right)\int_t^\infty \frac{dx}{\beta(x)} \geq \frac{p-1}{p}. \tag{19}$$

Since $\beta'(t) - \dfrac{2\beta(t)}{t} \geq \dfrac{h\beta(t)}{t} > 0$, we have

$$\left(\beta'(t) - \frac{2\beta(t)}{t}\right) \int\limits_t^\infty \frac{dx}{\beta(x)} \geq \left(\beta'(t) - \frac{2\beta(t)}{t}\right) \int\limits_t^{2t} \frac{dx}{\beta(x)} \geq \left(\beta'(t) - \frac{2\beta(t)}{t}\right) \frac{t}{\beta(t)} \geq h.$$

Therefore, choosing $p > 1$ such that $h - \dfrac{p-1}{p} \geq 0$, we get inequality (19), i. e. the function $\beta_1^{1/p}\left(\dfrac{1}{1-x}\right)$ is convex and in view of (18)

$$\int\limits_0^\infty \alpha'(x)\beta_1\left(\frac{1}{1 - \frac{1}{x}\int\limits_0^x v'(t)dt}\right) dx \leq \left(\frac{p}{p-1}\right)^p \int\limits_0^\infty \alpha'(x)\beta_1\left(\frac{1}{1-v'(x)}\right) dx. \tag{20}$$

Since $\displaystyle\int\limits_{x_0}^x v'(t)dt = \ln\frac{1}{W_F(x))}$, from (17) and (20) we obtain (15). Theorem 2 is proved.

□

References

1. Yu. V. Linnik and I. V. Ostrovskii, *Decompositon of random variables and vectors*, Nauka, Moscov, 1972 (Russian).
2. Yu. M. Gal' and M. M. Sheremeta, *Belonging of analytic functions to a covergence class*, Dop. AN URSR, Ser. A (1985), no. 7, 11–14 (Ukrainian).
3. G. R. MacLane, *Asymptotic values of holomorphic functions*, Rice Univ. Studies **49** (1963), no. 1, 1–83.
4. V. M. Sorokivskii, *On the growth of characteristic functions of probability laws*, Drogobych (1980), 20 p. Dep. in VINITI 17.12.80, N 5330-80 DEP (Russian).
5. L. V. Kulyavets', O. M. Mulyava, and M. M. Sheremeta, *On belonging of characteristic functions of probability laws to a convergence class*, Bull. Soc. Sci. Lett. Lodz., Ser Rech. Deform. **63** (2013), no. 2, 9–22.
6. M. M. Sheremeta, *On two classes of positive functions and the belonging to them of main characteristics of entire functions*, Mat. Stud. **19** (2003), no. 1, 75–82.
7. E. Seneta, *Regularly varying functions*, Lect. Notes Math. **508**, Springer-Verlag, Berlin, 1976.
8. О. М. Мулява, *Інтегральний аналог одного узагальнення нерівності Гарді та його застосування*, Укр. мат. журн. **58** (2006), no. 9, 1271–1275; **English version**: O. M. Mulyava, *Integral analog of one generalization of the Hardy inequality and its applications*, **58** (2006), no. 9, 1441–1447. DOI: 10.1007/s11253-006-0143-0

Стаття: надійшла до редколегії 23.10.2018
доопрацьована 03.12.2018
прийнята до друку 26.12.2018

ПРО НАЛЕЖНІСТЬ АНАЛІТИЧНИХ В ОДИНИЧНОМУ КРУЗІ ХАРАКТЕРИСТИЧНИХ ФУНКЦІЙ ЙМОВІРНІСНИХ ЗАКОНІВ ДО УЗАГАЛЬНЕНОГО КЛАСУ ЗБІЖНОСТІ

Оксана МУЛЯВА[1], **Мирослав ШЕРЕМЕТА**[2]

[1]*Київський національний університет харчових технологій*
Володимирівська 68, Київ, 01004
e-mail: info@nuft.edu.ua
[2]*Львівський національний університет ім. І. Франка,*
Університетська 1, 79000, Львів
e-mail: m.m.sheremeta@gmail.com

Для неперервних зростаючих до $+\infty$ на $[x_0, +\infty)$ функцій α і β будемо говорити, що аналітична в $\mathbb{D} = \{z : |z| < 1\}$ характеристична функція φ ймовірнісного закону F належить до узагальненого $\alpha\beta$-класу збіжності, якщо $\int\limits_{r_0}^{1} \frac{\alpha(\ln M(r,\varphi))}{(1-r)^2\beta(\frac{1}{1-r})}dr < +\infty$. Знайдено умови на α, β і F, за яких функція φ належить до узагальненого $\alpha\beta$-класу збіжності тоді і тільки тоді, коли $\int\limits_{x_0}^{\infty} \alpha'(x)\beta_1\left(\frac{x}{\ln(W_F(x)e^x)}\right)dx < +\infty$, де $\beta_1(x) = \int\limits_{x}^{\infty} \frac{dt}{\beta(t)}$ і $W_F(x) = 1 - F(x) + F(-x)$.

Ключові слова: аналітична функція, ймовірнісний закон, характеристична функція, узагальнений клас збіжності.