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A REMOVABILITY RESULT FOR SEPARATELY SUBHARMONIC FUNCTIONS

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Blanchet has shown that a $\mathcal{C}^2$ subharmonic function can be extended throught a $\mathcal{C}^1$ hypersurface provided the function satisfies certain $\mathcal{C}^1$ type continuity condtions on the exceptional hypersurface. Recently we improved Blanchet's result by measuring the exceptional set with the aid of Hausdorff measure. Now we give a related extension result for separately subharmonic functions.

Key words: subharmonic function, separately subharmonic function, Hausdorff measure, exceptional sets.

1. Introduction

1.1. We give an extension result for separately subharmonic $\mathcal{C}^2$ functions, see Theorem 2 below. Our proof is based on our previous extension result for $\mathcal{C}^2$ subharmonic functions, see [8, Theorem 1, p. 154], and on a general result, see [3, Proposition 1, p. 33]. Moreover, we need Federer's important results from the geometric measure theory, see e.g. [2, 9].

1.2. For the used notation, see [6, 7, 8]. However, for convenience of the reader we recall here the following: If $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $n \geqslant 2$, and $j \in \mathbb{N}$, $1 \leqslant j \leqslant n$, then we write $x = (x_j, X_j) = (X_j, x_j)$, where $X_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$. Moreover, if $A \subset \mathbb{R}^n$, $1 \leqslant j \leqslant n$, and $x_j^0 \in \mathbb{R}$, $X_j^0 \in \mathbb{R}^{n-1}$, we write

$$A(x_j^0) = \left\{X_j \in \mathbb{R}^{n-1} \colon x = (x_j^0, X_j) \in A\right\}, \quad A(X_j^0) = \left\{x_j \in \mathbb{R} \colon x = (x_j, X_j^0) \in A\right\}.$$

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2. Auxiliary results

2.1. **A result of Federer.** The following important result of Federer from the geometric measure theory will be used repeatedly.

Lemma 1 ([2, Theorem 2.10.25, p. 188], [9, Corollary 4, Lemma 2, p. 114]). *Suppose that $E \subset \mathbb{R}^n$, $n \geqslant 2$. Let $\alpha \geqslant 0$ and let $\pi_k : \mathbb{R}^n \to \mathbb{R}^k$ denote the projection onto the first k coordinates.*

(*i*) *If $\mathcal{H}^{k+\alpha}(E) = 0$, then $\mathcal{H}^{\alpha}(E \cap \pi_k^{-1}(x)) = 0$ for $\mathcal{H}^k$-almost all $x \in \mathbb{R}^k$.*
(*ii*) *If $\mathcal{H}^{k+\alpha}(E) < +\infty$, then $\mathcal{H}^{\alpha}(E \cap \pi_k^{-1}(x)) < +\infty$ for $\mathcal{H}^k$-almost all $x \in \mathbb{R}^k$.*

2.2. **Our previous extension result for subharmonic functions.** As pointed out above, we use our previous extension result [8, Theorem 1, p. 154], however, now in the following, only seemingly more general form. For our previous related results, see [4, Theorem 4, pp. 181-182], [6, Theorem, p. 568], and [7, Lemma 2, p. 51]. Let it be pointed out also here that Blanchet's results [1, Theorems 3.1, 3.2 and 3.3, pp. 312-313], have been the starting point of our cited results.

Theorem 1. *Suppose that Ω is a domain in $\mathbb{R}^n$, $n \geqslant 2$. Let $E \subset \Omega$ be closed in Ω and let $\mathcal{H}^{n-1}(E) < +\infty$. Let $u\colon \Omega \setminus E \to \mathbb{R}$ be subharmonic and such that the following conditions are satisfied:*

(*i*) $u \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$.
(*ii*) $u \in \mathcal{C}^2(\Omega \setminus E)$.
(*iii*) *For each j, $1 \leqslant j \leqslant n$, $\frac{\partial^2 u}{\partial x_j^2} \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$.*
(*iv*) *For each j, $1 \leqslant j \leqslant n$, and for $\mathcal{H}^{n-1}$-almost all $X_j \in \mathbb{R}^{n-1}$ such that $E(X_j)$ is finite, the following condition holds:*
For each $x_j^0 \in E(X_j)$ there exist sequences $x_{j,l}^{0,1}, x_{j,l}^{0,2} \in (\Omega \setminus E)(X_j)$, $l = 1, 2, \ldots$, such that $x_{j,l}^{0,1} \nearrow x_j^0$, $x_{j,l}^{0,2} \searrow x_j^0$ as $l \to +\infty$, and
(*iv*(a)) $\lim_{l \to +\infty} u(x_{j,l}^{0,1}, X_j) = \lim_{l \to +\infty} u(x_{j,l}^{0,2}, X_j) \in \mathbb{R}$,
(*iv*(b)) $-\infty < \liminf_{l \to +\infty} \frac{\partial u}{\partial x_j}(x_{j,l}^{0,1}, X_j) \leqslant \limsup_{l \to +\infty} \frac{\partial u}{\partial x_j}(x_{j,l}^{0,2}, X_j) < +\infty$.

Then u has a subharmonic extension to Ω.

2.3. In this connection and related to the above Theorem 1, we take the opportunity to state the following concise corollary. As a matter of fact, we have previously not stated it explicitly, and we feel that it might be of interest in itself.

Corollary 1. *Suppose that Ω is a domain in $\mathbb{R}^n$, $n \geqslant 2$. Let $E \subset \Omega$ be closed in Ω and let $\mathcal{H}^{n-1}(E) = 0$. Let $u\colon \Omega \setminus E \to \mathbb{R}$ be subharmonic and such that the following conditions hold:*

(*i*) $u \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$,
(*ii*) $u \in \mathcal{C}^2(\Omega \setminus E)$,
(*iii*) *for each j, $1 \leqslant j \leqslant n$, $\dfrac{\partial^2 u}{\partial x_j^2} \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$.*

Then u has a subharmonic extension to Ω.

2.4. In addition to Federer's above lemma and our above Theorem 1, we need also the following nice result. Observe here that the below used *hypoharmonic functions* are in our terminology just *subharmonic functions*.

Proposition 1 ([3, Proposition 1, p. 33]). *Suppose that Ω is a domain in $\mathbb{R}^{p+q}$, $p, q \geqslant 2$. Let $w\colon \Omega \to [-\infty, +\infty)$ be nearly subharmonic. Let $w^*\colon \Omega \to [-\infty, +\infty)$ be the regularized function of w, which is then subharmonic. Then the following properties are equivalent.*

(1) *The distribution $\Delta_x w = \Delta_x w^* =$ (sum of the square second order derivatives of w or w^* with respect to the p coordinates of x) is positive.*
(2) *For all $y \in \mathbb{R}^q$ the function $\Omega(y) \ni x \mapsto w^*(x, y) \in [-\infty, +\infty)$ is hypoharmonic.*
(3) *For almost every $y \in \mathbb{R}^q$ the function $\Omega(y) \ni x \mapsto w^*(x, y) \in [-\infty, +\infty)$ is subharmonic.*
(4) *For almost every $y \in \mathbb{R}^q$ the function $\Omega(y) \ni x \mapsto w^*(x, y) \in [-\infty, +\infty)$ is nearly subharmonic.*

3. An extension result for separately subharmonic functions

Our result is the following

Theorem 2. *Suppose that Ω is a domain in $\mathbb{R}^{p+q}$, $p, q \geqslant 2$. Let $E \subset \Omega$ be closed in Ω and let $\mathcal{H}^{p+q-1}(E) < +\infty$. Let $w\colon \Omega \setminus E \to \mathbb{R}$ be separately subharmonic, that is,*

for all $y \in \mathbb{R}^q$ the function $(\Omega \setminus E)(y) \ni x \mapsto w(x, y) \in \mathbb{R}$ is subharmonic,

and

for all $x \in \mathbb{R}^p$ the function $(\Omega \setminus E)(x) \ni y \mapsto w(x, y) \in \mathbb{R}$ is subharmonic,

and such that the following conditions are satisfied:

(i) $w \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$.
(ii) $w \in \mathcal{C}^2(\Omega \setminus E)$.
(iii) *For each j, $1 \leqslant j \leqslant p$, $\frac{\partial^2 w}{\partial x_j^2} \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$, and for each k, $1 \leqslant k \leqslant q$, $\frac{\partial^2 w}{\partial y_k^2} \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$.*
(iv) *For each j, $1 \leqslant j \leqslant p$, and for $\mathcal{H}^{p-1+q}$-almost all $(X_j, y) \in \mathbb{R}^{p-1+q}$ such that $E(X_j, y)$ is finite, the following condition holds:*
For each $x_j^0 \in E(X_j, y)$ there exist sequences $x_{j,l}^{0,1}, x_{j,l}^{0,2} \in (\Omega \setminus E)(X_j, y)$, $l = 1, 2, \ldots$, such that $x_{j,l}^{0,1} \nearrow x_j^0$, $x_{j,l}^{0,2} \searrow x_j^0$ as $l \to +\infty$, and
(iv(a)) $\lim_{l\to+\infty} w(x_{j,l}^{0,1}, X_j, y) = \lim_{l\to+\infty} w(x_{j,l}^{0,2}, X_j, y) \in \mathbb{R}$,
(iv(b)) $-\infty < \liminf_{l\to+\infty} \frac{\partial w}{\partial x_j}(x_{j,l}^{0,1}, X_j, y) \leqslant \limsup_{l\to+\infty} \frac{\partial w}{\partial x_j}(x_{j,l}^{0,2}, X_j, y) < +\infty$.
(v) *For each k, $1 \leqslant k \leqslant q$, and for $\mathcal{H}^{p+q-1}$-almost all $(x, Y_k) \in \mathbb{R}^{p+q-1}$ such that $E(x, Y_k)$ is finite, the following condition holds:*
For each $y_k^0 \in E(x, Y_k)$ there exist sequences $y_{k,l}^{0,1}, y_{k,l}^{0,2} \in (\Omega \setminus E)(x, Y_k)$, $l = 1, 2, \ldots$, such that $y_{k,l}^{0,1} \nearrow y_k^0$, $y_{k,l}^{0,2} \searrow y_k^0$ as $l \to +\infty$, and
(v(a)) $\lim_{l\to+\infty} w(x, y_{k,l}^{0,1}, Y_k) = \lim_{l\to+\infty} w(x, y_{k,l}^{0,2}, Y_k) \in \mathbb{R}$,
(v(b)) $-\infty < \liminf_{l\to+\infty} \frac{\partial w}{\partial y_k}(x, y_{k,l}^{0,1}, Y_k) \leqslant \limsup_{l\to+\infty} \frac{\partial w}{\partial y_k}(x, y_{k,l}^{0,2}, Y_k) < +\infty$.

Then w has a separately subharmonic extension to Ω.

Proof. By [5, Corollary 4.6, p. 412], w is subharmonic in $\Omega \setminus E$. Thus by Theorem 1 $w\colon \Omega \setminus E \to \mathbb{R}$ has a subharmonic extension $w^*\colon \Omega \to [-\infty, +\infty)$. By Proposition 1 it is therefore sufficient to show that

- for $\mathcal{H}^q$-almost all $y \in \mathbb{R}^q$ the subharmonic function $(\Omega \setminus E)(y) \ni x \mapsto w(x, y) \in \mathbb{R}$ has a subharmonic extension $\Omega(y) \ni x \mapsto w^*(x, y) \in [-\infty, +\infty)$, and

- for $\mathcal{H}^p$-almost all $x \in \mathbb{R}^p$ the subharmonic function $(\Omega \setminus E)(x) \ni y \mapsto w(x, y) \in \mathbb{R}$ has a subharmonic extension $\Omega(x) \ni y \mapsto w^*(x, y) \in [-\infty, +\infty)$.

We show that the first condition holds. The proof of the second is similar.

Fix j, $1 \leqslant j \leqslant p$, arbitrarily for a while.

By our assumption $\mathcal{H}^{p-1+q}(E) < +\infty$. From the above Lemma of Federer, it follows that for $\mathcal{H}^{p-1+q}$-almost all $(X_j, y) \in \mathbb{R}^{p-1+q}$ the set $E(y)(X_j)$ is finite. Write

$$A := \left\{(X_j, y) \in \mathbb{R}^{p-1+q} \colon E(y)(X_j) \text{ is finite}\right\}.$$

Thus

$$\mathcal{H}^{p-1+q}(A^c) = 0 \iff m_{p-1+q}(A^c) = 0 \iff \int\limits_{\mathbb{R}^{p-1+q}} \chi_{A^c}(X_j, y) dm_{p-1+q}(X_j, y) = 0,$$

where $\chi_{A^c}(\cdot, \cdot)$ is the characteristic function of the set A^c, the complement taken in $\mathbb{R}^{p-1+q}$.

Next use Fubini's theorem:

$$0 = \int\limits_{\mathbb{R}^{p-1+q}} \chi_{A^c}(X_j, y) dm_{p-1+q}(X_j, y) = \int\limits_{\mathbb{R}^q} \left[\int\limits_{\mathbb{R}^{p-1}} \chi_{A^c}(X_j, y) dm_{p-1}(X_j) \right] dm_q(y).$$

Since

$$\int\limits_{\mathbb{R}^{p-1}} \chi_{A^c}(X_j, y) dm_{p-1}(X_j) \geqslant 0,$$

we see that in fact

$$\int\limits_{\mathbb{R}^{p-1}} \chi_{A^c}(X_j, y) dm_{p-1}(X_j) = 0$$

for $\mathcal{H}^q$-almost all $y \in \mathbb{R}^q$.

Write

$$\begin{aligned} B_1^j :&= \left\{ y \in \mathbb{R}^q \colon \int\limits_{\mathbb{R}^{p-1}} \chi_{A^c}(X_j, y) dm_{p-1}(X_j) = 0 \right\}, \\ &= \left\{ y \in \mathbb{R}^q \colon \chi_{A^c}(X_j, y) = 0 \text{ for } \mathcal{H}^{p-1} \text{ almost all } X_j \in \mathbb{R}^{p-1} \right\}, \\ &= \left\{ y \in \mathbb{R}^q \colon \chi_{A}(X_j, y) = 1 \text{ for } \mathcal{H}^{p-1} \text{ almost all } X_j \in \mathbb{R}^{p-1} \right\}. \end{aligned}$$

Write $B_1 := B_1^1 \cap B_1^2 \cap \cdots \cap B_1^p$. Then for all $y \in B_1$ we have $(X_j, y) \in A$ for $\mathcal{H}^{p-1}$ almost all $X_j \in \mathbb{R}^{p-1}$, and this holds for all $j = 1, 2, \ldots, p$.

Next write

$$\begin{aligned} B_2 &:= \left\{ y \in \mathbb{R}^q \colon w(\cdot, y) \in \mathcal{L}^1_{\text{loc}}(\Omega(y)) \right\}, \\ B_3 &:= \left\{ y \in \mathbb{R}^q \colon w(\cdot, y) \in \mathcal{C}^2((\Omega \setminus E)(y)) \right\}, \\ B_4^j &:= \left\{ y \in \mathbb{R}^q \colon \tfrac{\partial^2}{\partial x_j^2} w(\cdot, y) \in \mathcal{L}^1_{\text{loc}}(\Omega(y)) \right\}, \\ B_4 &:= B_4^1 \cap B_4^2 \cap \cdots \cap B_4^p, \end{aligned}$$

and $B := B_1 \cap B_2 \cap B_3 \cap B_4$.

Then for all $y \in B$ the function $(\Omega \setminus E)(y) \ni x \mapsto w(x, y) \in \mathbb{R}$ satisfies the assumptions of Theorem 1. Therefore these functions have subharmonic extensions

$$\Omega(y) \ni x \mapsto w^*(x, y) \in [-\infty, +\infty).$$

But then our claim follows from Proposition 1. □

Example 1. The function $u : \mathbb{R}^4 \to \mathbb{R}$,

$$u(z_1, z_2) = u(x_1 + iy_1, x_2 + iy_2) = u(x_1, y_1, x_2, y_2) := \begin{cases} 1 + x_1, & \text{when } x_1 < 0, \\ 1 - x_1, & \text{when } x_1 \geqslant 0 \end{cases}$$

is continuous in $\mathbb{R}^4$ and separately subharmonic, even separately harmonic in $\mathbb{R}^4 \setminus (\{0\} \times \mathbb{R}^3)$, but not separately subharmonic in $\mathbb{R}^4$. Observe that u satisfies the above conditions (i), (ii), (iii), $(iv(\text{a}))$ and $(v(\text{a}))$ in $\mathbb{R}^4 \setminus (\{0\} \times \mathbb{R}^3)$. However, $u|\mathbb{R}^4 \setminus (\{0\} \times \mathbb{R}^3)$ does not satisfy the conditions $(iv(\text{b}))$ and $(v(\text{b}))$. Thus these conditions cannot be dropped in Theorem 2.

Corollary 2. *Suppose that Ω is a domain in $\mathbb{R}^{p+q}$, $p, q \geqslant 2$. Let $E \subset \Omega$ be closed in Ω and let $\mathcal{H}^{p+q-1}(E) = 0$. Let $w \colon \Omega \setminus E \to \mathbb{R}$ be separately subharmonic, that is,*

for all $y \in \mathbb{R}^q$ the function $(\Omega \setminus E)(y) \ni x \mapsto w(x, y) \in \mathbb{R}$ is subharmonic,

and

for all $x \in \mathbb{R}^p$ the function $(\Omega \setminus E)(x) \ni y \mapsto w(x, y) \in \mathbb{R}$ is subharmonic.

Suppose that the following conditions are satisfied:

(i) $w \in \mathcal{L}^1_{\text{loc}}(\Omega)$,
(ii) $w \in \mathcal{C}^2(\Omega \setminus E)$,
(iii) *for each j, $1 \leqslant j \leqslant p$, $\frac{\partial^2 w}{\partial x_j^2} \in \mathcal{L}^1_{\text{loc}}(\Omega)$ and for each k, $1 \leqslant k \leqslant q$, $\frac{\partial^2 w}{\partial y_k^2} \in \mathcal{L}^1_{\text{loc}}(\Omega)$.*

Then w has a separately subharmonic extension to Ω.

Proof. Follows directly from Theorem 2 and from the above Lemma of Federer. □

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ОДИН РЕЗУЛЬТАТ ПРО УСУНЕННЯ ОСОБЛИВОСТЕЙ ДЛЯ НАРІЗНО СУБГАРМОНІЧНИХ ФУНКЦІЙ

Юхані РІІГЕНТАУЗ

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Бланше довів, що субгармонічну функцію гладкості $\mathcal{C}^2$ можна продовжити через $\mathcal{C}^1$-гіперплощину, якщо функція задовольняє певну умову гладкості типу $\mathcal{C}^1$ на винятковій гіперплощині. Нещодавно ми покращили результат Бланше, розглянувши міру Гаусдорфа виняткової множини. Тепер ми наводимо подібне узагальнення для нарізно субгармонічних функцій.

Ключові слова: субгармонічна функція, нарізно субгармонічна функція, міра Гаусдорфа, виняткові множини.