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ON OLD AND NEW CLASSES OF FEEBLY COMPACT SPACES

Dedicated to the 75th birthday of Yaroslav Prytula

Oleg GUTIK[1], Oleksandr RAVSKY[2]

[1]*Ivan Franko National University of Lviv,*
Universitetska Str., 1, 79000, Lviv, Ukraine
[2]*Pidstryhach Institute for Applied Problems of Mechanics*
and Mathematics of NASU, Naukova Str., 3b, 79060, Lviv, Ukraine
e-mails: oleg.gutik@lnu.edu.ua, ovgutik@yahoo.com,
alexander.ravsky@uni-wuerzburg.de

We introduce three new classes of countably pracompact spaces, consider their basic properties and relations with another compact-like spaces.

Key words: compact, feebly compact, sequentially compact, ω-bounded, totally countably compact, countably compact, countably pracompact, pseudocompact, sequentially pseudocompact, sequentially pracompact, totally countably pracompact, ω-bounded-pracompact.

1. Definitions and relations

In general topology one often investigates different classes of compact-like spaces and relations between them, see, for instance, basic [11, Chap. 3] and general works [9], [19], [23], [22], [17]. We consider the present paper as a next small step in this quest.

We shall follow the terminology of [11]. By $\mathbb{N}$ we shall denote the set of all positive integers.

A subset of a topological space X is called *regular open* if it equals the interior of its closure. A space X is *quasiregular* if each nonempty open subset of X contains closure of some nonempty open subset of X.

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1.1. **Old classes.** We recall that a topological space X is said to be

- *semiregular* if X has a base consisting of regular open subsets;
- *compact* if each open cover of X has a finite subcover;
- *sequentially compact* if each sequence $\{x_n\}_{n\in\mathbb{N}}$ of X has a convergent subsequence in X;
- ω*-bounded* if each countable subset of X has compact closure;
- *totally countably compact* if each sequence of X contains a subsequence with compact closure;
- *countably compact* if each open countable cover of X has a finite subcover;
- *countably compact at a subset* $A \subseteq X$ if every infinite subset $B \subseteq A$ has an accumulation point x in X;
- *countably pracompact* if there exists a dense subset D in X such that X is countably compact at D;
- *feebly* ω*-bounded* if for each sequence $\{U_n\}_{n\in\mathbb{N}}$ of non-empty open subsets of X there is a compact subset K of X such that $K \cap U_n \neq \varnothing$ for each n;
- *selectively sequentially feebly compact* if for each sequence $\{U_n\}_{n\in\mathbb{N}}$ of non-empty open subsets of X we can choose a point $x_n \in U_n$ for each $n \in \mathbb{N}$ such that the sequence $\{x_n\}$ has a convergent subsequence;
- *selectively feebly compact*[1], if for each sequence $\{U_n\}_{n\in\mathbb{N}}$ of non-empty open subsets of X we can choose a point $x \in X$ and a point $x_n \in U_n$ for each $n \in \mathbb{N}$ such that the set $\{n \in \mathbb{N} \colon x_n \in W\}$ is infinite for every open neighborhood W of x.
- *sequentially feebly compact*[2] [10, Def. 1.4] if for each sequence $\{U_n \colon n \in \mathbb{N}\}$ of non-empty open subsets of the space X there exist a point $x \in X$ and an infinite set $I \subset \mathbb{N}$ such that for each neighborhood U of the point x the set $\{n \in I \colon U_n \cap U = \varnothing\}$ is finite;
- *feebly compact* if each locally finite family of nonempty open subsets of the space X is finite.
- k*-space* if X is Hausdorff and a subset $F \subset X$ is closed in X if and only if $F \cap K$ is closed in K for every compact subspace $K \subset X$.

[1]Selectively sequentially feebly compact Tychonoff spaces were recently introduced and studied by Dorantes-Aldama and Shakhmatov in [8]. Also they considered selectively feebly compact Tychonoff spaces under the name *selectively pseudocompact* spaces. An equivalent property appeared a few years earlier in papers by Garcıa-Ferreira with Ortiz-Castillo [12] and with Tomita [13] under the title "*strong pseudocompactness*", but since the term "strongly pseudocompact" is used in [3, 7] to denote two different properties, we stick to a name for this property which reflects its "selective" nature and also matches the name of the previous "selective" property.

[2]One of the authors introduced this notion a few years ago as a natural property intermediate between feeble and sequential compactness, which may be useful in some applications in topological algebra. Indeed, for instance, Proposition 1.10. by Artico et al. [4] combined with Theorem 1.1 by Lipparini [17] states that that each T_0 feebly compact topological group is sequentially feebly compact. But later we found that it is a known property, even with the same name. The oldest reference which we know (see [19, p. 15]) is Reznichenko's paper [21]. A similar notion had been given by Artico et al. in [4, Def. 1.8], where are used pairwise disjoint open sets instead. Lipparini proved in [17] that these notions are equivalent.

According to Theorem 3.10.22 of [11], a Tychonoff topological space X is feebly compact if and only if it is pseudocompact, that is, each continuous real-valued function on X is bounded. Also, a Hausdorff topological space X is feebly compact if and only if every locally finite family of non-empty open subsets of X is finite.

Relations between different classes of compact-like spaces are well-studied. Some of them are presented on Diagram 3 in [19, p.17], on Diagram 1 in [8, p. 58] (for Tychonoff spaces), and on Diagram 3.6 in [22, p. 611].

1.2. **New classes.** The notion of countable pracompactness has been studied by several authors under several names. According to Matveev [19] it "appeared in the literature under many different names". Matveev mentions that Baboolal, Backhouse and Ori [5] introduced an equivalent notion under the name e-countable compactness. In the recent paper [18] the authors study the notion using the expression "densely countably compact". A few references and a further name are recalled there [2] According to Arkhangel'skii [1] countable compactness at some subset and countable pracompactness "find important applications in C_p-theory".

In order to refine the stratification of countable pracompact spaces even more, we introduce the following definitions. In each of them we require that a space X contains a dense subset D with a special property. Namely,

- if each sequence of points of the set D has a convergent subsequence (in X) then X is *sequentially pracompact*;
- if each sequence of points of the set D has a subsequence with compact closure (in X) then X is *totally countably pracompact*;
- if each countable subset of the set D has compact closure (in X) then X is *ω-bounded-pracompact*.

Our main motivation to introduce the above spaces is their possible applications in topological algebra. In particular, we are going to use them in the paper [15].

Diagram 1 shows relations between different classes of compact-like spaces. All implications on the diagram are true and we *suggest* that they are either well-known or easy to prove and all non-marked arrows are not reversible without imposing additional conditions on spaces. In particular, in Section 4 of the present paper we construct a sequentially feebly compact space which is not selectively feebly compact (Example 2), a sequentially pracompact space which is not countably compact (Example 3), and a totally countably pracompact space which is nether ω-bounded-pracompact nor totally countably compact (Example 4).

2. Basic properties

2.1. **Extensions.** We recall that an *extension* of a space X is a space Y containing X as a dense subspace. It is easy to check that countable pracompactness, sequential pracompactness, feeble compactness, sequential feeble compactness, selective feeble compactness, selective sequential feeble compactness, and feeble ω-boundedness is preserved by extensions.

2.2. **Continuous images.** It is easy to check that sequential compactness, feeble compactness, sequential feeble compactness, countably pracompactness, and sequential pracompactness is preserved by continuous images and total countable compactness, total countable pracompactness, ω-boundedness, and ω-bounded-pracompactness is preserved by continuous Hausdorff images.

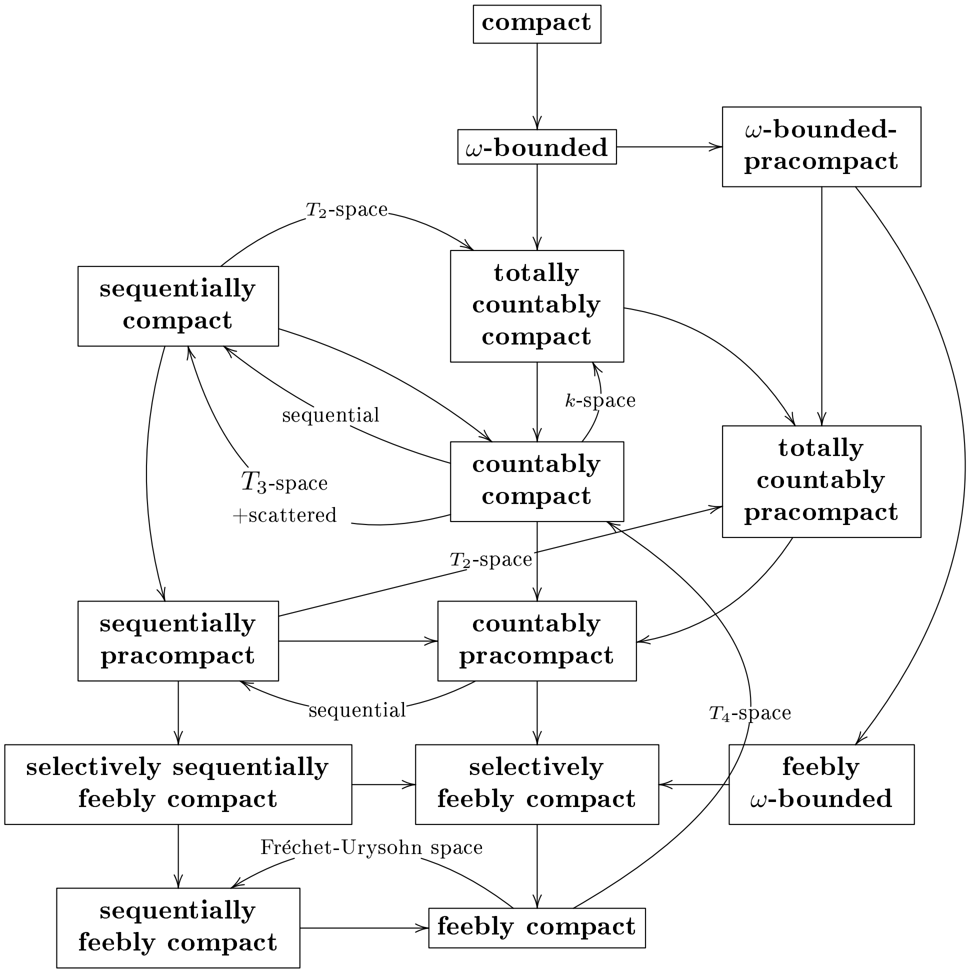

Diagram 1

2.3. **Products.** The investigation of productivity of compact-like spaces is motivated by the fundamental Tychonoff theorem, stating that a product of a family of compact spaces is compact, On the other hand, there are two countably compact spaces whose product is not feebly compact (see [11], the paragraph before Theorem 3.10.16). The product of a countable family of sequentially compact spaces is sequentially compact [11, Theorem 3.10.35]. But already the Cantor cube $D^{\mathfrak{c}}$ is not sequentially compact (see [11], the paragraph after Example 3.10.38). On the other hand, some compact-like spaces are also preserved by products, see [23, §3-4] (especially Theorem 3.3, Proposition 3,4, Example 3.15, Theorem 4.7, and Example 4.15) and §7 for the history, and [22, §5]. Among more recent results we note that Dow et al. in Theorem 4.1 of [10] proved that a product of a family of sequentially feebly compact spaces is again sequentially feebly compact, and in Theorem 4.3 that every product of feebly compact spaces, all but one of which are sequentially feebly compact, is feebly compact.

In the next propositions we show that sequentially pracompact, T_1 totally countably compact, and ω-bounded-pracompact spaces are preserved by products. The proofs are easy and straightforward but we provide them because a theorem should have a proof.

Let X be a product of a family $\{X_\alpha\colon \alpha\in A\}$ of spaces. For each subset B of the set A by π_B we denote the projection from $X=\prod\{X_\alpha\colon \alpha\in A\}$ to $\prod\{X_\alpha\colon \alpha\in B\}$. If $B=\{\alpha\}$ then π_B we shall denote also by π_α. A space $Y\subset X$ is called a Σ-product of the family $\{X_\alpha\}$ provided there exists a point $y\in X$ such that

$$Y=\{x\in X\colon x_\alpha=y_\alpha \text{ for all but countably many } \alpha\in A\}.$$

In this case Y is also called the *Corson Σ-subspace of X based at y.*

Proposition 1. *The* (Σ-) *product of a family of sequentially pracompact spaces is sequentially pracompact.*

Proof. Let X be the non-empty product of a family $\{X_\alpha\colon \alpha\in A\}$ of sequentially pracompact spaces and $Y\subset X$ be the Corson Σ-subspace of X based at a point $y=(y_\alpha)\in X$. For each index $\alpha\in A$ fix a dense subset $D_\alpha\ni y_\alpha$ of the space X_α such that each sequence of points of the set D_α has a convergent subsequence and fix a point $a_\alpha\in D_\alpha$. Put $D=Y\cap\prod_{\alpha\in A}D_\alpha$. Then the set D is a dense subset of the space X. Let $C=\{x_n\colon n\in\mathbb{N}\}$ be a sequence of points of the set D and $B=\{\alpha_m\colon m\in\mathbb{N}\}$ be an enumeration of the countable set $\{\alpha\in A\colon \exists\, x\in C(x_\alpha\neq y_\alpha)\}$. By induction we can build a sequence $\{x_{\alpha_m}\in X_{\alpha_m}\}$ of points and a sequence $\{S_m\}$ of *infinite* subsets of $\mathbb{N}$ such that $S_m\supset S_{m'}$ for each $m\le m'$ and for each neighborhood $U_{\alpha_m}\subset X_{\alpha_m}$ of the point x_{α_m} the set $\{n\in S_m\colon x_{n\alpha_m}\notin U_{\alpha_m}\}$ is finite. We can easily construct an infinite set $S\subset\mathbb{N}$ such that the set $S\setminus S_m$ is finite for each $m\in\mathbb{N}$. Choose a point $x=(x_\alpha)\in Y$ such that x_α is already defined for $\alpha\in B$ and $x_\alpha=y_\alpha$ for $\alpha\in A\setminus B$. Let U be an arbitrary neighborhood of the point x. There exist a finite subset F of the set A and a family

$$\{U_\alpha\colon \alpha\in F, U_\alpha\subset X_\alpha \text{ is an open neighborhood of } x_\alpha\}$$

such that $x\in U'=\pi_F^{-1}\left(\prod\{U_\alpha\colon \alpha\in F\}\right)\subset U$. The inductive construction implies that the set $T_\alpha=\{n\in S\colon x_{n\alpha}\notin U_\alpha\}$ is finite for each $\alpha\in F$. Then $x_n\in U'\subset U$ for each $n\in S\setminus\bigcup\{T_\alpha\colon \alpha\in F\}$. □

Proposition 2. *The* (Σ-) *product of a family of totally countably pracompact T_1 spaces is totally countably pracompact.*

Proof. Let X be the non-empty product of a family $\{X_\alpha\colon \alpha\in A\}$ of totally countably pracompact spaces and $Y\subset X$ be the Corson Σ-subspace of X based at a point $y=(y_\alpha)\in X$. For each index $\alpha\in A$ fix a dense subset $D_\alpha\ni y_\alpha$ of the space X_α such that each sequence of points of the set D_α has a subsequence with compact closure in X_α. Put $D=Y\cap\prod_{\alpha\in A}D_\alpha$. Then the set D is a dense subset of the space X. Let $C=\{x_n\colon n\in\mathbb{N}\}$

be a sequence of points of the set D and $\{\alpha_m : m \in \mathbb{N}\}$ be an enumeration of the countable set $\{\alpha \in A : \exists x \in C(x_\alpha \neq y_\alpha)\}$. By induction we can build a sequence $\{S_m\}$ of *infinite* subsets of $\mathbb{N}$ such that $S_m \supset S_{m'}$ for each $m \le m'$ and the set $\{x_{n\alpha_m} : n \in S_m\}$ has compact closure in X_{α_m}. We can easily construct an infinite set $S \subset \mathbb{N}$ such that the set $S \setminus S_m$ is finite for each $m \in \mathbb{N}$. Then the set $\{x_n : n \in S\}$ has compact closure in X, which is contained in Y. □

Remark 1. The referee remarked that in the case of Cartesian product in Proposition 2 T_1 condition can be weakened to that for each $\alpha \in A$ a set $\{y_\alpha\}$ has compact closure in X_α. The proof remains almost the same, only the final words "which is contained in Y" should be dropped.

It motivates to define a class of spaces in which every singleton (that is, one-point set) has compact closure. The referee suggested to investigate which classes of compact-like spaces belong to the class. By definition, each T_1 space belong to the class. Each totally countably compact space X also belongs to the class because for any point $x \in X$ the set $\overline{\{x\}}$ is the closure of any subsequence of the constant sequence $\{x_n\}$, where $x_n = x$ for each n.

On the other hand, the referee proposed to endow ω with the topology of left intervals, whose open sets are the intervals $[0, n)$, plus the whole of ω. Here the closure of 0 is the noncompact space ω. We extend this construction as follows. Let $X = \omega_1 + \omega$ endowed with a topology with a subbase consisting of halfintervals $[0, \alpha)$, where $\alpha < \omega_1 + \omega$ and $(\alpha, \omega_1 + \omega)$, where $\alpha < \omega_1$. Then the closure of ω_1 is a noncompact set $[\omega_1, \omega_1 + \omega)$. Now put $D = \omega_1$. Then D is dense in X and each countable subset C of D is contained in a closed compact set $[0, \sup C]$ of D. Thus X is both sequentially and ω-bounded-pracompact.

A sequentially compact example of a space not belonging to the class is more complicated, but, luckily, already known. Namely, in [20, Example 5] the second author constructed a group $G = \bigoplus_{\alpha \in \omega_1} \mathbb{Z}$, which is the direct sum of the groups $\mathbb{Z}$ and its subgroup

$$S = \{0\} \cup \{(x_\alpha) \in G : (\exists \beta \in \omega_1)((\forall \alpha > \beta)(x_\alpha = 0) \& (x_\beta > 0))\}.$$

Let G_S be the group G endowed with a topology with a base $\{x + S : x \in G\}$. Then G_S is a paratopological group, that is the group operation $+ : G \times G \to G$ is continuous. In [20, Example 5] it is shown that the group G_S is sequentially compact. On the other hand, by [20, Lemma 17] the set $S \subset G_S$ is compact. Since $\overline{\{0\}} = \{x \in G : x + S \ni 0\} = -S$, if the set $-S$ is compact then $G = S \cup (-S)$ is compact too, which contradicts [20, Proposition 12].

Proposition 3. *The product of a family of ω-bounded-pracompact spaces is ω-bounded-pracompact. Moreover, if all spaces of the family are T_1 then a Σ-product of the family is ω-bounded-pracompact too.*

Proof. Let X be the non-empty product of a family $\{X_\alpha : \alpha \in A\}$ of ω-bounded-pracompact spaces and $Y \subset X$ be the Corson Σ-subspace of X based at a point $y = (y_\alpha) \in X$. For each index $\alpha \in A$ fix a dense subset $D_\alpha \ni y_\alpha$ of the space X_α such that each countable subset of the set D_α has compact closure in X_α. Put $D = Y \cap \prod_{\alpha \in A} D_\alpha$. Then the set D is a dense subset of the space X. Let C be a countable subset of the set

D. Then C is a subset of a closed compact subset $C' = \prod_{\alpha\in A} \overline{\pi_\alpha(C)}$ of the space X. Now assume that all spaces X_α are T_1. Put $B = \{\alpha \in A : \exists x \in C(x_\alpha \neq y_\alpha)\}$. The set B is countable and so $C' = \prod_{\alpha\in B} \overline{\pi_\alpha(C)} \times \prod_{\alpha\in A\setminus B}\{y_\alpha\} \subset Y$. □

Example 1. This example shows that T_1 condition is essential in the Σ-product case of Propositions 2 and 3. Let X' be a space consisting of two distinct points a and b endowed with the topology $\{\varnothing, \{a\}, X'\}$. Let A be an uncountable subset, X be the product of a family $\{X_\alpha : \alpha \in A\}$, $Y \subset X$ be the Corson Σ-subspace of X based at a point $y = (a_\alpha) \in X$, where $X_\alpha = X'$ and $a_\alpha = a$ for each $\alpha \in A$. Since the space X' is compact, it is easy to check that the space Y is countably compact. On the other hand, the space Y is not totally countably pracompact. For this purpose it suffices to show that for any point $x = (x_\alpha) \in Y$ a set $\overline{\{x\}}$ (everywhere in this example we by $\overline{S}$ we mean the closure in Y of its subset S) is not compact, because $\overline{\{x\}}$ is the closure (in Y) of any subsequence of a constant sequence $\{x_n\}$, where $x_n = x$ for each n. By [11, Proposition 2.3.3], $\overline{\{x\}} = \overline{\{(x_\alpha)\}} = Y \cap \prod_{\alpha\in A} \overline{\{x_\alpha\}}$. Remark that $b \in \overline{\{x_\alpha\}}$ for each $\alpha \in A$. Now for each $\alpha \in A$ put $Y_\alpha = \{y = (y_\beta) \in Y : y_\alpha = a\}$. Since for each point $z = (z_\alpha) \in Y$, there exists an index α such that $z_\alpha = a$, the family $\{Y_\alpha : \alpha \in A\}$ is an open cover of the set Y, and hence of $\overline{\{x\}}$. Let C be any finite subset of A. Let $t = (t_\alpha) \in Y$ be such that $t_\alpha = b$ if $x_\alpha = b$ or $\alpha \in C$ and $t_\alpha = a$, otherwise. Then $t \in \overline{\{x\}} \setminus \bigcup\{Y_\alpha : \alpha \in C\}$. Thus the set $\overline{\{x\}}$ is not compact.

Since the sequential feebly compactness is preserved by extensions, the following proposition strengthens Theorem 4.1 of [10] a bit.

Proposition 4. *The Σ-product of a family of sequentially feebly compact spaces is sequentially feebly compact.*

Proof. Let X be a non-empty product of a family $\{X_\alpha : \alpha \in A\}$ of sequentially feebly compact spaces, $Y \subset X$ be the Corson Σ-subspace of X based at a point $y = (y_\alpha) \in X$, and $\{V_n : n \in \mathbb{N}\}$ be a sequence of non-empty open subsets of the space Y. For each index n choose a finite subset B_n of the set A and a family

$$\{U_{n\alpha} : \alpha \in B_n, U_{n\alpha} \text{ is a non-empty open subset of } X_\alpha\}$$

such that $U_n \cap Y \subset V_n$, where $U_n = \pi_{B_n}^{-1}\left(\prod\{U_{n\alpha} : \alpha \in B_n\}\right)$. Put $B = \bigcup B_n$. By Theorem 4.1 of [10], the space $X' = \{X_\alpha : \alpha \in B\}$ is sequentially feebly compact. Since $\{\pi_B(U_n)\}$ is a sequence of its non-empty open subsets, there exist a point $x' \in X'$ and an infinite set $I \subset \mathbb{N}$ such that for each neighborhood U' of the point $x' = (x'_\alpha)_{\alpha\in B}$ the set $\{n \in I : \pi_B(U_n) \cap U' = \varnothing\}$ is finite. Define a point $x = (x_\alpha)_{\alpha\in A} \in Y$ by putting $x_\alpha = x'_\alpha$ for each $\alpha \in B$ and $x_\alpha = y_\alpha$ for each $\alpha \in A \setminus B$. Let V be an arbitrary neighborhood of the point x in the space Y. Pick a canonical neighborhood U of the point x in the space X such that $U \cap Y \subset V$. Then there exists a subset I' of the set I such that a set $I \setminus I'$ is finite and $\pi_B(U_n) \cap \pi_B(U) \neq \varnothing$ for each $n \in I'$. Fix any such n and pick a point $z' = (z'_\alpha)_{\alpha\in B} \in \pi_B(U_n) \cap \pi_B(U)$. Define a point $z = (z_\alpha)_{\alpha\in A} \in Y$ by putting $z_\alpha = z'_\alpha$ for each $\alpha \in B$ and $z_\alpha = y_\alpha$ for each $\alpha \in A \setminus B$. It is easy to check that $z \in U_n \cap U \cap Y \subset V_n \cap V$. □

3. Backward implications

In [6], Banakh and Zdomskyy defined a topological space X to be an α_7*-space* if for any family $\{S_n : n \in \mathbb{N}\}$ of countable infinite subsets of the space X such that a set $S_n \setminus U$ is finite for any n and any neighborhood U of x there exist a countable infinite subset S of the space X and a point $y \in X$ such that a set $S \setminus V$ is finite for any neighborhood V of y and $S_n \cap S \neq \varnothing$ for infinitely many n.

Proposition 5. *Let X be a Fréchet-Urysohn feebly compact space. Then X is sequentially feebly compact. Moreover, if X is either quasiregular or α_7 then X is selectively sequentially feebly compact.*

Proof. Let X be a Fréchet-Urysohn feebly compact space and $\{V_n : n \in \mathbb{N}\}$ be a sequence of non-empty open subsets of the space X. For each n choose a non-empty open set $U_n \subset V_n$ such that $\overline{U}_n \subset V_n$ provided the space X is quasiregular. Since the space X is feebly compact, there exists a point $x \in X$ such that each neighborhood of the point x intersects infinitely many sets of the sequence $\{U_n\}$. Put $I_0 = \{n \in \mathbb{N} : x \in \overline{U}_n\}$.

Suppose that the set I_0 is infinite. Then $U \cap U_n \neq \varnothing$ for each $n \in I_0$ and each neighborhood U of the point x. If the space X is quasiregular then $x \in V_n$ for each $n \in I_0$, thus the constant sequence $\{x_n = x : n \in I_0\}$ converges to x. Assume that X is an α_7-space. Since the space X is Fréchet-Urysohn, for each $n \in I_0$ there exists a sequence $S'_n = \{x^n_k : k \in \mathbb{N}\}$ of points of U_n convergent to a point x. Considering its subsequence, if necessarily, we can assume that the sequence S'_n either consists of distinct points or it is constant. In the latter case we have $x^n_k = x^n \in U_n$ for each k for some point $x^n \in U_n$ such that $x \in \overline{\{x^n\}}$. Put $I'_0 = \{n \in I_0 : S'_n \text{ is constant}\}$. If the set I'_0 is infinite then a sequence $\{x^n : n \in I'_0\}$ converges to the point x. So we suppose that the set I'_0 is finite. Since X is an α_7-space, there exist a countable infinite subset S of the space X and a point $y \in X$ such that a set $S \setminus V$ is finite for any neighborhood V of y and a set

$$I''_0 = \left\{n \in I_0 \setminus I'_0 : \text{ there exists a natural } k(n) \text{ such that } x^n_{k(n)} \in S\right\}$$

is infinite. For each $n \in I''_0$ put $x_n = x^n_{k(n)} \in U_n$. If there exists a point $z \in X$ such that the set $I_1 = \{n \in I''_0 : x_n = z\}$ is infinite then the sequence $\{x_n : n \in I_1\}$ converges to the point z. Otherwise the sequence $\{x_n : n \in I''_0\}$ converges to the point y. Indeed, let V be an arbitrary neighborhood of the point y. Then the set $S \setminus V$ is finite and $x_n \in V$ for each $n \in I''_0 \setminus \{n : x_n \in S \setminus V\}$.

Suppose that the set I_0 is finite. Since $x \in \overline{\bigcup\{U_n : n \in \mathbb{N} \setminus I_0\}}$ and X is a Fréchet-Urysohn space, there exists a sequence $\{x'_m : m \in \mathbb{N}\}$ of points of the set $\bigcup\{U_n : n \in \mathbb{N} \setminus I_0\}$ converging to the point x. For each index $m \in \mathbb{N}$ choose an index $n(m) \in \mathbb{N} \setminus I_0$ such that $x'_m \in U_{n(m)}$. Put $I_1 = \{n(m) : m \in \mathbb{N}\}$. Since $x \notin \overline{U_n}$ for each $n \in \mathbb{N} \setminus I_0$, the set I_1 is infinite. For each $r \in I_1$ pick a point $x_r = x'_{m(r)}$, where $n(m(r)) = r$. Then $x_r \in U_r$ and the sequence $\{x_r : r \in I_1\}$ converges to the point x. Indeed, let U be an arbitrary neighbourhood of the point x. Since the sequence $\{x'_m\}$ converges to the point x, there exists $N \in \mathbb{N}$ such that $x'_m \in U$ for each $m > N$. Then $x_r \in U$ for each $r \in I_1 \setminus \{n(m) : 0 \le m \le N\}$. □

Proposition 6. *Each sequential countably pracompact space is sequentially pracompact.*

Proof. Let X be a sequential countably pracompact space. There exists a dense subset D of the space X such that each infinite subset of the set D has an accumulation point in X. Let $\{x_n : n \in \mathbb{N}\}$ be a sequence of points of the set D. If there exists a point $x \in X$ such that $x \in \overline{\{x_n\}}$ for infinitely many indices $n \in \mathbb{N}$ then the $\{x_n : x_n = x\}$ is a convergent subsequence of the sequence $\{x_n : n \in \mathbb{N}\}$. So we suppose that there is no such point x. Then the set $B = \{x_n : n \in \mathbb{N}\}$ is infinite. The set B has an accumulation point y in X. Then $y \in \overline{B \setminus \{y\}}$. Therefore the set $B \setminus \{y\}$ is not sequentially closed and there exists a sequence $\{z_m : m \in \mathbb{N}\}$ of points of the set $B \setminus \{y\}$ converging to a point $z \notin B \setminus \{y\}$. Then the sequence $\{z_m : m \in \mathbb{N}\}$ contains infinitely many distinct points of the set $B \setminus \{y\}$. $\square$

Proposition 7. *Each countably pracompact k-space X is totally countably pracompact.*

Proof. There exists a dense subset D of the space X such that each infinite subset of the set D has an accumulation point in X. Let $\{x_n : n \in \mathbb{N}\}$ be a sequence of points of the set D. Put $B = \{x_n : n \in \mathbb{N}\}$. If the set B is finite then there exists a point $x \in X$ such that $x_n = x$ for infinitely many indices $n \in \mathbb{N}$. Then a subsequence $\{x_n : x_n = x\}$ of the sequence $\{x_n : n \in \mathbb{N}\}$ has compact closure $\{x\}$ in X. Thus we suppose that the set B is infinite. The set B has an accumulation point y in X. Then $y \in \overline{B \setminus \{y\}}$. Therefore the set $B \setminus \{y\}$ is not closed and there exists a compact subset K of the space X such that a set $B \cap K$ is not closed in K. Then the set $B \cap K$ is infinite, the sequence $\{x_n : x_n \in B \cap K\}$ is infinite too and $\overline{\{x_n : x_n \in B \cap K\}} \subset K$. $\square$

Proposition 8. *Each sequentially feebly compact space containing a dense set D of isolated points is sequentially pracompact.*

Proof. It is easy to check that each sequence of points of the set D has a convergent subsequence. $\square$

4. Examples

Example 2. Let X_0 be a non-empty T_1 space. Determine a topology on the set $X = (X_0 \times \omega) \cup \{y_0\}$, where $y_0 \notin X_0 \times \omega$ by the following base

$$\begin{aligned}\mathscr{B} = & \{U \times \{n\} : U \text{ is an open subset of the space } X_0, n \in \omega\} \cup \\ & \cup \bigcup \Big\{ \{y_0\} \cup \bigcup_{m \geq n} X_0 \times \{m\} \setminus F_m : n \in \omega, F_m \text{ is a finite subset of } X_0 \\ & \qquad \text{for each } m \in \omega \text{ such that } m \geq n \Big\}.\end{aligned}$$

It is easy to check the following:

- the space X is Hausdorff provided the space X_0 is Hausdorff;
- the space X is feebly compact provided the space X_0 is a feebly compact space without isolated points;
- the space X is sequentially feebly compact provided the space X_0 is a sequentially feebly compact space without isolated points.

Now we take the standard unit segment $[0, 1]$ as X_0. Then X is a sequentially feebly compact space containing a closed discrete infinite subspace $\{1\} \times \omega$. Now for each $n \in \omega$ put $U_n = X_0 \times \{n\}$. Let $\{x_n\}$ be a sequence of points of the space X such that $x_n \in U_n$.

Then the set $\{x_n\}$ has no accumulation points, thus the space X is not selectively feebly compact.

We recall that the Stone-Čech compactification of a Tychonoff space X is a compact Hausdorff space βX containing X as a dense subspace so that each continuous map $f\colon X \to Y$ to a compact Hausdorff space Y extends to a continuous map $\overline{f}\colon \beta X \to Y$ (see [11]).

Example 3 ([11, Exer. 3.6.I], [8, Ex. 2.6])**.** Let $\{N_\alpha\}_{\alpha\in A}$, where $A\cap\mathbb{N}=\varnothing$, be an infinite family of infinite subsets of $\mathbb{N}$ such that the intersection $N_\alpha\cap N_\beta$ is finite for every pair α,β of distinct elements of A and that $\{N_\alpha\}_{\alpha\in A}$ is maximal with respect to the last property. Generate a topology on the set $X=\mathbb{N}\cup S$ by the neighborhood system $\{\mathcal{B}(x)\}_{x\in X}$, where $\mathcal{B}(x)=\{\{n\}\}$, if $x=n\in\mathbb{N}$ and $\mathcal{B}(x)=\{\{\alpha\}\cup(N_\alpha\setminus\{1,2,\ldots,n\})\}_{n=1}^{\infty}$ if $x=\alpha\in A$.

Since A is a closed discrete infinite subset of X, X is not countably compact. On the other hand, the set $D=\mathbb{N}$ is dense in X. Let $\{x_n\colon n\in\mathbb{N}\}$ be an arbitrary sequence of points of the set D. If the *set* $S=\{x_n\colon n\in\mathbb{N}\}$ is finite then the sequence $\{x_n\colon n\in\mathbb{N}\}$ has a constant subsequence. If the set S is infinite then by maximality of A there exists $\alpha\in A$ such that $N_\alpha\cap S$ is infinite. Note that the enumeration $\{x_{n_k}\colon k\in\mathbb{N}\}$ of $N_\alpha\cap S$ in the increasing order is a subsequence of the sequence $\{x_n\colon n\in\mathbb{N}\}$ converging to the point α. Thus the space X is sequentially pracompact.

Example 4. Endow the set $\mathbb{N}$ with the discrete topology. Let $\mathscr{A}(\mathbb{N})=\mathbb{N}\cup\{\infty\}$ be a one-point Alexandroff compactification of $\mathbb{N}$ with the remainder ∞. We define on $\mathscr{A}(\mathbb{N})\times\mathbb{N}$ the product topology τ_p and extend the topology τ_p onto $X=\mathscr{A}(\mathbb{N})\times\mathbb{N}\cup\{a\}$, where $a\notin\mathscr{A}(\mathbb{N})\times\mathbb{N}$, to a topology τ^* in the following way: bases of the topologies τ_p and τ^* coincide at x for any $x\in\mathscr{A}(\mathbb{N})\times\mathbb{N}$ and the family

$$\mathscr{B}^*(a)=\{U_a(i_1,\ldots,i_n)\colon i_1,\ldots,i_n\in\mathbb{N}\},$$

where

$$U_a(i_1,\ldots,i_n)=X\setminus((\{\infty\}\times\mathbb{N})\cup(\mathscr{A}(\mathbb{N})\times\{i_1,\ldots,i_n\})),$$

determines a set of neighbourhood systems for τ^* at the point a.

The definition of the topology τ^* on X implies that $\mathbb{N}\times\mathbb{N}$ is the maximum discrete subspace of (X,τ^*) and $\mathbb{N}\times\mathbb{N}$ is dense in (X,τ^*). Hence every dense subset D of (X,τ^*) contains $\mathbb{N}\times\mathbb{N}$. However, $\overline{\mathbb{N}\times\mathbb{N}}=X$ is not compact, and hence (X,τ^*) is not an ω-bounded-pracompact space.

Now we shall show that (X,τ^*) is totally countably pracompact. Especially we shall prove that $\mathbb{N}\times\mathbb{N}$ is the requested dense subset of the space (X,τ^*). Fix an arbitrary sequence $\{x_n\}_{n\in\mathbb{N}}\subset\mathbb{N}\times\mathbb{N}$. If there exists a positive integer i such that the set $\{x_n\}_{n\in\mathbb{N}}\cap(\mathscr{A}(\mathbb{N})\times\{i\})$ is infinite then the subsequence $\left\{x_{i_j}\right\}_{j\in\mathbb{N}}=\{x_n\}_{n\in\mathbb{N}}\cap(\mathscr{A}(\mathbb{N})\times\{i\})$ with the corresponding renumbering has compact closure in (X,τ^*). In the other case the set $\{x_n\}_{n\in\mathbb{N}}\cap(\mathscr{A}(\mathbb{N})\times\{i\})$ is finite for any positive integer i. Then the definition of (X,τ^*) implies that $\overline{\{x_n\}_{n\in\mathbb{N}}}=\{a\}\cup\{x_n\}_{n\in\mathbb{N}}$ is a compact subset of (X,τ^*).

We observe that by Proposition 19 of [14], (X,τ^*) is Hausdorff non-semiregular countably pracompact non-countably compact space, and hence (X,τ^*) is not totally countably compact.

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ПРО СТАРІ ТА НОВІ КЛАСИ СЛАБКО КОМПАКТНИХ ПРОСТОРІВ

Олег ГУТІК[1], Олександр РАВСЬКИЙ[2]

[1]*Львівський національний університет імені Івана Франка, вул. Університетська 1, 79000, Львів*
[2]*Інститут прикладних проблем механіки і математики ім. Я. С. Підстригача, вул. Наукова, 3б, 79060, Львів*
e-mail: oleg.gutik@lnu.edu.ua, ovgutik@yahoo.com, alexander.ravsky@uni-wuerzburg.de

Введено три нових класи зліченно пракомпактних просторів, вивчаються їхні загальні властивості та відношення з іншими компактно-близькими просторами.

Ключові слова: компактний, слабко компактний, секвенціально компактний, ω-обмежений, цілком зліченно компактний, зліченно компактний, зліченно пракомпактний, псевдокомпактний, секвенціально псевдокомпактний, секвенціально пракомпактний, цілком зліченно пракомпактний, ω-обмежений пракомпактний.