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FUNCTORS AND MANIFOLDS MODELED ON SOME k_ω-SPACE

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There is a universal space K^∞ for the class of compact metric spaces of finite f.-d. derivative. We consider the question of preservation of K^∞-manifolds by some functorial constructions. We also consider a universal map $\varphi_K\colon \mathbb{R}^\infty \to K^\infty$ and discuss some of its properties, in particular, its preservation by some functors.

Key words: universal space, infinite-dimensional manifold, functor.

Introduction

Recall that the k_ω-spaces are the countable direct limits of compact Hausdorff spaces. By $\mathbb{R}^\infty$ we denote the direct limit of the sequence

$$\mathbb{R} \to \mathbb{R}^2 \to \mathbb{R}^3 \to \dots$$

and by Q^∞ the direct limit of the sequence

$$Q \to Q^2 \to Q^3 \to \dots,$$

where $Q = [0,1]^\omega$ is the Hilbert cube.

Let A be a topological space. The f.d.-derivative of A is the set

$$A^{(1)} = A \setminus \{x \in A \mid \text{ there is a neighborhood } U \text{ of } x \text{ such that } \dim U < \infty\}.$$

Clearly, $A^{(1)}$ is a closed subset in A. By induction, we define $A^{(n)} = (A^{(n-1)})^{(1)}$, for $n > 1$.

Let $\mathcal{K}$ denote the class of compact metrizable spaces A such that $A^{(n)} = \emptyset$, for some natural n. Let $\mathcal{M}(\omega)$ denote the class of finite-dimensional compact metrizable spaces.

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T. Banakh [1] considered the following construction. Given $n \in \mathbb{N}$, let

$$K_n = \left[-\frac{1}{n}, \frac{1}{n}\right]^n \times \{(0, 0, \dots)\} \subset \ell^2.$$

Let $K = \bigcup_{i\in\mathbb{N}} K_n \subset \ell^2$. Clearly, K is a compact metric space and $K^{(1)} = * = (0, 0, \dots)$.

Let $\mathcal{C}$ be a class of topological spaces. A topological space X is said to be *strongly $\mathcal{C}$-universal* if, for every pair (A, B) of topological spaces, where $A \in \mathcal{C}$ and B is a closed subset of A, every embedding $f\colon B \to X$, there is an embedding $\bar{f}\colon A \to X$ that extends f.

It is proved in [1] that the space $K^\infty = \varinjlim K^n$ is strongly $\mathcal{K}$-universal.

Recall that a space X is called a *K^∞-manifold* if X is locally homeomorphic to K^∞. We assume that all K^∞-manifolds under consideration are k_ω-spaces.

The theory of K^∞-manifolds is developed by T. Banakh [1]. It turned out that this theory is parallel to the theories of $\mathbb{R}^\infty$- and Q^∞-manifolds.

We will consider the question of preservation of K^∞-manifolds by some functorial constructions. Similar questions for another classes of infinite-dimensional manifolds were considered in publications of different authors (see, e.g., [10] and the bibliography therein).

The functors are assumed to be close to being normal. The notion of normal functor acting in the category of compact Hausdorff spaces is introduced by Shchepin [9]. In the sequel, we will use the terminology of [9]. In particular, the notions of monomorphic functor and functor that preserves intersections as well as of degree and support can be found in [9] (see, e.g., [10]).

By $F_k(X)$ we denote the set of points of degree $\leq k$ in $F(X)$. Note that F_k is a subfunctor of F.

If F is a normal functor, X is a k_ω-space, $X = \varinjlim X_i$, then we define $F(X) = \varinjlim F(X_i)$.

1. Results

Lemma 1. *Let F be a monomorphic functor that preserves intersections and the degree of F equals $n \in \mathbb{N}$. If $F(n)$ is a finite-dimensional space, then $F(X) \in \mathcal{K}$, for every $X \in \mathcal{K}$.*

Proof. We will prove by induction on the degree of the functor. If $n = 1$, then $F(X) = X$ and there is nothing to prove. Consider $a \in F(X)$ with $\deg(a) = n$. Then there is a neighborhood U of a in $F(X)$ homeomorphic to $V_1 \times \cdots \times V_n \times W$, where V_i, $i = 1, \dots, n$, are open subsets of X and W is an open subset of $F(n)$ (see [5]). Therefore, there exists $m \in \mathbb{N}$ such that $F(X)^{(m)} \subset F_{n-1}(X)$. Then we can apply induction. □

The following lemma is proved in [1].

Lemma 2. *Let X be a K^∞-manifold. If $A \subset X$ be a compact subset, then there is an embedding $i\colon A \times K \to X$ such that $i(a, *) = a$ for every $a \in A$.*

We will need the following its corollary.

Corollary 1. *Let X be a K^∞-manifold. If $A \subset X$ be a compact subset, then for every $n \in \mathbb{N}$ there is an embedding $i\colon A \times K^n \to X$ such that $i(a, *, \dots, *) = a$ for every $a \in A$.*

Theorem 1. *Let X be a K^∞-manifold and let F be a functor of finite degree that preserves ANR-spaces and finite-dimensional spaces. Then $F(X)$ is a K^∞-manifold.*

Proof. Represent X as $\varinjlim X_i$, where X_i are compact ANR-spaces. Clearly, $X_i \in \mathcal{K}$. Then $F(X) = \varinjlim F(X_i)$. By Lemma 1, $F(X_i) \in \mathcal{K}$.

We are going to demonstrate the strong local $\mathcal{K}$-universality of $F(X)$. Let (A, B) be a pair of compact metrizable spaces with $A \in \mathcal{K}$ and let $\beta\colon B \to F(X)$ be an embedding. Then there exists $n \in \mathbb{N}$ such that $\beta(B) \subset F(X_n)$. Since $F(X_n)$ is an ANR-space, there is an extension $\alpha'\colon U \to F(X_n)$ of β onto a closed neighborhood U of B in A.

Let U/B denote the quotient space of U obtained when we identify all the points of U to a single equivalence class leaving all the other points of B equivalent only to themselves. Note that $U/B \in \mathcal{K}$. Let $q\colon U \to U/B$ denote the quotient map. There exists an embedding $\gamma\colon U/B \to K^m$, for some $m \in \mathbb{N}$, such that $\gamma(A) = (*, \dots, *)$ (see [1]). Let $q\colon U \to U/B$ denote the quotient map.

Using Corollary 1, one may assume that $X_n \times K^m \subset X$ and $x = (x, *, \cdots *) \in X_n \times K^m$ for every $x \in X_n$. For any $y \in K^m$, denote by $i_y\colon X_n \to X_n \times K^m$ the map defined by the formula $i_y(x) = (x, y)$, $x \in X_n$.

Given $x \in U$, define $\alpha(x) = F(i_{\gamma(q(x))})(\alpha'(x))$.

Let us verify that α is an embedding that extends β. Indeed, the continuity of α is a consequence of Propositions 2.2 and 2.4 from [7]. Next, if $x \in B$, $y \in U \setminus B$, then $\mathrm{supp}(\alpha(x)) \subset X_n$ and

$$\mathrm{supp}(\alpha(y)) \subset X_n \times (K^m \setminus \{(*, \dots, *)\})$$

and we conclude that $\alpha(x) \neq \alpha(y)$. If $x, y \in U \setminus B$, $x \neq y$, then

$$\mathrm{supp}(\alpha(x)) \subset X_n \times \{\gamma q(x)\}, \qquad \mathrm{supp}(\alpha(y)) \subset X_n \times \{\gamma q(y)\},$$

and therefore $\alpha(x) \neq \alpha(y)$. □

Recall that a *free topological group* of a Tychonov space X is a topological group $F(X)$ satisfying the following conditions:

(1) X is a subspace of $F(X)$;
(2) for any continuous map $f\colon X \to G$, where G is a topological group, there exists a unique continuous homomorphism $\bar{f}\colon F(X) \to G$ that extends f.

It is well-known (see, e.g., [6]) that a topological group exists and is unique up to isomorphism.

One can similarly prove the following theorems. Note that its proof relies on the fact that, for any compact Hausdorff space X, the free topological group $F(X)$ is homeomorphic to the countable direct limit $\varinjlim F_n(X)$, where $F_n(X)$ stands for the words of length $\leq n$ in $F(X)$. Actually, F_n is a functor for which the assumptions of Lemma 1 are satisfied (see [11]).

Theorem 2. *Let $X \in \mathcal{K}$ be a compact metric space such that X contains a topological copy of K. The free topological group of the space X is a K^∞-manifold.*

Theorem 3. *The free topological group of the space K is a K^∞-manifold.*

The previous results are counterparts of the results of the second number author on free topological groups of ANR-spaces (see [11]).

By $(\mathcal{M}(\omega), \mathcal{K})$ we denote the class of continuous maps $f\colon X \to Y$, where $X \in \mathcal{M}(\omega)$, $Y \in \mathcal{K}$.

Given two maps, $f\colon X' \to X''$ and $g\colon Y' \to Y''$, we say that a pair $i = (i', i'')$, where $i'\colon X' \to Y'$, $i''\colon X'' \to Y''$ are maps, is a morphism in the category of maps if the diagram

$$\begin{array}{ccc} X' & \xrightarrow{i'} & Y' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X'' & \xrightarrow[i'']{} & Y'' \end{array}$$

is commutative. In this case we say that i is an embedding if both i', i'' are embeddings. Also, if $X'' \subset X'$, $Y'' \subset Y'$, $g = f|X''\colon X'' \to Y''$, then we say that a pair of maps (f, g) is given.

A map $f\colon \mathbb{R}^\infty \to K^\infty$ is said to be $(\mathcal{M}(\omega), \mathcal{K})$-*universal* if, for every pair (g, h) of maps with $g, h \in (\mathcal{M}(\omega), \mathcal{K})$ and every embedding $i\colon h \to f$, there exists an embedding $j\colon g \to f$ that extends i.

Proposition 1. *There is a $(\mathcal{M}(\omega), \mathcal{K})$-universal map $\mathbb{R}^\infty \to K^\infty$.*

Proof. Let $\varphi\colon \mathbb{R}^\infty \to Q^\infty$ be a universal map described in [12]. We suppose that $K^\infty \subset Q^\infty$ and let

$$\psi = \varphi|\varphi^{-1}(K^\infty)\colon \varphi^{-1}(K^\infty) \to K^\infty.$$

Consider the composition

$$\mathbb{R}^\infty \xrightarrow{\quad\simeq\quad} \varphi^{-1}(K^\infty) \to K^\infty$$

(that $\varphi^{-1}(K^\infty)$ is homeomorphic to $\mathbb{R}^\infty$ easily follows from the fact that K^∞ is a k_ω-space which is an absolute retract as well as from the universality of the map φ.)

Let us denote the obtained map by φ_K. We are going to prove that φ_K is $(\mathcal{M}(\omega), \mathcal{K})$-universal. Suppose that we have a pair (g, h) of maps $g\colon X' \to X''$, $h\colon Y' \to Y''$ with $g, h \in (\mathcal{M}(\omega), \mathcal{K})$ and an embedding $i = (i', i'')\colon h \to \varphi_K$.

By the characterization theorem for K^∞ (see [1]), there exists an embedding $j''\colon X'' \to K^\infty \subset Q^\infty$ which is an extension of the embedding $i''\colon Y'' \to K^\infty \subset Q^\infty$. By the universality property of φ, there exists an embedding $i'\colon X' \to \mathbb{R}^\infty$ such that $j'|Y' = j''$ and $\varphi j' = j''$. Then $i'(X') \subset \varphi^{-1}(K^\infty)$ and this finishes the proof. □

The following is a characterization theorem for the map φ_K.

Theorem 4. *Let $f\colon X \to Y$ be a map of k_ω-spaces. Then the following are equivalent:*

(1) *X is a countable union of finite-dimensional compact metrizable spaces, Y is a countable union of spaces from the class $\mathcal{K}$, and f is $(\mathcal{M}(\omega), \mathcal{K})$-universal.*
(2) *f is homeomorphic to φ_K.*

Proof. Follows the proof of the main result in [12] concerning characterization of the map $\varphi\colon \mathbb{R}^\infty \to Q^\infty$ by its universality condition. □

Let P_n denote the functor of probability measures supported on the sets of cardinality $\leq n$. This functor acts in the category of compact Hausdorff spaces. Any $\mu \in P_n(X)$

admits a representation of the form $\sum_{i=1}^{n}\alpha_i\delta_{x_i}$, where $\alpha_i \in [0,1]$, $\sum_{i=1}^{n}\alpha_i = 1$, and δ_{x_i} is the Dirac measure concentrated at $x_i \in X$, $i = 1,\dots,n$.

By P_∞ we denote the functor of probability measures of finite support. Recall that for any k_ω-space $X = \varinjlim X_i$, we put $P_\infty(X) = \varinjlim P_i(X_i)$.

Proposition 2. *The map $P_\infty(\varphi_K)$ is homeomorphic to φ_K.*

Proof. We modify the proof of Theorem 2 from [13], namely, we replace every copy Q_n of the Hilbert cube Q by K^n, $n \in \mathbb{N}$, and thus Q^∞ by K^∞. The rest of the proof remains unchanged. □

One can also prove a similar result for the functor of idempotent measures of finite support (see [14]) for the definition of idempotent measures).

2. Remarks and open questions

Is there a map from K^∞ into Q^∞ which is a counterpart of the universal map $\mathbb{R}^\infty \to Q^\infty$?

T. Banakh and O. Hryniv [2] characterized the spaces whose free topological semigroups in some classes are $\mathbb{R}^\infty$-manifolds. Whether a counterpart of their result is valid for the K^∞-manifolds remains an open question. See also [3] for some related results concerning free topological universal algebras.

The notion of f.d.-derivative can be extended over transfinite numbers. This allows us to consider the theories of infinite-dimensional manifolds modeled over countable direct limits of universal spaces for compacta whose f.d.-derivative of order $< \alpha$ is empty, for given countable ordinal α. The case of finite α corresponds to the theory of manifolds modeled on the countable direct limits of $(n-1)$-dimensional Menger compacta (injectively-Menger manifolds; see [8]).

The following questions arise in connection with the results of [4].

Question 1. Is there a linear realization of the map φ_K, i.e., a linear map of linear topological spaces which is homeomorphic to φ_K?

Question 2. Is the map φ_K locally self-similar? Recall that a map $f\colon X \to Y$ is called *locally self-similar* if, for any $x \in X$ and any neighborhoods U, V of x and $f(x)$ respectively, there exists a neighborhood W of x such that $W \subset U$, $f(V) \subset U$, and the restriction $f|W\colon W \to f(W)$ is homeomorphic to the map f.

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ФУНКТОРИ ТА МНОГОВИДИ, МОДЕЛЬОВАНІ НА ДЕЯКИХ k_ω-ПРОСТОРАХ

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Існує універсальний простір K^∞ для класу компактних метричних просторів скінченної скінченновимірної (f.-d.) похідної. Ми розглядаємо питання збереження K^∞-многовидів деякими функторіальними конструкціями. Ми розглядаємо також універсальне відображення $\varphi_K : \mathbb{R}^\infty \to K^\infty$ і обговорюємо деякі його властивості, зокрема, його збереження деякими функторами.

Ключові слова: універсальний простір, нескінченновимірний многовид, функтор.