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ON AUTOMORPHISMS OF THE SEMIGROUP OF ENDOMORPHISMS OF A FREE ABELIAN DIMONOID

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We describe all isomorphisms between the endomorphism semigroups of free abelian dimonoids and prove that that all automorphisms of the endomorphism semigroup of a free abelian dimonoid are inner.

Key words: dimonoid, free abelian dimonoid, endomorphism semigroup, automorphism.

1. **Introduction**

The notion of a dimonoid was introduced by J.-L. Loday in [1]. Recall that a nonempty set D with two binary associative operations $\dashv$ and $\vdash$ is called a *dimonoid* if for all $x, y, z \in D$ the following conditions hold:

$$(D_1) \qquad (x \dashv y) \dashv z = x \dashv (y \vdash z),$$
$$(D_2) \qquad (x \vdash y) \dashv z = x \vdash (y \dashv z),$$
$$(D_3) \qquad (x \dashv y) \vdash z = x \vdash (y \vdash z).$$

It is not hard to see that a dimonoid becomes a semigroup if the operations of a dimonoid coincide. Dimonoids play a prominent role in the theory of Leibniz algebras, these structures and related systems have been studied by many authors (see, e.g., [2]–[5]).

The problem of studying automorphisms of the endomorphism semigroup for free algebras in a certain variety was raised by B. I. Plotkin in his papers on universal algebraic geometry (see, e.g., [6], [7]). Now there are quite a lot papers devoted to studying automorphisms of endomorphism semigroups of free finitely generated algebras of different varieties (see, e.g., [8]–[13]). In this paper, we investigate automorphisms of endomorphism semigroups for free algebras in the variety of abelian dimonoids [14].

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The paper is organized as follows. In Section 2, we give necessary definitions and auxiliary statements. In Section 3, we describe automorphisms of the endomorphism semigroup of a free abelian dimonoid of rank 1. In Section 4, we establish that all automorphisms of the endomorphism semigroup of a free abelian dimonoid are inner.

2. Auxiliary statements

A dimonoid $(D, \dashv, \vdash)$ is called *abelian* [14] if for all $x, y \in D$,

$$x \dashv y = y \vdash x.$$

Let X be an arbitrary nonempty set and $\mathbb{N}$ be the set of all positive integers. Denote by $\mathrm{FCm}(X)$ the free commutative monoid on X with the identity ε. Words of $\mathrm{FCm}(X)$ are written as $w = w_1^{\alpha_1} w_2^{\alpha_2} \dots w_n^{\alpha_n}$, where $w_1, w_2, \dots, w_n \in X$ are pairwise distinct, and $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{N} \cup \{0\}$. Here $w^0 = \varepsilon$ and $w^1 = w$ for all $w \in X$.

We put

$$\mathrm{FAd}(X) = X \times \mathrm{FCm}(X)$$

and define two binary operations $\dashv$ and $\vdash$ on $FAd(X)$ as follows:

$$(x, u) \dashv (y, v) = (x, uyv),$$

$$(x, u) \vdash (y, v) = (y, xuv).$$

Theorem 1 ([14])**.** *The algebra $(FAd(X), \dashv, \vdash)$ is the free abelian dimonoid of rank $|X|$.*

Further, for the sake of convenientce, the free abelian dimonoid $(\mathrm{FAd}(X), \dashv, \vdash)$ will be denoted also by $\mathfrak{F}_X$.

Let $(S, \circ)$ be an arbitrary semigroup and $a \in S$. Define on S a new binary operation $\circ_a$ as follows:

$$x \circ_a y = x \circ a \circ y$$

for all $x, y \in S$.

Clearly, $(S, \circ_a)$ is a semigroup, it is called a *variant* of $(S, \circ)$.

Proposition 1 ([14])**.** *The operations of the free abelian dimonoid $\mathfrak{F}_X$ of rank 1 coincide, and the semigroup $(FAd(X), \dashv), |X| = 1$, is isomorphic to the variant $(\mathbb{N}^0, +_1)$ of the additive semigroup of all nonnegative integers.*

Let $\mathfrak{D}_1 = (D_1, \dashv_1, \vdash_1)$ and $\mathfrak{D}_2 = (D_2, \dashv_2, \vdash_2)$ be arbitrary dimonoids. A mapping $\varphi : D_1 \to D_2$ is called a *homomorphism* of $\mathfrak{D}_1$ into $\mathfrak{D}_2$ if for all $x, y \in D_1$,

$$(x \dashv_1 y)\varphi = x\varphi \dashv_2 y\varphi, \;\; (x \vdash_1 y)\varphi = x\varphi \vdash_2 y\varphi.$$

A bijective homomorphism $\varphi : D_1 \to D_2$ is called an *isomorphism* of $\mathfrak{D}_1$ into $\mathfrak{D}_2$.

The following lemma is obvious.

Lemma 1. *Let $\mathfrak{F}_X$ and $\mathfrak{F}_Y$ be free abelian dimonoids on X and Y, respectively. Every bijection $\varphi : X \to Y$ induces an isomorphism π_φ of $\mathfrak{F}_X$ into $\mathfrak{F}_Y$ such that*

$$(x, \varepsilon)\pi_\varphi = (x\varphi, \varepsilon), \;\; (y, \omega)\pi_\varphi = (y\varphi, (w_1\varphi)^{\alpha_1}(w_2\varphi)^{\alpha_2} \dots (w_n\varphi)^{\alpha_n})$$

for all $(x, \varepsilon), (y, \omega) \in FAd(X)$, where $w = w_1^{\alpha_1} w_2^{\alpha_2} \dots w_n^{\alpha_n} \neq \varepsilon$.

For an arbitrary dimonoid $\mathfrak{D} = (D, \dashv, \vdash)$, we denote the endomorphism semigroup of $\mathfrak{D}$ and the automorphism group of $\mathfrak{D}$ by $\mathrm{End}(\mathfrak{D})$ and $\mathrm{Aut}(\mathfrak{D})$, respectively.

Let $(t, u) \in \mathrm{FAd}(X)$, $u = u_1^{\alpha_1} u_2^{\alpha_2} \ldots u_n^{\alpha_n}$. An arbitrary endomorphism $\Xi \in \mathrm{End}(\mathfrak{F}_X)$ has the form:

$$(t, u)\Xi = (t, \varepsilon)\xi \dashv ((u_1, \varepsilon)\xi)^{\alpha_1} \dashv \ldots \dashv ((u_n, \varepsilon)\xi)^{\alpha_n},$$

where $\xi : X \times \varepsilon \to \mathrm{FAd}(X)$ is any mapping.

In particular, an endomorphism Φ of $\mathfrak{F}_X$ is an automorphism iff a restriction Φ on $X \times \varepsilon$ belong to the symmetric group $S(X \times \varepsilon)$. So, the automorphism group $\mathrm{Aut}(\mathfrak{F}_X)$ is isomorphic to the group $S(X)$.

Let $F(X)$ be a free algebra in a variety V with a generating set X and $u \in F(X)$. An endomorphism $\theta_u \in \mathrm{End}(F(X))$ is called *constant* if $x\theta_u = u$ for all $x \in X$.

Let $\Psi : \mathrm{End}(\mathfrak{F}_X) \to \mathrm{End}(\mathfrak{F}_Y)$ be an arbitrary isomorphism. From Theorem 3 of [14] it follows that for every $x \in X$ there exists $y \in Y$ such that $\theta_{(x,\varepsilon)}\Psi = \theta_{(y,\varepsilon)}$. Define a bijection $\psi : X \to Y$ putting $x\psi = y$ if $\theta_{(x,\varepsilon)}\Psi = \theta_{(y,\varepsilon)}$. In this case, we say that ψ is induced by the isomorphism Ψ.

3. The automorphism group of $\mathrm{End}(\mathfrak{F}_X), |X| = 1$

We denote by $\mathfrak{F}_n$ the free abelian dimonoid $\mathfrak{F}_X = (\mathrm{FAd}(X), \dashv, \vdash)$ on an arbitrary finite n-element set X.

From Proposition 1 it follows that the dimonoid $\mathfrak{F}_1$ is isomorphic to the variant $(\mathbb{N}^0, +_1, +_1)$. Therefore, we identify the elements of $\mathrm{FAd}(X), |X| = 1$, with the corresponding elements of $\mathbb{N}^0$.

Lemma 2. *The endomorphism monoid End*$(\mathfrak{F}_1)$ *of the free abelian dimonoid* $\mathfrak{F}_1$ *is isomorphic to the semigroup* $(\mathbb{N}^0, *)$*, where* $x * y = x + y + xy$ *for all* $x, y \in \mathbb{N}^0$.

Proof. It is obvious that $\mathrm{End}(\mathbb{N}^0, +_1, +_1) = \mathrm{End}(\mathbb{N}^0, +_1)$. Let φ be an arbitrary endomorphism of $(\mathbb{N}^0, +_1)$ and $\varphi(0) = k$ for some $k \in \mathbb{N}^0$. Then for any $a \in \mathbb{N}^0$ we have

$$\begin{aligned} a\varphi &= (\underbrace{0 +_1 0 +_1 \ldots +_1 0}_{a+1})\varphi = \\ &= \underbrace{0\varphi +_1 0\varphi +_1 \ldots +_1 0\varphi}_{a+1} = \\ &= \underbrace{k +_1 k +_1 \ldots +_1 k}_{a+1} = \\ &= (a+1)k + a. \end{aligned}$$

On the other hand, any transformation $\varphi_k, k \in \mathbb{N}^0$, of $\mathbb{N}^0$ defined by

$$a\varphi_k = (a+1)k + a$$

for all $a \in \mathbb{N}^0$, is an endomorphism of $(\mathbb{N}^0, +_1)$. Thus,

$$\mathrm{End}(\mathbb{N}^0, +_1) = \left\{\varphi_k \mid k \in \mathbb{N}^0\right\}.$$

Define a mapping Θ of $\mathrm{End}(\mathbb{N}^0, +_1)$ into $(\mathbb{N}^0, *)$ by $\varphi_k\Theta = k$ for all $\varphi_k \in \mathrm{End}(\mathbb{N}^0, +_1)$. It is clear that Θ is a bijection, moreover,

$$a(\varphi_k \circ \varphi_l) = (a\varphi_k)\varphi_l = ((a+1)k + a)\varphi_l = (ak + k + a + 1)l + ak + k + a = a\varphi_{kl+k+l}$$

for all $a \in \mathbb{N}^0$, where $\circ$ is the usual composition of transformations. Thus,

$$(\varphi_k \circ \varphi_l)\Theta = \varphi_{kl+k+l}\Theta = \varphi_{k*l}\Theta = k * l = \varphi_k\Theta * \varphi_l\Theta$$

for all $\varphi_k, \varphi_l \in \mathrm{End}(\mathbb{N}^0, +_1)$. □

Lemma 3. *The semigroup* $(\mathbb{N}^0, *)$ *is isomorphic to the multiplicative semigroup* $(\mathbb{N}, \cdot)$ *of all positive integers.*

Proof. Define a mapping θ of $(\mathbb{N}^0, *)$ into $(\mathbb{N}, \cdot)$ by $n\theta = n + 1$ for all $n \in \mathbb{N}^0$. Clearly, θ is a bijection. In addition,

$$(n * m)\theta = (n + m + nm)\theta = n + m + nm + 1 = (n + 1) \cdot (m + 1) = n\theta \cdot m\theta$$

for all $n, m \in \mathbb{N}^0$. □

By $\mathbb{P}$ we denote the set of all prime numbers.

Proposition 2. *Let X be a singleton set, Y be an arbitrary nonempty set and $End(\mathfrak{F}_X) \cong End(\mathfrak{F}_Y)$. Then $|Y| = 1$ and the isomorphisms of $End(\mathfrak{F}_X)$ onto $End(\mathfrak{F}_Y)$ are in a natural one-to-one correspondence with permutations of $\mathbb{P}$.*

Proof. The fact that $|Y| = 1$ follows from Theorem 3 of [14], where it was proved that free abelian dimonoids are determined by their endomorphism semigroups. By Lemmas 2 and 3, $\mathrm{End}(\mathfrak{F}_X), |X| = 1$, is isomorphic to $(\mathbb{N}, \cdot)$. The monoid $(\mathbb{N}, \cdot)$ is the free commutative monoid with the countably infinite set of free generators that are prime numbers. Thus, every isomorphism $\varphi : \mathrm{End}(\mathfrak{F}_X) \to \mathrm{End}(\mathfrak{F}_Y)$ is uniquely determined by a bijection between the free generators of $\mathrm{End}(\mathfrak{F}_X)$ and $\mathrm{End}(\mathfrak{F}_Y)$. These bijections, and hence isomorphisms, are in a natural one-to-one correspondence with permutations of the set $\mathbb{P}$ of all prime numbers. In addition, the automorphisms of the monoid $\mathrm{End}(\mathfrak{F}_X)$ correspond to the permutations of free generators of $\mathbb{P}$. □

Recall that the symmetric group on a set X is denoted by $S(X)$. From Proposition 2 immediately follows

Corollary 1. *The automorphism group $Aut(End(\mathfrak{F}_1))$ of the endomorphism monoid $End(\mathfrak{F}_1)$ is isomorphic to the symmetric group $S(\mathbb{P})$ on a countably infinite set $\mathbb{P}$.*

4. The automorphism group of End$(\mathfrak{F}_X), |X| \geqslant 2$

Let $F(X)$ be a free algebra in a variety V over a set X. An automorphism Ψ of the endomorphism monoid $\mathrm{End}(F(X))$ is called *stable* if Ψ induces the identity permutation of X, that is, $\theta_x\Psi = \theta_x$ for all $x \in X$.

An endomorphism θ of the free algebra $F(X)$ is called *linear* if $x\theta \in X$ for all $x \in X$.

For every $\omega \in \mathrm{FCm}(X)$ by $l(\omega)$ and $c(\omega)$ we denote the *length* and the *content* of ω, respectively. Recall that the content of a non-identity word $\omega = x_1x_2 \ldots x_n \in \mathrm{FCm}(X)$ is the set $\{x_1, x_2, \ldots, x_n\}$, and $c(\varepsilon) = \varnothing$ and $l(\varepsilon) = 0$. For all $(x, u) \in \mathfrak{F}_X$ we put $|(x, u)| = l(u) + 1$ and $c(x, u) = c(u) \cup \{x\}$.

Lemma 4. *Let Ψ be a stable automorphism of $End(\mathfrak{F}_X)$, $g \in End(\mathfrak{F}_X)$ and $x \in X$. The following equalities hold:*

(*i*) $c(t, u) = c(s, v)$ *if* $\theta_{(t,u)}\Psi = \theta_{(s,v)}$,
(*ii*) $|(x, \varepsilon)g| = |(x, \varepsilon)(g\Psi)|$.

Proof. (i) Let $z \in c(t,u) \setminus c(s,v)$, $x \in X, x \neq z$ and $\varphi, g \in \mathrm{End}(\mathfrak{F}_X)$ such that $(x,\varepsilon)\varphi = (t,u), (z,\varepsilon)g = (x,\varepsilon)$ and $(y,\varepsilon)g = (y,\varepsilon)$ for all $y \in X, y \neq z$. Then g is linear, $(s,v)g = (s,v)$ and, in addition,

$$\theta_{(x,\varepsilon)(g\Psi)} = \theta_{(x,\varepsilon)}\ (g\Psi) = \theta_{(x,\varepsilon)}\Psi\ g\Psi = (\theta_{(x,\varepsilon)}g)\Psi = \theta_{(x,\varepsilon)g}\Psi = \theta_{(x,\varepsilon)g}.$$

From here, $g\Psi = g$ and then

$$\theta_{(t,u)}\Psi = \theta_{(s,v)} = \theta_{(s,v)g} = \theta_{(s,v)}g = \theta_{(t,u)}\Psi g\Psi = (\theta_{(t,u)}g)\Psi = (\theta_{(t,u)g})\Psi.$$

By injectivity of Ψ, we have $\theta_{(t,u)} = \theta_{(t,u)g}$. From here $(t,u) = (t,u)g$, which contradicts to the definition of g, so $c(t,u) \setminus c(s,v) = \varnothing$. Similarly we can prove that $c(s,v) \setminus c(t,u) = \varnothing$. It means that $c(t,u) = c(s,v)$.

(ii) Let $g_1, g_2 \in \mathrm{End}(\mathfrak{F}_X)$ such that $|(x,\varepsilon)g_1| = |(x,\varepsilon)g_2| = m$ and

$$|(x,\varepsilon)(g_1\Psi)| = k,\ \ |(x,\varepsilon)(g_2\Psi)| = l.$$

For all $(y,v) \in \mathrm{FAd}(X)$ with $|v| = r-1$ we obtain

$$(y,v)(\theta_{(x,\varepsilon)}g_1\theta_{(x,\varepsilon)}) = ((x,\varepsilon)^r g_1)\theta_{(x,\varepsilon)} = (x,\varepsilon)^{rm}\theta_{(x,\varepsilon)} = (x,\varepsilon)^{rm} = (y,v)\theta_{(x,\varepsilon)^m}.$$

Thus, $\theta_{(x,\varepsilon)}g_1\theta_{(x,\varepsilon)} = \theta_{(x,\varepsilon)^m}$. Analogously it is proved that

$$\theta_{(x,\varepsilon)}g_2\theta_{(x,\varepsilon)} = \theta_{(x,\varepsilon)^m},$$
$$\theta_{(x,\varepsilon)}(g_1\Psi)\theta_{(x,\varepsilon)} = \theta_{(x,\varepsilon)^k},$$
$$\theta_{(x,\varepsilon)}(g_2\Psi)\theta_{(x,\varepsilon)} = \theta_{(x,\varepsilon)^l}.$$

Using that Ψ is stable, we have

$$\theta_{(x,\varepsilon)^m}\Psi = (\theta_{(x,\varepsilon)}g_1\theta_{(x,\varepsilon)})\Psi = \theta_{(x,\varepsilon)}(g_1\Psi)\theta_{(x,\varepsilon)} = \theta_{(x,\varepsilon)^k},$$
$$\theta_{(x,\varepsilon)^m}\Psi = (\theta_{(x,\varepsilon)}g_2\theta_{(x,\varepsilon)})\Psi = \theta_{(x,\varepsilon)}(g_2\Psi)\theta_{(x,\varepsilon)} = \theta_{(x,\varepsilon)^l},$$

from here $k = l$.

Assume that A is a nonempty finite subset of X, $m \in \mathbb{N}$ and

$$\mathrm{End}_A^m(x,\varepsilon) = \{g \in \mathrm{End}(\mathfrak{F}_X) : |(x,\varepsilon)g| = m,\ c((x,\varepsilon)g) = A\}.$$

Let $g \in \mathrm{End}_A^m(x,\varepsilon)$, then $\theta_{(x,\varepsilon)g} \in \mathrm{End}_A^m(x,\varepsilon)$. It is not hard to check that $\theta_{(x,\varepsilon)g}\Psi = \theta_{(x,\varepsilon)(g\Psi)}$. By (i), $c((x,\varepsilon)g) = c((x,\varepsilon)(g\Psi))$, that is, $g\Psi \in \mathrm{End}_A^k(x,\varepsilon)$ for some natural k. So, $\mathrm{End}_A^m(x,\varepsilon)\Psi \subseteq \mathrm{End}_A^k(x,\varepsilon)$. Since Ψ is bijective, $k = m$. Hence $|(x,\varepsilon)g| = |(x,\varepsilon)(g\Psi)|$ for all $g \in \mathrm{End}(\mathfrak{F}_X)$ and $x \in X$. □

Corollary 2. *Let Ψ be a stable automorphism of End$(\mathfrak{F}_X)$ and $a, b \in X$, $a \neq b$. Then*

$$\theta_{(a,b)}\Psi \in \{\theta_{(a,b)}, \theta_{(b,a)}\}.$$

Proof. Since $\theta_{(x,\varepsilon)g}\Psi = \theta_{(x,\varepsilon)(g\Psi)}$, then $\theta_{(a,b)}\Psi = \theta_{(t,u)}$ for some $(t,u) \in \mathrm{FAd}(X)$. By Lemma 4$(i)$, $c(t,u) = \{a,b\}$. In according to (ii) of Lemma 4, we have $|(t,u)| = 2$, whence $l(u) = 1$. It is means that $(t,u) = (a,b)$ or $(t,u) = (b,a)$. □

By Φ_0 we denote the identity automorphism of $\mathrm{End}(\mathfrak{F}_X)$.

Lemma 5. *Let Ψ be a stable automorphism of End$(\mathfrak{F}_X)$ and $a, b \in X$ are distinct. If $\theta_{(a,b)}\Psi = \theta_{(a,b)}$, then $\Psi = \Phi_0$.*

The proof of this lemma is similar to Lemma 5 of [13].

Lemma 6. *Let $a, b \in X$ be distinct. There is no stable automorphism Ψ of End($\mathfrak{F}_X$) such that $\theta_{(a,b)}\Psi = \theta_{(b,a)}$.*

Proof. Assume that there exists a stable automorphism Ψ of the monoid $\mathrm{End}(\mathfrak{F}_X)$ such that $\theta_{(a,b)}\Psi = \theta_{(b,a)}$. According to condition (i) of Lemma 4, $\theta_{(b,a)}\Psi = \theta_{(a,b)}$.

Let $g \in \mathrm{End}(\mathfrak{F}_X)$ be such that $(a,\varepsilon)g = (a,\varepsilon)$, $(b,\varepsilon)g = (a,b)$ and $(x,\varepsilon)g = (x,\varepsilon)$ for all $x \in X \setminus \{a,b\}$. It is easy to check that $\theta_{(b,\varepsilon)}g = \theta_{(a,b)}$ and then

$$\theta_{(a,b)}\Psi = (\theta_{(b,\varepsilon)}g)\Psi = \theta_{(b,\varepsilon)}\Psi g\Psi = \theta_{(b,\varepsilon)}g\Psi = \theta_{(b,\varepsilon)g\Psi}.$$

Since $\theta_{(a,b)}\Psi = \theta_{(b,a)}$, then $\theta_{(b,a)} = \theta_{(b,\varepsilon)g\Psi}$ and therefore we have

$$(b,a) = (b,\varepsilon)g\Psi. \tag{1}$$

Using equality $\theta_{(a,\varepsilon)}g = \theta_{(a,\varepsilon)}$, we obtain

$$\theta_{(a,\varepsilon)}\Psi = (\theta_{(a,\varepsilon)}g)\Psi = \theta_{(a,\varepsilon)}g\Psi = \theta_{(a,\varepsilon)g\Psi} = \theta_{(a,\varepsilon)},$$

and therefore

$$(a,\varepsilon)g\Psi = (a,\varepsilon). \tag{2}$$

Further, for all $x \in X$,

$$(x,\varepsilon)\theta_{(a,b)}g = (a,b)g = ((a,\varepsilon) \dashv (b,\varepsilon))g = (a,\varepsilon) \dashv (a,b) = (a,ab) = (x,\varepsilon)\theta_{(a,ab)}.$$

Then

$$\theta_{(a,ab)}\Psi = (\theta_{(a,b)}g)\Psi = \theta_{(a,b)}\Psi g\Psi = \theta_{(b,a)}g\Psi = \theta_{(b,a)g\Psi}.$$

Using equalities (1) and (2) we obtain

$$(b,a)g\Psi = ((b,\varepsilon) \dashv (a,\varepsilon))g\Psi = (b,\varepsilon)g\Psi \dashv (a,\varepsilon)g\Psi = (b,a) \dashv (a,\varepsilon) = (b,aa).$$

Thus,

$$\theta_{(a,ab)}\Psi = \theta_{(b,aa)}. \tag{3}$$

It is clear that $\theta_{(b,a)}g = \theta_{(a,ba)}$. Then

$$\theta_{(a,ba)}\Psi = (\theta_{(b,a)}g)\Psi = \theta_{(a,b)}g\Psi = \theta_{(a,b)g\Psi},$$

where $(a,b)g\Psi = (a,\varepsilon)g\Psi \dashv (b,\varepsilon)g\Psi = (a,\varepsilon) \dashv (b,a) = (a,ba)$, that is,

$$\theta_{(a,ba)}\Psi = \theta_{(a,ba)}. \tag{4}$$

Since $ab = ba$ in $\mathrm{FCm}(X)$, then $\theta_{(a,ba)} = \theta_{(a,ab)}$ and according to (3), (4), we have $(b,aa) = (a,ba)$ that contradicts the condition $a \neq b$. □

Theorem 2. *Every isomorphism $\Phi : End(\mathfrak{F}_X) \to End(\mathfrak{F}_Y)$ is induced by the isomorphism π_f of $\mathfrak{F}_X$ to $\mathfrak{F}_Y$ for a uniquely determined bijection $f : X \to Y$.*

Proof. Let $|X| > 1$ and $\Phi : \mathrm{End}(\mathfrak{F}_X) \to \mathrm{End}(\mathfrak{F}_Y)$ be an arbitrary isomorphism. Then Φ induces a uniquely determined bijection $f : X \to Y$ such that $\theta_{(x,\varepsilon)}\Phi = \theta_{(xf,\varepsilon)}$ for every $x \in X$ (see Section 2). By Lemma 1, f induces the isomorphism $\pi_f : \mathfrak{F}_X \to \mathfrak{F}_Y$. It is not hard to check that the mapping $E_f : \eta \mapsto \pi_f^{-1}\eta\pi_f$ is an isomorphism of $\mathrm{End}(\mathfrak{F}_X)$ onto $\mathrm{End}(\mathfrak{F}_Y)$. From here, $\Omega = \Phi E_f^{-1}$ is an automorphism of $\mathrm{End}(\mathfrak{F}_X)$. Moreover, for all $x \in X$,

$$\theta_{(x,\varepsilon)}\Omega = (\theta_{(x,\varepsilon)}\Phi)E_f^{-1} = \theta_{(xf,\varepsilon)}E_f^{-1} = \theta_{(xff^{-1},\varepsilon)} = \theta_{(x,\varepsilon)},$$

therefore Ω is stable.

By Corollary 2, Lemma 5 and Lemma 6, Ω is an identity automorphism Φ_0. From $\Phi E_f^{-1} = \Phi_0$ we obtain $\Phi = E_f$, i.e., Φ is an isomorphism induced by π_f. □

Let $F(X)$ be a free algebra in a variety V over a set X. An automorphism Φ of $\mathrm{End}(F(X))$ is called *inner* if there exists an automorphism α of $F(X)$ such that $\beta\Phi = \alpha^{-1}\beta\alpha$ for all $\beta \in \mathrm{End}(F(X))$.

Now we characterize the automorphism group of the endomorphism monoid of a free abelian dimonoid.

Theorem 3. *All automorphisms of End($\mathfrak{F}_X$) are inner. In addition, the automorphism group Aut(End($\mathfrak{F}_X$)) is isomorphic to the symmetric group $S(X)$.*

Proof. For the case $X = Y$, Theorem 2 will be the first part of the given theorem. By Theorem 2, every automorphism Φ of $\mathrm{End}(\mathfrak{F}_X)$ has the form $\Phi = E_f$, where $\eta\Phi = \pi_f^{-1}\eta\pi_f$ for all $\eta \in \mathrm{End}(\mathfrak{F}_X)$ and some bijection $f : X \to X$. According to Lemma 1 (see Section 2), $\pi_f \in \mathrm{Aut}(\mathfrak{F}_X)$ for all $f \in S(X)$. Consequently, all automorphisms of $\mathrm{End}(\mathfrak{F}_X)$ are inner.

It is clear that the groups $\mathrm{Aut}(\mathrm{End}(\mathfrak{F}_X))$ and $S(X)$ are isomorphic. □

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ПРО АВТОМОРФІЗМИ НАПІВГРУПИ ЕНДОМОРФІЗМІВ ВІЛЬНОГО АБЕЛЕВОГО ДІМОНОЇДА

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Описано всі ізоморфізми між напівгрупами ендоморфізмів вільних абелевих дімоноїдів і доведено, що всі автоморфізми напівгрупи ендоморфізмів вільного абелевого дімоноїда є внутрішніми.

Ключові слова: дімоноїд, вільний абелевий дімоноїд, напівгрупа ендоморфізмів, автоморфізм.