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EXTENSION OF BOUNDED BAIRE-ONE FUNCTIONS VS EXTENSIONS OF UNBOUNDED BAIRE-ONE FUNCTIONS

Dedicated to the 60th birthday of M. M. Zarichnyi

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We compare possibilities of extension of bounded and unbounded Baire-one functions from subspaces of topological spaces.

Key words: extension, Baire-one function, B_1-embedded set, B_1^*-embedded set.

1. Introduction

Let X be a topological space. A function $f\colon X\to\mathbb{R}$ belongs *to the first Baire class*, if it is a pointwise limit of a sequence of real-valued continuous functions on X. We will denote by $\mathrm{B}_1(X)$ and $\mathrm{B}_1^*(X)$ the collections of all Baire-one and bounded Baire-one functions on X, respectively.

A subset E of X is B_1-*embedded* (B_1^*-*embedded*) in X, if every (bounded) function $f\in\mathrm{B}_1(E)$ can be extended to a function $g\in\mathrm{B}_1(X)$. We will say that a space X has the *property* ($\mathrm{B}_1^*=\mathrm{B}_1$) if every B_1^*-embedded subset of X is B_1-embedded in X.

Characterizations of B_1- and B_1^*-embedded subsets of topological spaces were obtained in [3] and [4].

This short note is devoted to the following interesting problem: to find topological spaces with the property ($\mathrm{B}_1^*=\mathrm{B}_1$).

In the second section of this note we extend results from [4, Section 6] and show that every hereditarily Lindelöff hereditarily Baire space X which hereditarily has a σ-discrete π-base has the property ($\mathrm{B}_1^*=\mathrm{B}_1$). In Section 3 we show that any countable completely

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2010 Mathematics Subject Classification: Primary 54C20, 26A21; Secondary 54C30, 54C50 regular hereditarily irresolvable space X without isolated points is B_1^*-embedded and is not B_1-embedded in βX.

2. Spaces with the property $(\mathrm{B}_1^* = \mathrm{B}_1)$

Recall that a set A in a topological space X is *functionally G_δ (functionally F_σ)*, if A is an intersection (a union) of a sequence of functionally open (functionally closed) subsets of X. We say that a subset A of a topological space X is *functionally ambiguous* if A is functionally F_σ and functionally G_δ simultaneously.

Lemma 1. *Let X be a completely regular topological space of the first category with a σ-discrete π-base. Then there exist disjoint functionally ambiguous sets A and B such that*

$$X = A \cup B = \overline{A} = \overline{B}.$$

Proof. We fix a π-base $\mathscr{V} = (\mathscr{V}_n : n \in \omega)$ of X, where each family $\mathscr{V}_n$ is discrete and consists of functionally open sets in X. Denote $V_n = \bigcup\{V : V \in \mathscr{V}_n\}$ for all $n \in \omega$.

Let us observe that every open set $G \subseteq X$ contains a functionally open subset U such that $U \subseteq G \subseteq \overline{U}$. Indeed, for every $n \in \omega$ we put $U_n = \cup\{V \in \mathscr{V}_n \colon V \subseteq G\}$ and $U = \bigcup_{n\in\omega} U_n$. Then each U_n is functionally open as a union of a discrete family of functionally open sets. Hence, U is functionally open. It is easy to see that U is dense in G.

Keeping in mind the previous fact, we may assume that there exists a covering $(F_n \colon n \in \omega)$ of the space X by nowhere dense functionally closed sets $F_n \subseteq X$. Let $X_0 = F_0$ and $X_n = F_n \setminus \bigcup_{k<n} F_k$ for all $n \geq 1$. Then $(X_n \colon n \in \omega)$ is a partition of X by nowhere dense functionally ambiguous sets X_n.

Fix $n \in \omega$ and $V \in \mathscr{V}_n$. Since X is regular, we can choose two open sets H_1 and H_2 in V such that $\overline{H_1} \cap \overline{H_2} = \varnothing$ and $\overline{H_i} \subseteq V$ for $i = 1, 2$. Let G_i and O_i be functionally open sets such that $G_i \subseteq H_i \subseteq \overline{G_i}$ and $O_i \subseteq X \setminus \overline{H_i} \subseteq \overline{O_i}$, $i = 1, 2$. We put $A_{V,n} = X \setminus (G_1 \cup O_1)$ and $B_{V,n} = X \setminus (G_2 \cup O_2)$ and obtain disjoint nowhere dense functionally closed subsets of V.

We put $m_0 = 0$ and choose numbers $n_1 \geq 0$ and $m_1 > n_1$ such that $X_{n_1} \cap V_1 \neq \varnothing$ and $X_{m_1} \cap V_1 \neq \varnothing$. Notice that $A_1' = \bigcup_{n=0}^{n_1} X_n$ and $B_1' = \bigcup_{n=n_1+1}^{m_1} X_n$ are nowhere dense functionally ambiguous sets in X. Now we consider the set

$$\mathscr{W}_1 = \{V \in \mathscr{V}_1 : V \cap (A_1' \cup B_1') = \varnothing\}$$

and observe that the sets $A_1'' = \cup\{A_{V,1} \colon V \in \mathscr{W}_1\}$ and $B_1'' = \cup\{B_{V,1} \colon V \in \mathscr{W}_1\}$ are functionally closed and nowhere dense in X. Let $A_1 = A_1' \cup A_1''$ and $B_1 = B_1' \cup B_1''$. Notice that A_1 and B_1 are functionally ambiguous nowhere dense disjoint subsets of X.

Since $\overline{X \setminus (A_1 \cup B_1)} = X$, there exists a number $n_2 > m_1$ such that $(X_{n_2} \setminus (A_1 \cup B_1)) \cap V_2 \neq \varnothing$. We put $A_2' = \bigcup_{n=m_1+1}^{n_2} (X_n \setminus (A_1 \cup B_1))$. Moreover, there exists $m_2 > n_2$ such that $(X_{m_2} \setminus (A_1 \cup B_1)) \cap V_2 \neq \varnothing$. Let $B_2' = \bigcup_{n=n_2+1}^{m_2} (X_n \setminus (A_1 \cup B_1))$. We put $\mathscr{W}_2 = \{V \in \mathscr{V}_2 \colon V \cap (A_2' \cup B_2') = \varnothing\}$ and observe that the sets $A_2'' = \{A_{V,2} \colon V \in \mathscr{W}_2\}$ and $B_2'' = \{b_{V,2} \colon V \in \mathscr{W}_2\}$ are functionally closed and nowhere dense in X. We denote $A_2 = A_2' \cup A_2''$ and $B_2 = B_2' \cup B_2''$. Then A_2 and B_2 are functionally ambiguous nowhere dense disjoint subsets of X.

Proceeding this process inductively we obtain sequences $(A_k)_{k=1}^{\infty}$ and $(B_k)_{k=1}^{\infty}$ of functionally ambiguous sets such that $A_k \cap V \neq \varnothing \neq B_k \cap V$, $A_k \cap B_k = \varnothing$ for all $k \in \mathbb{N}$ and $V \in \mathscr{V}_k$. It remains to put $A = \bigcup_{k=1}^{\infty} A_k$, $B = \bigcup_{k=1}^{\infty} B_k$ and observe that $A \cup B = X$.

In addition, note that Borel resolvability of topological spaces was also studied in [1, 2].

We say that a topological space X *hereditarily has a σ-discrete π-base* if every its closed subspace has a σ-discrete π-base. It is easy to see that if a space X hereditarily has a σ-discrete π-base, then each subspace of X has a σ-discrete π-base.

Recall that a subspace E of a topological space X is *z-embedded in X*, if any functionally closed subset F of E can be extended to a functionally closed subset of X.

Lemma 2. *Let X be a normal space such that X hereditarily has a σ-discrete π-base. If X is a B_1^*-embedded subset of a hereditarily Baire space Y, then X is hereditarily Baire.*

Proof. Assume that X is not hereditarily Baire and find a closed subset $F \subseteq X$ of the first category. According to Lemma 1, there exist disjoint functionally ambiguous subsets A and B in F such that $F = A \cup B = \overline{A} = \overline{B}$. Since F is a closed subset of a normal space, F is z-embedded in X. Therefore, there are two functionally ambiguous disjoint sets $\widetilde{A}$ and $\widetilde{B}$ in X such that $\widetilde{A} \cap F = A$ and $\widetilde{B} \cap F = B$ (see [4, Proposition 4.3]). Let us observe that the characteristic function $\chi : X \to [0,1]$ of the set $\widetilde{A}$ belongs to the first Baire class. Then there exists an extension $f \in \mathrm{B}_1(Y)$ of χ. The sets $f^{-1}(0)$ and $f^{-1}(1)$ are disjoint G_δ-sets which are dense in $\overline{X}$. We obtain a contradiction, because $\overline{X}$ is a Baire space as a closed subset of a hereditarily Baire space. □

Remark 1. There exist a metrizable separable Baire space X and its B_1^*-embedded subspace E which is not a Baire space. Let $X = (\mathbb{Q} \times \{0\}) \cup (\mathbb{R} \times (0,1])$ and $E = \mathbb{Q} \times \{0\}$. Then E is closed in X. Therefore, any F_σ- and G_δ-subset C of E is also F_σ- and G_δ- in X. Hence, E is B_1^*-embedded in X.

Theorem 1. *Let Y be a hereditarily Baire completely regular space and $X \subseteq Y$ be a Lindelöf space which hereditarily has a σ-discrete π-base. The following are equivalent:*

(1) *X is B_1^*-embedded in Y;*
(2) *X is B_1-embedded in Y.*

Proof. We need only to show 1) $\Rightarrow$ 2). By Lemma 2, X is hereditarily Baire. Then X is B_1-embedded in Y by [3, Theorem 13]. □

Corollary 1. *Every hereditarily Lindelöff hereditarily Baire space X which hereditarily has a σ-discrete π-base has the property ($\mathrm{B}_1^* = \mathrm{B}_1$).*

3. Spaces without the property ($\mathrm{B}_1^* = \mathrm{B}_1$)

A subset A of a topological space X is called *(functionally) resolvable in the sense of Hausdorff* or *(functionally) H-set* if

$$A = (F_1 \setminus F_2) \cup (F_3 \setminus F_4) \cup \cdots \cup (F_\xi \setminus F_{\xi+1}) \cup \ldots,$$

where $(F_\xi)_{\xi<\alpha}$ is a decreasing chain of (functionally) closed sets in X.

It is well-known [5, §12.I] that a set A is an H-set if and only if for any closed nonempty set $F \subseteq X$ there is a nonempty relatively open set $U \subseteq F$ such that $U \subseteq A$ or $U \subseteq X \setminus A$.

A topological space without isolated points is called *crowded*.

A topological space X is *irresolvable* if it is not a union of two disjoint dense subsets. A space X is *hereditarily irresolvable* if every subspace of X is irresolvable.

Lemma 3. *Every subset of a hereditarily irresolvable space is an H-set.*

Proof. Assume that there is a closed nonempty set F in a hereditarily irresolvable space X and a set $A \subseteq X$ such that $\overline{F \cap A} \cap \overline{F \setminus A} = F$. Then

$$\overline{F \cap A} = \overline{F \setminus A} = F = (F \cap A) \cup (F \setminus A),$$

which contradicts to irresolvability of F. □

A function $f : X \to Y$ from a topological space X to a metric space (Y, d) is called *fragmented* if for every $\varepsilon > 0$ and for every closed nonempty set $F \subseteq X$ there exists a relatively open nonempty set $U \subseteq F$ such that $\mathrm{diam} f(U) < \varepsilon$.

Proposition 1. *Every bounded function $f : X \to \mathbb{R}$ on a hereditarily irresolvable space X is fragmented.*

Proof. To obtain a contradiction we assume that there exists a bounded function $f : X \to \mathbb{R}$ which is not fragmented. Then there is $\varepsilon > 0$ and a closed nonempty set $F \subseteq X$ such that for every relatively open set $U \subseteq F$ we have $\mathrm{diam} f(U) \geq \varepsilon$.

Since $f(X)$ is a compact set, we take a finite partition $\{B_1, \dots, B_n\}$ of $f(X)$ by sets of diameter $< \varepsilon$. Let $H_k = f^{-1}(B_k) \cap F$ for every $k \in \{1, \dots, n\}$. Then each H_k has empty interior in F, because f is not fragmented. By Lemma 3, each H_k is an H-set and, therefore, is nowhere dense in F. Hence, $\{H_1, \dots, H_n\}$ is a finite partition of F by nowhere dense sets, which is impossible. □

Lemma 4. *Let E be a z-embedded countable subspace of a topological space X and $A \subseteq E$ be a functionally H-set in E. Then there exists a functionally H-set $B \subseteq X$ such that B is F_σ and $B \cap E = A$.*

Proof. We take a decreasing transfinite sequence $(A_\xi : \xi < \alpha)$ of functionally closed subsets of E such that $A = \bigcup_{\xi<\alpha}(A_\xi \setminus A_{\xi+1})$ (every ordinal ξ is odd). Since $|A| \leq \aleph_0$, we may assume that $|(A_\xi : \xi < \alpha)| \leq \aleph_0$. The subspace E is z-embedded in X and we choose a decreasing sequence $(B_\xi : \xi < \alpha)$ of functionally closed sets in X such that $A_\xi = B_\xi \cap E$ for all $\xi < \alpha$. We put

$$B = \bigcup_{\xi<\alpha,\, \xi \text{ is odd}} (B_\xi \setminus B_{\xi+1}).$$

Then B is functionally F_σ-set in X and $B \cap E = A$. □

Lemma 5. *Let X be a compact space and $B \subseteq X$ be functionally Borel measurable H-set. Then B is functionally ambiguous in X.*

Proof. Since B is functionally Borel measurable, there exists a sequence $(f_n)_{n\in\omega}$ of continuous functions $f_n : X \to [0,1]$ such that B belongs to the σ-algebra generated by the system of sets $\{f_n^{-1}(0) : n \in \omega\}$. We consider a continuous map $f : X \to [0,1]^\omega$, $f(x) = (f_n(x))_{n\in\omega}$ for all $x \in X$, and a compact metrizable space $Y = f(X) \subseteq [0,1]^\omega$.

We show that the set $B' = f(B)$ is an H-set in Y. Suppose to the contrary that there is a closed nonempty set Y' in Y such that $\overline{Y' \cap B'} = \overline{Y' \setminus B'} = Y'$. We put $X' = f^{-1}(Y')$ and $g = f|_{X'}$. Since X' is a compact space and $f(X') = Y'$, we apply Zorn's Lemma and find a closed nonempty set $Z \subseteq X'$ such that the restriction $g|_Z : Z \to Y'$ of the continuous map $g : X' \to Y'$ is irreducible. Keeping in mind that the preimage of any everywhere dense set remains everywhere dense under an irreducible map, we obtain that

$$\overline{g^{-1}(Y' \cap B')} = \overline{g^{-1}(Y' \setminus B')} = Z = \overline{Z \cap B} = \overline{Z \setminus B},$$

which contradicts to resolvability of B.

By [5, §30, X, Theorem 5] the set $f(B)$ is F_σ and G_δ in a compact metrizable space Y. Since $B = f^{-1}(f(B))$ and f is continuous, we have that B is functionally ambiguous subset of X. □

Proposition 2. *Let X be a countable hereditarily irresolvable completely regular space. Then X is B_1^*-embedded in βX.*

Proof. Since X is countable and completely regular, it is perfectly normal. Therefore, every subsets of X is functionally ambiguous.

Fix an arbitrary $A \subseteq X$. By Lemma 3 the set A is an H-set. We apply Lemma 4 and find a functionally H-set $B \subseteq \beta X$ such that B is F_σ and $B \cap X = A$. Notice that B is functionally ambiguous by Lemma 5. Hence, B is a B_1^*-embedded subspace of βX according to [4, Proposition 5.1]. □

Let us observe that examples of countable hereditarily irresolvable completely regular spaces can be found, for instance, in [6, p. 536].

Proposition 3. *Let X be a countable completely regular space without isolated points. Then X is not B_1-embedded in βX.*

Proof. Observe that X is a functionally F_σ-subset of βX. Now assume that X is B_1-embedded in βX. According to [3, Proposition 8(iii)] there should be a function $f \in \mathrm{B}_1(\beta X)$ such that $X \subseteq f^{-1}(0)$ and $\beta X \setminus X \subseteq f^{-1}(1)$. Then the set X is G_δ in βX. Therefore, X is a Baire space, which implies a contradiction, since X is of the first category in itself. □

Propositions 2 and 3 imply the following fact.

Theorem 2. *Let X be a countable hereditarily irresolvable completely regular space without isolated points. Then X is B_1^*-embedded in βX and is not B_1-embedded in βX.*

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ПРОДОВЖЕННЯ ОБМЕЖЕНИХ І НЕОБМЕЖЕНИХ ФУНКЦІЙ ПЕРШОГО КЛАСУ БЕРА

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Порівнюються можливості продовження обмежених і необмежених функцій першого класу Бера з підпросторів топологічних просторів.

Ключові слова: продовження, функція першого класу Бера, B_1-вкладена множина, B_1^*-вкладена множина.