ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2017. Випуск 84. С. 80–86
Visnyk of the Lviv Univ. Series Mech. Math. 2017. Issue 84. P. 80–86

УДК 517.53

PROXIMITY OF THE ENTIRE FUNCTIONS OF ZERO ORDER WITH υ-DENSITY OF ZEROS

Mykola ZABOLOTSKYI, Yuliia BASIUK

Ivan Franko National University of Lviv,
Universitetska Str., 1, Lviv, 79000, Ukraine
e-mail: m_zabol@franko.lviv.ua, yuliya.basyuk.92@gmail.com

The behavior of the difference of logarithms of the entire functions of zero order, whose zeros have equal modules and have density with respect to slowly increasing function υ, and the arguments of zeros differ by a small number, is investigated.

Key words: entire function, density of zeros, approximation

1. Introduction. The construction of the theory of entire functions of completely regular growth (c.r.g.) by Levin and Pfluger was the main event in the 1930s in the theory of analytic functions. The works of Valiron [1] and Titchmarsh [2] on connection between the regular behavior of the logarithm of an entire function f of positive order ρ with zeros on one ray and the existence of density of these zeros with repsect to the comparison function r^ρ became the first step towards the creation of this theory. From these results it was easy to obtain analogues of such connections between $\ln f$ and zeros of f in case of the arrangement of zeros on a finite system of rays. The transition to the general case of the location of the zeros required appropriate statements on proximity of $\ln f$ and $\ln g$, where g is an entire function with zeros on a finite system of rays and moduli of its zeros equal to the moduli of zeros of f, and arguments of zeros differ on a small number δ (see, for example, [3, p. 130, 160]). Also, in works [4] and [5] the results on proximity of logarithmic derivatives of f and g played an important role in the study of the asymptotic behavior of the logarithmic derivative of entire functions of c.r.g.

In this paper we obtain similar results for $\ln f$ and $\ln g$ for entire functions of zero order.

2. Definitions and main of result. Let L be the set of all growth functions υ such that $r\upsilon'(r)/\upsilon(r) \to 0$ as $r \to +\infty$ where a growth function $\upsilon\colon [0;+\infty) \to \mathbb{R}_+$ is a continuously differentiable increasing to $+\infty$ function. It is clear that the set L coincides up to equivalent functions with a set of slow by growing functions in the sense

---

2010 Mathematics Subject Classification: 30E99 of Karamata. For $\upsilon \in L$ we denote by $H_0(\upsilon)$ the class of entire functions of zero order that satisfy the condition

$$n(r) = O(\upsilon(r)),\ r \to +\infty,$$

where $n(r) = n(r, 0, f)$ is a counting function of zeros of a function f. By $\tilde{H}_0(\upsilon)$ denote the subclass of class $H_0(\upsilon)$ consisting of the functions such that

$$\lim_{r\to+\infty} \frac{n(r)}{\upsilon(r)} = \Delta > 0.$$

Let E be the set of circles $\{z : |z - z_j| < r_j\}$, $j \in \mathbb{N}$. The number

$$\rho^*(E) := \varlimsup_{r\to+\infty} \frac{1}{r} \sum_{|z_j|\leqslant r} r_j$$

is called the *upper linear density* of a set E. The family of sets E, $\rho^*(E) \leqslant \eta$, is denoted by C_η.

Let $\upsilon \in L$, $\delta > 0$, $f(z) = \prod_{n=1}^{+\infty} \left(1 - \frac{z}{a_n}\right) \in \tilde{H}_0(\upsilon)$, $g(z) = \prod_{n=1}^{+\infty} \left(1 - \frac{z}{a'_n}\right)$, where $|a'_n| = |a_n|$ and $|\arg a'_n - \arg a_n| < \delta$.

Let $D = \mathbb{C} \setminus \bigcup_{n=1}^{+\infty} \{z : |z| \geqslant |a^*_n|,\ \arg z = \arg a^*_n\}$, where (a^*_n) is the union of the sequences (a_n) and (a'_n). By $\ln f$ and $\ln g$ denote the single-valued in the domain D branches of multi-valued functions $\mathrm{Ln}\ f$ and $\mathrm{Ln}\ g$, respectively, such that $\ln f(0) = 0$, $\ln g(0) = 0$. From now on we make the notation $z = re^{i\varphi}$.

Theorem 1. *Let $\upsilon \in L$, $\varepsilon > 0$ and $\eta > 0$. Then there exists $\delta > 0$ such that for every functions $f, g \in H_0(\upsilon)$ defined as above we have*

i) $|\ln|f(z)| - \ln|g(z)|| < \varepsilon\, \upsilon(r)$, $z \notin E$, *where* $E \in C_\eta$ *;*

ii) $|\arg f(z) - \arg g(z)| < \varepsilon\, \upsilon(r)$, $z \in D$, $r \geqslant r_0$.

Proof. Let $\varepsilon > 0$ and $\eta \in (0; 1)$ are given arbitrary numbers,

$$n(r) = n(r, 0, f) = n(r, 0, g) = \Delta\upsilon(r) + o(\upsilon(r)),$$

$r \to +\infty$,

$$\xi(r) = \sup_{t\geqslant r} \frac{n(2t) - n(t/2)}{\upsilon(t)} \searrow 0 \ \text{ as } \ r \to +\infty.$$

Note that

$$\int_{2r}^{+\infty} \frac{\upsilon(t)}{t^2}\, dt \sim \frac{\upsilon(2r)}{2r},\ r \to +\infty.$$

Choose $0 < \delta < \frac{1}{4}$, $r_0 > 0$ so that

$$r \int_{2r}^{+\infty} \frac{\upsilon(t)}{t^2}\, dt < \upsilon(2r), \quad \upsilon(2r) < 2\upsilon(r), \quad n(r) < 2\Delta\upsilon(r), \quad r \geqslant r_0,$$

$$5\pi\xi(r_0)\ln(432e/\eta) < \frac{\varepsilon}{3}, \quad 16\Delta\delta < \frac{\varepsilon}{3}.$$

Let us represent $f \in \tilde{H}_0(\upsilon)$ in the form of the following three functions:

$$f_1(z) = \prod_{|a_n|\leqslant r/2} \left(1 - \frac{z}{a_n}\right),$$

$$f_2(z) = \prod_{|a_n|>2r} \left(1 - \frac{z}{a_n}\right),$$

$$f_3(z) = \prod_{r/2<|a_n|\leqslant 2r} \left(1 - \frac{z}{a_n}\right).$$

In the same way the designate the functions g_1, g_2, g_3 that correspond to the function g. Then

$$|\ln|f(z)| - \ln|g(z)|| \leqslant |\ln f(z) - \ln g(z)| \leqslant |\ln f_1(z) - \ln g_1(z)| +$$

$$+ |\ln f_2(z) - \ln g_2(z)| + |\ln|f_3(z)| - \ln|g_3(z)|| = \tag{1}$$

$$= \sum\nolimits_{1/2} + \sum\nolimits_2 + |\ln|f_3(z)|| + |\ln|g_3(z)||,$$

where

$$\sum\nolimits_{1/2} = \left|\sum_{|a_n|\leqslant r/2} \left(\ln\left(1-\frac{z}{a_n}\right) - \ln\left(1-\frac{z}{a'_n}\right)\right)\right| = \left|\sum_{|a_n|\leqslant r/2} \ln\left(1 - \frac{(a'_n-a_n)z}{a_n(a'_n-z)}\right)\right|,$$

$$\sum\nolimits_2 = |\ln f_2(z) - \ln g_2(z)| = \left|\sum_{|a_n|>2r} \ln\left(1 - \frac{(a'_n-a_n)z}{a_n(a'_n-z)}\right)\right|.$$

Similarly,

$$|\arg f(z) - \arg g(z)| \leqslant |\ln f(z) - \ln g(z)| \leqslant \sum\nolimits_{1/2} + \sum\nolimits_2 + |\arg f_3(z) - \arg g_3(z)|. \tag{2}$$

For $|a_n| \leqslant r/2$ and $|a_n| \geqslant 2r$, $|\arg a'_n - \arg a_n| = |\alpha_n| < \delta$ we have

$$\left|\frac{(a'_n-a_n)z}{a_n(a'_n-z)}\right| \leqslant \frac{r|1-e^{i\alpha_n}|}{||a_n|-r|} \leqslant \frac{r\delta}{||a_n|-r|} \leqslant 2\delta < \frac{1}{2}$$

Using the inequality (see [3, p. 87])

$$|\ln(1-u)| \leqslant 2|u| \text{ for } |u| \leqslant \frac{1}{2},$$

we get

$$|\ln f_1(z) - \ln g_1(z)| \leqslant \sum_{|a_n|\leqslant r/2} \left|\ln\left(1-\frac{z}{a_n}\right) - \ln\left(1-\frac{z}{a'_n}\right)\right| =$$

$$= \sum_{|a_n|\leqslant r/2} \left|\ln\left(1 - \frac{(a'_n-a_n)z}{a_n(a'_n-z)}\right)\right| \leqslant 2\sum_{|a_n|\leqslant r/2} \left|\frac{(a'_n-a_n)z}{a_n(a'_n-z)}\right| \leqslant \tag{3}$$

$$\leqslant 2\delta\, n(r/2) < 4\delta\Delta\upsilon(r) < \frac{\varepsilon}{3}\upsilon(r),\ r \geqslant r_0.$$

Since

$$\int_{2r}^{+\infty} \frac{dn(t)}{t} \leqslant \int_{2r}^{+\infty} \frac{n(t)}{t^2}\,dt \leqslant 2\Delta \int_{2r}^{+\infty} \frac{\upsilon(t)}{t^2}\,dt \leqslant 2\Delta \frac{\upsilon(2r)}{r} = 4\Delta \frac{\upsilon(2r)}{r}, \qquad r \geqslant r_0,$$

we have that

$$|\ln f_2(z) - \ln g_2(z)| \leqslant \sum_{|a_n|>2r} \left| \ln\left(1 - \frac{(a_n' - a_n)z}{a_n(a_n' - z)}\right)\right| \leqslant 2 \sum_{|a_n|>2r} \left| \frac{(a_n' - a_n)z}{a_n(a_n' - z)}\right| \leqslant$$

$$\leqslant 2\delta r \sum_{|a_n|\geqslant 2r} \frac{1}{|a_n| - r} = 2\delta r \sum_{|a_n|\geqslant 2r} \frac{1}{|a_n|\left(1 - \dfrac{r}{|a_n|}\right)} \leqslant 4\delta r \int_{2r}^{+\infty} \frac{dn(t)}{t} \leqslant \tag{4}$$

$$\leqslant 16\delta\Delta\upsilon(r) < \frac{\varepsilon}{3}\upsilon(r), \qquad r \geqslant r_0.$$

Whereas $\left|\arg\left(1 - \dfrac{z}{a_n^*}\right)\right| < 2\pi$, then for $r \geqslant r_0$ we have

$$|\arg f_3(z)| + |\arg g_3(z)| \leqslant 4\pi\,(n(2r) - n(r/2)) \leqslant 4\pi\xi(r)\upsilon(r) \leqslant 4\pi\xi(r_0)\upsilon(r) < \frac{\varepsilon}{3}\,\upsilon(r). \tag{5}$$

Taking into account (3)–(5) from (2) we obtain

$$|\arg f(z) - \arg g(z)| < \varepsilon\upsilon(r), \ r \geqslant r_0$$

that proves statement *ii*) of Theorem 1.

Whereas $\left|1 - \dfrac{z}{a_n}\right| \leqslant 1 + \left|\dfrac{z}{a_n}\right| \leqslant 3$ for $|a_n| \geqslant \dfrac{r}{2}$ then

$$\begin{aligned} &\ln|f_3(z)| + \ln|g_3(z)| \leqslant 2\ln 3\,(n(2r) - n(r/2)) \leqslant \\ &\leqslant 4\xi(r)\upsilon(r) \leqslant 4\xi(r_0)\upsilon(r) < \frac{\varepsilon}{6}\,\upsilon(r), \ r \geqslant r_0 \end{aligned} \tag{6}$$

In order to evaluate $\ln|f_3(z)|$ from below, let us consider the function $(R > 0)$

$$\psi(z; R) = \prod_{R/2<|a_n|\leqslant 2R} \left(1 - \frac{z}{a_n}\right)$$

and, according to the Cartan estimation from [3, p. 31], for an arbitrary $H > 0$ the polynomial

$$P(z) = \prod_{R/2<|a_n|\leqslant 2R} (z - a_n)$$

satisfies the inequality

$$\ln|P(z)| > n(R/2; 2R)\ln\frac{H}{e}$$

outside the circles with the total sum of radii $2H$, where $n(R/2; 2R) = n(2R) - n(R/2)$.

Choose $H = \dfrac{\eta}{24}R$ then

$$\ln|\psi(z;R)| = \ln|P(z)| - \sum_{R/2<|a_n|<2R} \ln|a_n| \geqslant$$
$$\geqslant \ln|P(z)| - n(R/2;2R)\ln 2R > n(R/2;2R)\ln\frac{\eta}{48e}$$

outside the exceptional circles such that the sum of their radii equals $\eta R/12$.

Now we construct an exceptional set E_1 in the entire plane. Let

$$R_j = 2^{2j},$$
$$K_j = K(R_j/2;2R_j) = \{t\colon R_j/2 < |t| \leqslant 2R_j\},$$
$$H_j = \frac{\eta}{24}R_j,$$
$$n_j = n(2R_j) - n(R_j/2).$$

for $j = 0,1,2,\ldots$. It is obvious that $\mathbb{C} = \left\{z : |z| \leqslant \dfrac{1}{2}\right\} \cup \left(\bigcup_{j=0}^{+\infty} K_j\right)$. In each annulus K_j we can choose exceptional circles such that the sum of their radii equals $\eta R/12$ and outside these circles the inequality

$$\ln|\psi(z;R_j)| > n_j \ \ln\frac{\eta}{48e} \tag{7}$$

holds.

Since each exceptional circle contains a point a_n from the annulus K_j, the centers b_k of these circles are either in the annulus K_j or in the one of the adjacent annuli K_{j-1} and K_{j+1}. Denote by E_1 the set of all exceptional circles that correspond to different annuli K_j $(j = 0,1,2,\ldots)$ and denote by r_k $(k = 1,2,\ldots)$ the radii of these circles.

Let $R_m \leqslant r < R_{m+1}$. We estimate the upper linear density of the set E_1. We have

$$\sum_{|b_k|\leqslant r} r_k \leqslant \sum_{|b_k|\leqslant R_{m+1}} r_k = \sum_{j=0}^{m+1}\frac{\eta R_j}{12} = \frac{\eta}{12}\sum_{j=0}^{m+1}2^{2j} \leqslant \frac{\eta}{12}2^{2m+2}\cdot\frac{4}{3} \leqslant \frac{4\eta}{9}r,$$

therefore, $\rho^*(E_1) < \dfrac{4\eta}{9} < \dfrac{\eta}{2}$.

Inequality (7) gives a lower estimation of the function $|\psi(z;R_j)|$ in the annulus K_j outside the exceptional set E_1. Estimate $|\psi(z;r)| = |f_3(z)|$ outside this set for any r. For $R_m \leqslant r < R_{m+1}$ represent $\psi(z;r)$ in the form

$$f_3(z) = \psi(z;r) = \frac{\psi(z;R_m)\cdot\psi(z;R_{m+1})}{\psi_1(z;R_m/2,r/2)\cdot\psi_1(z;2r,2R_{m+1})}, \tag{8}$$

where $\psi_1(z;A,B) = \prod\limits_{A<|a_n|\leqslant B}\left(1-\dfrac{z}{a_n}\right)$.

Since, $|a_n| > \dfrac{R_m}{2} \geqslant \dfrac{r}{8}$, then $\left|1-\dfrac{z}{a_n}\right| \leqslant 1+\dfrac{r}{|a_n|} \leqslant 9$, and therefore,

$$\ln|\psi_1(z;R_m/2,r/2)| \leqslant n(R_m/2;R_{m+1}/2)\ \ln 9 = n(R_m/2;2R_m)\ \ln 9 = n_m\ln 9,$$

$$\ln|\psi_1(z;2r,2R_{m+1})| \leqslant n(2R_m;2R_{m+1})\,\ln 9 = n(R_{m+1}/2;2R_{m+1})\,\ln 9 = n_{m+1}\ln 9.$$

Using these inequalities and (7), (8), outside the set E_1 we obtain

$$\ln|f_3(z)| > (n_m+n_{m+1})\ln\frac{\eta}{48e} - (n_m+n_{m+1})\ln 9 \geqslant$$
$$\geqslant \left(\frac{n(2R_m)-n(R_m/2)}{\upsilon(R_m)} + \frac{n(2R_{m+1})-n(R_{m+1}/2)}{\upsilon(R_{m+1})}\cdot\frac{\upsilon(R_{m+1})}{\upsilon(r)}\right)\cdot\upsilon(r)\ln\frac{\eta}{432e} \geqslant$$
$$\geqslant (\xi(R_m)+4\xi(R_{m+1}))\cdot\upsilon(r)\ln\frac{\eta}{432e} \geqslant 5\xi(r_0)\upsilon(r)\ln\frac{\eta}{432e} > -\frac{\varepsilon}{6}\upsilon(r), \quad r\geqslant r_0,$$

and considering (6), the inequality

$$|\ln|f_3(z)|| < \frac{\varepsilon}{6}\,\upsilon(r),\ r\geqslant r_0, \tag{9}$$

holds outside the set E_1, $\rho^*(E_1) < \dfrac{\eta}{2}$.

Similarly we get that

$$|\ln|g_3(z)|| < \frac{\varepsilon}{6}\,\upsilon(r),\ r\geqslant r_0, \tag{10}$$

outside the set E_1^δ, $\rho^*(E_1^\delta) < \dfrac{\eta}{2}$.

Taking into account (3), (4), (9), (10) from (1) we obtain

$$|\ln|f(z)| - \ln|g(z)|| < \varepsilon\,\upsilon(r),\ r\geqslant r_0,$$

outside the set $E = E_1 \bigcup E_1^\delta \bigcup \{z : |z|\leqslant r_0\}$, $\rho^*(E) < \eta$ and statement (i) of the Theorem 1 is proved. □

References

1. G. Valiron, *Sur les fonctions entieres d'ordre nul et d'ordre fini, et en particulier sur les fonctions a correspondance reguliere*, Ann. Fac. Sci. Univ. Toulouse, III. Ser. **5** (1913), 117–257.
2. E. C. Titchmarsh, *On integral functions with real negative zeros*, Proc. London Math. Soc. (2) **26** (1927), no. 2, 185–200.
3. B. Ya. Levin, *Distribution of the roots of the entire function*, GITTL, Moskov, 1956 (Russian).
4. A. A. Goldberg and N. E. Korenkov, *Asymptotic behavior of logarithmic derivative of entire function of completely regular growth*, Sib. Mat. Zh. **21** (1980), no. 3, 63–79 (Russian); **English version in**: Sib. Math. J. **21** (1980), no. 3, 363–375.
5. M. V. Zabolotskyj and M. R. Mostova, *Asymptotic behavior of the logarithmic derivative of entire functions of zero order*, Carpathian Math. Publ. **6** (2014), no. 2, 237–241 (Ukrainian).

Стаття: надійшла до редколегії 30.11.2017
доопрацьована 07.02.2018
прийнята до друку 24.04.2018

БЛИЗЬКІСТЬ ЦІЛИХ ФУНКЦІЙ НУЛЬОВОГО ПОРЯДКУ З v-ЩІЛЬНІСТЮ НУЛІВ

Микола ЗАБОЛОЦЬКИЙ, Юлія БАСЮК

Львівський національний університет імені Івана Франка,
вул. Університетська, 1, м. Львів, 79000
e-mail:m_zabol@franko.lviv.ua, yuliya.basyuk.92@gmail.com

Досліджено поводження різниці логарифмів цілих функцій нульового порядку, модулі нулів яких однакові і мають щільність стосовно повільно зростаючої функції v, а аргументи нулів відрізняються на мале число.

Ключові слова: ціла функція, щільність нулів, апроксимація.