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# BOUNDED $l$-INDEX AND INFINITE PRODUCTS OF INFINITE GENUS

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We established conditions on zeros of infinite product of infinite genus providing $l$-index boundedness of the product. Our investigation uses the known logarithmic criterion of $l$-index boundedness which describes the behavior of logarithmic derivative outside some exceptional set and uniform distribution of zeros in some sense. The main results are solutions of Skaskiv's problem on sufficient conditions of $l$-index boundedness for infinite products of infinite genus.

*Key words:* entire function, bounded $l$-index, infinite product, infinite genus, infinite order

**Introduction.** Let $l : \mathbb{C} \to \mathbb{R}_+$ be a fixed positive continuous function, where $\mathbb{R}_+ = (0, +\infty)$. An entire function $f$ is said to be of bounded $l-$index [20] if there exists an integer $m$, independent of $z$, such that for all $p$ and all $z \in \mathbb{C}$ $\frac{|f^{(p)}(z)|}{l^p(z)p!} \leqslant \max\{\frac{|f^{(s)}(z)|}{l^s(z)s!} : 0 \leqslant s \leqslant m\}$. The least such integer $m$ is called the $l$-index of $f$ and is denoted by $N(f;l)$. If $l(z) \equiv 1$ then the function $f$ is of bounded index [21]. Let $Q$ be the class of positive continuous functions $l$ on $[0, +\infty)$ such that

$$\lambda(r) = \sup\left\{\frac{l(t_1)}{l(t_2)} : |t_1 - t_2| < \frac{r}{\min\{l(t_1), l(t_2)\}}\right\}$$

is finite for all $r \geq 0$.

In particular, these functions have the following properties [6, 8, 24]: their upper growth estimates are sharp, they have uniform in some sense distribution of its zeros, etc.

Let us consider infinite products of infinite genus, i.e.

$$\pi(z) = \prod_{n=1}^{\infty}\left(1 - \left(\frac{z}{c_n}\right)^{m_n}\right), \tag{1}$$

*2010 Mathematics Subject Classification*: 30D05, 30D15 where $m_n \in \mathbb{N}$, $m_n \to +\infty$, $c_n \to \infty$, $c_n \in \mathbb{C}$ and for every $p \in \mathbb{N}$ $\sum\limits_{n=1}^{\infty} \frac{1}{|c_n|^p} = \infty$. Obviously, the function $\pi(z)$ is entire and has non-complete regular growth. G. H. Fricke, A. A. Goldberg, M. M. Sheremeta, I. E. Chyzhykov, M. T. Bordulyak, Yu. S. Trukhan [11–15,18,26–28] studied $l$-index boundedness of infinite products of finite genus $p$. They established necessary and sufficient conditions providing that an infinite product of finite order has bounded $l$-index for some continuous function $l : \mathbb{C} \to \mathbb{R}_+$. Furthermore, for every positive continuous function $l(|z|)$ such that $|z|l(|z|) \to +\infty$ as $|z| \to \infty$ they constructed an infinite product of bounded $l$-index. It leads us to the following problem

**Problem 1** (O. B. Skaskiv). *What is a continuous function $l : \mathbb{C} \to \mathbb{R}_+$ such that infinite product of form* (1) *has bounded l-index?*

Direct application of methods from [11–15,18,26–28] to product (1) is not impossible. In these papers, the authors estimated modulus of logarithmic derivative and split an infinite product into three groups. The second group contained one fraction $\frac{1}{z-c_n}$ for $|z - c_n| \geq q/l(|c_n|)$, $q > 0$. However, for infinite product (1) the corresponding second group consists of $\dfrac{m_n z^{m_n-1}}{(c_n)^{m_n} - z^{m_n}}$ for $|z - c_n e^{2\pi i k/m_n}| \geq q/l(|c_n|)$, $q > 0$, $k \in \{0, 1, \ldots, \mu_n\}$. Then an upper estimate of the fraction equals $m_n l(|c_n|)/q$. Since $m_n \to +\infty$ as $n \to \infty$, it means that $m_n l(|c_n|) \neq O(l(|c_n|)$. Thus, Problem 1 requires another sharper method of investigation.

Note that similar infinite products are considered in [10]. In particular, there is constructed an infinite product which is an entire function of completely regular growth, of order $\rho$, of unbounded $l_\rho$-index and its zeros do not satisfy known Levin's conditions (C) and (C′) ($l_\rho(r) = r^{\rho-1}$ for $r \geq 1$). More details concerning properties of entire functions of completely regular growth are in monographs [1, 17, 22, 23].

In multidimensional case sufficient conditions of index boundedness for infinite products are also known (see more results and open problems in [2–5]).

**Main result.** Let us denote $G_r(f) = \bigcup_n \left\{z : |z - c_n| < \frac{r}{l(|c_n|)}\right\}$ and $n(r, z, 1/f) = \sum_{|c_k - z| < r} 1$ be a counting zero function, where $(c_k)_{k\in\mathbb{N}}$ is a zero sequence of the function $f$, $z$ is a fixed point.

To find conditions by $c_n$ providing boundedness of $l$-index we need an auxiliary assertion:

**Proposition 1** ([24, 25]). *Suppose $l \in Q$. An entire function $f$ has bounded $l$-index if and only if:*

1) *for any $r > 0$ there exists $P = P(r) > 0$ such that $|f'(z)/f(z)| \leq Pl(z)$ for each $z \in \mathbb{C} \setminus G_r$;*
2) *for any $r > 0$ there exists $\tilde{n} = \tilde{n}(r) \in \mathbb{Z}_+$ such that $n(r/l(z), z, 1/f) \leq \tilde{n}$ for each $z \in \mathbb{C}$.*

Weaker sufficient conditions of $l$-index boundedness are obtained in [7, 9]. This criterion is very convenient to investigate infinite products [11–16, 18, 26–28]. Also it is applicable to differential equations [8, 19].

**Theorem 1.** *Let $l \in Q$ be an non-decreasing function and a positive sequence $(c_n)$ of infinite genus satisfy the following conditions:*

1) *for some* $q_0 > 0$ *and all* $n \geq 1$ $c_{n+1} - c_n > \dfrac{2q_0}{l(c_n)}$ *and* $l(c_{n+1}) = O(l(c_n))$, $n \to \infty$;
2) $c_n/m_n > q_1/l(c_n)$ *for all* $n \geq 1$ *and some* $q_1 > 0$;
3) $\displaystyle\sum_{k=1}^{n-1} \frac{m_k(c_n + c_k)^{m_k-1}}{(c_n + c_k)^{m_k} - (2c_k)^{m_k}} = O(l(c_n))$ *as* $n \to \infty$;
4) $\displaystyle\sum_{k=n+2}^{\infty} \frac{m_k(c_n + c_k)^{m_k-1}}{(2c_k)^{m_k} - (c_n + c_k)^{m_k}} = O(l(c_n))$ *as* $n \to \infty$;
5) $\displaystyle\sum_{k=1}^{n-1} \frac{m_k c_n^{m_k-1}}{c_n^{m_k} - c_k^{m_k}} = O(l(c_n))$ *as* $n \to \infty$;
6) $\displaystyle\sum_{k=n+2}^{\infty} \frac{m_k c_{n+1}^{m_k-1}}{c_k^{m_k} - c_{n+1}^{m_k}} = O(l(c_n))$ *as* $n \to \infty$.

*Then function* (1) *has bounded $l$-index.*

*Proof.* From the definition of the class $Q$ it follows that $l(r+q_0/l(r))/l(r) \leq \lambda(q_0)$, where $q_0$ is chosen in condition 1). Then $l(r + q_0/l(r)) \leq \lambda(q_0)l(r)$. Choose $q_2 < \frac{q_0}{\lambda(q_0)}$. Hence, we obtain that $l(r + q_0/l(r)) < \frac{q_0}{q_2} l(r)$.

Next, we prove that $n\left(\frac{q_3}{l(|z_0|)}, z_0, 1/f\right) \leq 1$ for all $|z| \geq r_0$, where $r_0$ is a sufficiently large radius and $q_3\lambda(q_3) < \min\left\{q_0, q_1\pi\sqrt{2}\right\}$.

On the contrary, assume that $n\left(\frac{q_3}{l(|z_0|)}, z_0, 1/f\right) \geq 2$. It generates two cases: either a) $|z_0 - c_n| \leq \frac{q_3}{l(|z_0|)}$ and $|z_0 - c_{n+1}| \leq \frac{q_3}{l(|z_0|)}$,
or b) $|z_0 - c_n e^{i2\pi j/m_n}| \leq \frac{q_3}{l(|z_0|)}$ and $|z_0 - c_n e^{i2\pi(j+1)/m_n}| \leq \frac{q_3}{l(|z_0|)}$.

In case a) the following inequalities hold:

$$|z_0| - \frac{q_3}{l(|z_0|)} \leq c_n \leq |z_0| + \frac{q_3}{l(|z_0|)}$$

and

$$|z_0| - \frac{q_3}{l(|z_0|)} \leq c_{n+1} \leq |z_0| + \frac{q_3}{l(|z_0|)}.$$

Hence, in view of condition 1) we have

$$\frac{2q_3}{l(|z_0|)} \geq c_{n+1} - c_n > \frac{2q_0}{l(c_n)} \geq \frac{2q_0}{l\left(|z_0| + \frac{q_3}{l(|z_0|)}\right)} \geq \frac{2q_0}{\lambda(q_3)l(|z_0|)} > \frac{2q_3}{l(|z_0|)}.$$

This is a contradiction.

Next, we consider case b). Therefore,

$$(2)\quad |c_n e^{i2\pi j/m_n} - c_n e^{i2\pi(j+1)/m_n}| = 2c_n \left|\cos\frac{\pi}{m_n}\sin\frac{\pi(2j+1)}{m_n} + i\sin\frac{\pi}{m_n}\cos\frac{\pi(2j+1)}{m_n}\right| =$$

$$= (1 + o(1))2\pi c_n \left|\frac{(2j+1)+i}{m_n}\right| \geq (1+o(1))2\pi\sqrt{2}\frac{c_n}{m_n} \geq 2\pi\sqrt{2}\frac{q_1}{l(c_n)} \geq$$

$$(3)\quad \geq \frac{2\pi\sqrt{2}q_1}{l(|z_0| + \frac{q_3}{l(|z_0|)})} \geq \frac{2\pi\sqrt{2}q_1}{\lambda(q_3)l(|z_0|)} \text{ as } n \to \infty.$$

Also it can be deduced that

$$|c_n e^{i2\pi j/m_n} - c_n e^{i2\pi (j+1)/m_n}| \leq |z_0 - c_n e^{i2\pi(j+1)/m_n}| + |c_n e^{i2\pi j/m_n} - z_0| \leq$$
$$\leq \frac{2q_3}{l(|z_0|)} < \frac{2\pi\sqrt{2}q_1}{\lambda(q_3)l(|z_0|)}.$$

But this contradicts (3). Thus, we prove $n(\frac{q_3}{l(|z_0|)}, z_0, 1/f) \leq 1$ for all $|z| \geq r_0$. It implies validity of condition 2) of Proposition 1 because in view of results from [7,9] it is sufficient to check validity of the inequality for some value of radius, but not for all.

Next, we prove $\forall k \neq n: |c_k - c_n| \geq \dfrac{2q_0}{l(c_n)}$. Let $k > n$. Then in view of condition 1)

$$|c_k - c_n| = c_k - c_{k-1} + c_{k-1} - c_{k-2} + \ldots + c_{n+1} - c_n >$$
$$> 2q_0\left(\frac{1}{l(c_{k-1})} + \frac{1}{l(c_{k-2})} + \ldots + \frac{1}{l(c_n)}\right) > \frac{2q_0}{l(c_n)}.$$

Similarly, for $k < n$ we have

$$|c_k - c_n| = c_n - c_{n-1} + c_{n-1} - c_{n-2} + \ldots + c_{k+1} - c_k >$$
$$> 2q_0\left(\frac{1}{l(c_{n-1})} + \frac{1}{l(c_{n-2})} + \ldots + \frac{1}{l(c_k)}\right) > \frac{2q_0}{l(c_{n-1})} > \frac{2q_0}{l(c_n)}.$$

In view of results from [7,9] it is sufficient to show that condition 1) of Proposition 1 holds for some $q \leq q_0$. Denote

$$A_n = \left\{z \in \mathbb{C}:\ ||z| - c_n| \leq \frac{q}{l(c_n)},\ |z - c_n e^{i2\pi j/m_n}| \geq \frac{q}{l(c_n)},\ j \in \{0, 1, 2, \ldots, m_n - 1\}\right\},$$
$$B_n = \left\{z \in \mathbb{C}:\ c_n + \frac{q}{l(c_n)} \leq |z| \leq c_{n+1} - \frac{q}{l(c_{n+1})}\right\}.$$

Obviously that $\mathbb{C} \setminus G_q(f) = \bigcup_{n=1}^{\infty} A_n \cup B_n$.

For the function $f(z)$ given in (1) the logarithmic derivative equals

$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty} -\frac{m_n z^{m_n - 1}}{c_n^{m_n} - z^{m_n}}.$$

Thus, for $z \in A_n$ we have

$$\left|\frac{f'(z)}{f(z)}\right| \leq \sum_{k=1}^{n-1} \frac{m_k |z|^{m_k - 1}}{|z|^{m_k} - c_k^{m_k}} + \frac{m_n |z|^{m_n - 1}}{|c_n^{m_n} - z^{m_n}|} + \sum_{k=n+1}^{\infty} \frac{m_k |z|^{m_k - 1}}{c_k^{m_k} - |z|^{m_k}} \leq$$

$$\leq \sum_{k=1}^{n-1} \frac{m_k (c_n - q/l(c_n))^{m_k - 1}}{(c_n - q/l(c_n))^{m_k} - c_k^{m_k}} + \sum_{k=n+1}^{\infty} \frac{m_k (c_n + q/l(c_n))^{m_k - 1}}{c_k^{m_k} - (c_n + q/l(c_n))^{m_k}} + \frac{m_n |z|^{m_n - 1}}{|c_n^{m_n} - z^{m_n}|} \leq$$

$$\leq \sum_{k=1}^{n-1} \frac{m_k (c_n - (c_n - c_k)/2)^{m_k - 1}}{(c_n - (c_n - c_k)/2)^{m_k} - c_k^{m_k}} + \sum_{k=n+1}^{\infty} \frac{m_k (c_n + (c_k - c_n)/2)^{m_k - 1}}{c_k^{m_k} - (c_n + (c_k - c_n)/2)^{m_k}} + \frac{m_n |z|^{m_n - 1}}{|c_n^{m_n} - z^{m_n}|} \leq$$

$$\leq \sum_{k=1}^{n-1} \frac{2m_k(c_n+c_k)^{m_k-1}}{(c_n+c_k)^{m_k}-(2c_k)^{m_k}} + \sum_{k=n+1}^{\infty} \frac{2m_k(c_n+c_k)^{m_k-1}}{(2c_k)^{m_k}-(c_n+c_k)^{m_k}} + \frac{m_n|z|^{m_n-1}}{|c_n^{m_n}-z^{m_n}|} =$$

$$= O(l(c_n)) + \frac{m_n|z|^{m_n-1}}{|c_n^{m_n}-z^{m_n}|} \tag{4}$$

because $w = \dfrac{t^{\alpha-1}}{t^\alpha - b}$ is a decreasing function and $w = \dfrac{t^{\alpha-1}}{b-t^\alpha}$ is an increasing function $(b \in \mathbb{R}_+,\ \alpha \geq 1)$.

By the maximum modulus principle, the maximum of $\dfrac{m_n|z|^{m_n-1}}{|c_n^{m_n}-z^{m_n}|}$ for $z \in A_n$ is attained on the boundary of $A_n$, that is for

$$z \in \partial A_n = \left\{z \in \mathbb{C}:\ |z| = c_n + \frac{q}{l(c_n)}\right\} \cup \left\{z \in \mathbb{C}:\ |z| = c_n + \frac{q}{l(c_n)}\right\} \cup$$
$$\cup \bigcup_{j=0}^{m_n-1} \left\{z \in \mathbb{C}:\ |z - c_n e^{i2\pi j/m_n}| = \frac{q}{l(c_n)}\right\}.$$

We consider three cases respectively.

Let $|z| = c_n + \dfrac{q}{l(c_n)}$. Taking into account condition 2) we deduce

$$\frac{m_n|z|^{m_n-1}}{|c_n^{m_n}-z^{m_n}|} = \frac{m_n(c_n+q/l(c_n))^{m_n-1}}{|c_n^{m_n}-(c_n+q/l(c_n))^{m_n}e^{i\theta m_n}|} \leq \frac{m_n(c_n+q/l(c_n))^{m_n-1}}{(c_n+q/l(c_n))^{m_n}-c_n^{m_n}} \leq$$
$$\leq \frac{m_n(c_n+q/l(c_n))^{m_n-1}}{m_n c_n^{m_n-1} q/l(c_n)} = \frac{l(c_n)}{q}\left(1+\frac{q}{c_n l(c_n)}\right)^{m_n-1} < \frac{l(c_n)}{q} e^{\frac{q m_n}{c_n l(c_n)}} <$$
$$< \frac{l(c_n)}{q} e^{q/q_1} = O(l(c_n))$$

as $n \to \infty$.

Let $|z| = c_n - \dfrac{q}{l(c_n)}$. It follows

$$\frac{m_n|z|^{m_n-1}}{|c_n^{m_n}-z^{m_n}|} = \frac{m_n(c_n-q/l(c_n))^{m_n-1}}{|c_n^{m_n}-(c_n-q/l(c_n))^{m_n}e^{i\theta m_n}|} \leq \frac{m_n(c_n-q/l(c_n))^{m_n-1}}{c_n^{m_n}-(c_n-q/l(c_n))^{m_n}} =$$
$$= \frac{m_n}{c_n(1-q/(c_n l(c_n)))((1-q/(c_n l(c_n))^{-m_n}-1)} < \frac{m_n}{c_n(1-q/(c_1 l(c_1)))(e^{q m_n/(c_n l(c_n))}-1)}.$$

Assume that $\dfrac{m_n}{c_n l(c_n)} \geq K \neq 0$. Then in view of condition 2) we have

$$\frac{m_n|z|^{m_n-1}}{|c_n^{m_n}-z^{m_n}|} < \frac{m_n}{c_n(1-q/(c_1 l(c_1)))(e^K-1)} < \frac{l(c_n)}{q_1(1-q/(c_1 l(c_1)))(e^K-1)} = O(l(c_n))$$

as $n \to \infty$. If $\varliminf\limits_{n\to\infty} \dfrac{m_n}{c_n l(c_n)} = 0$ then from $e^\alpha - 1 \sim \alpha$ as $\alpha \to 0$ we conclude

$$\frac{m_n|z|^{m_n-1}}{|c_n^{m_n}-z^{m_n}|} < \frac{m_n}{c_n(1-q/(c_1 l(c_1)))q m_n/(c_n l(c_n))} = \frac{l(c_n)}{(1-q/(c_1 l(c_1)))q} = O(l(c_n))$$

as $n \to \infty$.

Now, let $|z-c_ne^{i2\pi j/m_n}|=q/l(c_n)$ for some $j\in\{0,1,\ldots,m_n-1\}$. Hence, $z=c_ne^{i2\pi j/m_n}+q/l(c_n)e^{i\theta}$, $\theta\in[0,2\pi]$. Then

$$\frac{m_n|z|^{m_n-1}}{|c_n^{m_n}-z^{m_n}|}=\frac{m_n|c_ne^{i2\pi j/m_n}+q/l(c_n)e^{i\theta}|^{m_n-1}}{|c_n^{m_n}-(c_ne^{i2\pi j/m_n}+q/l(c_n)e^{i\theta})^{m_n}|}=$$
$$=\frac{m_n|1+q/(c_nl(c_n))e^{i(\theta-2\pi j/m_n)}|^{m_n-1}}{c_n|1-e^{i2\pi j}(1+q/(c_nl(c_n))e^{i(\theta-2\pi j/m_n)})^{m_n}|}=$$
$$=\frac{m_n}{c_n|1+q/(c_nl(c_n))e^{i(\theta-2\pi j/m_n)}||(1+q/(c_nl(c_n))e^{i(\theta-2\pi j/m_n)})^{-m_n}-1|}=$$
$$=\frac{(1+o(1))m_n}{c_n|1-(1+q/(c_nl(c_n)))^{-m_n}|}\le\frac{(1+o(1))m_n}{c_n|1-e^{-m_nq/(c_nl(c_n))}|}.$$

As above, the assumption $\dfrac{m_n}{c_nl(c_n)}\ge K\neq 0$ implies

$$\frac{m_n|z|^{m_n-1}}{|c_n^{m_n}-z^{m_n}|}\leqslant\frac{(1+o(1))m_n}{c_n(1-e^{-qK})}=\frac{(1+o(1))l(c_n)}{q_1(1-e^{-qK})}=O(l(c_n))$$

as $n\to\infty$. If $\varliminf\limits_{n\to\infty}\dfrac{m_n}{c_nl(c_n)}=0$ then we deduce

$$\frac{m_n|z|^{m_n-1}}{|c_n^{m_n}-z^{m_n}|}\le\frac{(1+o(1))m_n}{c_n|1-e^{-m_nq/(c_nl(c_n))}|}=\frac{(1+o(1))m_n}{c_nm_nq/(c_nl(c_n))}=O(l(c_n))$$

as $n\to\infty$. Since $l\in Q$ we have $l(c_n)\le\lambda(q)l(|z|)$ for $z\in A_n$. Therefore, from (4) it follows

$$\left|\frac{f'(z)}{f(z)}\right|=O(l(|z|))\text{ for }z\in A_n, n\to\infty.$$

Let $z\in B_n$. We obtain the following logarithmic derivative estimate

$$\left|\frac{f'(z)}{f(z)}\right|\le\sum_{k=1}^{n-1}\frac{m_k|z|^{m_k-1}}{|z|^{m_k}-c_k^{m_k}}+\frac{m_n|z|^{m_n-1}}{|z|^{m_n}-c_n^{m_n}}+\sum_{k=n+1}^{\infty}\frac{m_k|z|^{m_k-1}}{c_k^{m_k}-|z|^{m_k}}\le$$
$$\le\sum_{k=1}^{n-1}\frac{m_k(c_n+q/l(c_n))^{m_k-1}}{(c_n+q/l(c_n))^{m_k}-c_k^{m_k}}+\sum_{k=n+1}^{\infty}\frac{m_k(c_{n+1}-\frac{q}{l(c_{n+1})})^{m_k-1}}{c_k^{m_k}-(c_{n+1}-\frac{q}{l(c_{n+1})})^{m_k}}+\frac{m_n|z|^{m_n-1}}{|z|^{m_n}-c_n^{m_n}|}\le$$
$$\le\sum_{k=1}^{n-1}\frac{m_kc_n^{m_k-1}}{c_n^{m_k}-c_k^{m_k}}+\sum_{k=n+2}^{\infty}\frac{m_kc_{n+1}^{m_k-1}}{c_k^{m_k}-c_{n+1}^{m_k}}+\frac{m_{n+1}(c_{n+1}-\frac{q}{l(c_{n+1})})^{m_{n+1}-1}}{c_{n+1}^{m_{n+1}}-(c_{n+1}-\frac{q}{l(c_{n+1})})^{m_{n+1}}}+$$
$$+\frac{m_n(c_n+\frac{q}{l(c_n)})^{m_n-1}}{(c_n+\frac{q}{l(c_n)})^{m_n}-c_n^{m_n}}\le O(l(c_n))+O(l(c_{n+1}))+\frac{m_{n+1}(c_{n+1}-\frac{q}{l(c_{n+1})})^{m_{n+1}-1}}{c_{n+1}^{m_{n+1}}-(c_{n+1}-\frac{q}{l(c_{n+1})})^{m_{n+1}}}+$$
$$+\frac{m_n(c_n+\frac{q}{l(c_n)})^{m_n-1}}{(c_n+\frac{q}{l(c_n)})^{m_n}-c_n^{m_n}}. \tag{5}$$

As above, we will estimate every fraction

$$\frac{m_{n+1}\left(c_{n+1}-\frac{q}{l(c_{n+1})}\right)^{m_{n+1}-1}}{c_{n+1}^{m_{n+1}}-\left(c_{n+1}-\frac{q}{l(c_{n+1})}\right)^{m_{n+1}}}=\frac{m_{n+1}\left(1-\frac{q}{c_{n+1}l(c_{n+1})}\right)^{m_{n+1}-1}}{c_{n+1}\left(1-\left(1-\frac{q}{c_{n+1}l(c_{n+1})}\right)^{m_{n+1}}\right)}=$$

$$=\frac{m_{n+1}}{c_{n+1}\left(1-\frac{q}{c_{n+1}l(c_{n+1})}\right)\left(\left(1-\frac{q}{c_{n+1}l(c_{n+1})}\right)^{-m_{n+1}}-1\right)}=$$

$$=\frac{(1+o(1))m_{n+1}}{c_{n+1}\left(\left(1-\frac{q}{c_{n+1}l(c_{n+1})}\right)^{-m_{n+1}}-1\right)}=\frac{(1+o(1))m_{n+1}}{c_{n+1}\left(e^{\frac{m_{n+1}q}{c_{n+1}l(c_{n+1})}}-1\right)}.$$

If $\varliminf\limits_{n\to\infty}\frac{m_{n+1}q}{c_{n+1}l(c_{n+1})}=0$ then from $e^{\alpha}-1\sim\alpha$ $(\alpha\to 0)$ it follows

$$\frac{m_{n+1}}{c_{n+1}\left(e^{\frac{m_{n+1}q}{c_{n+1}l(c_{n+1})}}-1\right)}=\frac{(1+o(1))m_{n+1}}{c_{n+1}\frac{m_{n+1}q}{c_{n+1}l(c_{n+1})}}=(1+o(1))l(c_{n+1})/q\ (n\to\infty).$$

In other cases it can be proved that

$$\frac{m_{n+1}\left(c_{n+1}-\frac{q}{l(c_{n+1})}\right)^{m_{n+1}-1}}{c_{n+1}^{m_{n+1}}-\left(c_{n+1}-\frac{q}{l(c_{n+1})}\right)^{m_{n+1}}}=O(l(c_{n+1})).$$

By analogy,

$$\frac{m_n\left(c_n+\frac{q}{l(c_n)}\right)^{m_n-1}}{\left(c_n+\frac{q}{l(c_n)}\right)^{m_n}-c_n^{m_n}}=O(l(c_n))$$

as $n\to\infty$. Thus, (5) and condition 1) imply

$$\left|\frac{f'(z)}{f(z)}\right|=O(l(c_n))+O(l(c_{n+1}))=O(l(c_n))\le P(q)l(z)\text{ for }z\in B_n.$$

Hence, by Proposition 1 in view of results from [7,9], function (1) has bounded $l$-index. □

For example, if $c_n=\ln n$, $m_n=n$ then infinite product (1) has bounded $l$-index with $l(r)=\exp\{3r\}$. This follows from direct validation of conditions of Theorem 1.

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## ОБМЕЖЕНИЙ *l*-ІНДЕКС ТА НЕСКІНЧЕННІ ДОБУТКИ НЕСКІНЧЕННОГО РОДУ

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З'ясовано умови на нулі деякого нескінченного добутку нескінченного роду, які забезпечують обмеженість *l*-індексу. Доведення ґрунтується на використанні відомого логарифмічного критерію обмеженості *l*-індексу, який описує поводження логарифмічної похідної зовні деякої виняткової множини та рівномірний розподіл у деякому сенсі цих нулів. Основний результат — вирішення проблеми Скасківа про достатні умови обмеженості *l*-індексу для нескінченних добутків нескінченного роду.

*Ключові слова:* ціла функція, обмежений *l*-індекс, нескінченний добуток, нескінченний рід, нескінченний порядок.