ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2017. Випуск 84. С. 67–70
Visnyk of the Lviv Univ. Series Mech. Math. 2017. Issue 84. P. 67–70

УДК 515.12

ON COARSE EQUIVALENCE OF THE HYPERSPACES OF EUCLIDEAN SPACES

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We prove that the hyperspace and the hyperspace of continua of the euclidean space $\mathbb{R}^n$ are not coarsely equivalent. Also, we show that the hyperspace of continua in $\mathbb{R}^n$, $n \geq 2$, is not geodesic.

Key words: coarse equivalence, hyperspace, geodesic space.

1. Introduction. Coarse geometry deals with the properties "at infinity" of metric spaces and more general structures (coarse spaces), see, e.g., [2, 4].

There are two important categories in the coarse geometry of metric spaces. The Roe category has the proper metric spaces as its objects and the metrically proper coarse uniform maps as its morphisms. A metric space (X, d) is proper if every closed ball in X is compact. Two spaces X, Y are coarsely equivalent if there are morphisms $f\colon X \to Y$ and $g\colon Y \to X$ such that $f \circ g$ and $g \circ f$ are of finite distance to the identity maps 1_X and 1_Y respectively.

One of the most important general problems of coarse geometry is that of classification of metric spaces up to coarse equivalence.

Continuing the study of [5], we consider here the hyperspaces of the euclidean spaces. Let us start with some necessary definitions. For every metric space X we denote by $\exp X$ the set of all nonempty compact subsets of X (see, e.g., [1]). The metric d on X induces the Hausdorff metric d_H on $\exp X$:

$$d_H(A, B) = \inf\{\varepsilon > 0\colon A \subset O_\varepsilon(B), B \subset O_\varepsilon(A)\}.$$

For every $n \in \mathbb{N}$, by $\exp_n X$ we denote the subspace of $\exp X$ consisting of all sets of cardinality $\leqslant n$.

The hyperspace of all connected compact subsets of a space X is denoted by $\exp^c X$.

The hyperspace of compact convex subsets in the space $\mathbb{R}^n$ is denoted by $cc(\mathbb{R}^n)$.

Note that the hyperspaces of convex and connected subsets of the trees coincide.

2010 Mathematics Subject Classification: 54B20, 54B30, 54E40.

It was proved in [5] that the hyperspaces $\mathrm{cc}(\mathbb{R}^n)$ and $\exp\mathbb{R}^n$ are geodesic but not coarsely equivalent.

In the present note we will show that the hyperspace $\exp^c\mathbb{R}^n$ is not geodesic. Note that the property to be a geodesic space is an important ingredient of the proof in [5]. Nevertheless, we are able to prove hat the hyperspaces $\exp\mathbb{R}^n$ and $\exp^c\mathbb{R}^n$ are not coarsely equivalent.

2. Terminology and notation. We provide here some necessary definitions; see, e.g., [3] for details. A map $f\colon (X,d)\to(Y,\rho)$ is called asymptotically Lipschitz if there are $\lambda>0$ and $s\geqslant 0$, such that

$$\rho(f(x),f(y))\leqslant\lambda d(x,y)+s,\quad x,y\in X.$$

A map $f\colon (X,d)\to(Y,\rho)$ is a bi-Lipschitz embedding if there exists $\lambda>0$ such that

$$\frac{1}{\lambda}d(x,y)\leqslant\rho(f(x),f(y))\leqslant\lambda d(x,y),\quad x,y\in X.$$

A map $f\colon (X,d)\to(Y,\rho)$ is a coarse embedding if there exist non-decreasing functions φ_1, $\varphi_2\colon[0,\infty)\to[0,\infty)$ such that

$$\varphi_1(d(x,y))\leqslant\rho(f(x),f(y))\leqslant\varphi_2(d(x,y)),\quad x,y\in X.$$

A metric space (X,d) is geodesic if for every $x,y\in X$ there exists an isometric embedding $\alpha\colon[0,d(x,y)]\to X$ such that $\alpha(0)=x$, $\alpha(d(x,y))=y$. It is known (see [3]) that for any geodesic metric space X every coarsely uniform map of X is asymptotically Lipschitz.

Let $C\geqslant 1$, $r,R>0$. For a metric space X, let

$$\Phi_{C,r}(R)=\Phi^X_{C,r}(R)=\max\left\{|A|\colon A\subset\overline{B}_{Cr}(x)\text{ is }r\text{-discrete},\ \|x\|\leqslant R\right\}.$$

Let $D>0$. Recall that a subset A of a metric space X is called D-discrete if $d(x,y)\geqslant D$ for all $x,y\in X,\ x\neq y$.

3. Main results.

Theorem 1. *The hyperspace* $\exp^c\mathbb{R}^2$ *is not a geodesic space.*

Proof. Let $A,B\in\exp^c\mathbb{R}^2$ be defined as follows:

$$A=\left\{e^{i\varphi}\colon\frac{\pi}{8}\leqslant\varphi\leqslant\frac{\pi}{8}\right\},$$
$$B=\left\{\frac{3}{2}\,e^{i\varphi}\colon -\frac{7\pi}{8}\leqslant\varphi\leqslant\frac{7\pi}{8}\right\},$$

then

$$d_H(A,B)=\sqrt{\frac{13}{4}-3\cos\frac{\pi}{8}}=K.$$

If the hyperspace $\exp^c\mathbb{R}^2$ is geodesic then for all $0\leqslant R\leqslant K$ there exists $C\in\exp^c\mathbb{R}^2$ such that $d_H(A,C)=R$, $d_H(C,B)=K-R$.

Clearly, $C\subseteq\overline{O_R(A)}\bigcap\overline{O_{K-R}(B)}$.

Put $R=\frac{K}{2}$. The intersection $\overline{O_R(A)}\bigcap\overline{O_{K-R}(B)}$ consists of two connected components. Any connected subset $C\subseteq\overline{O_R(A)}\bigcap\overline{O_{K-R}(B)}$ is of distance greater than R from A, i.e. $d_H(A,C)>R$. $\square$

One can similarly prove that the hyperspace $\exp^c \mathbb{R}^n$ is not geodesic for any $n \geq 2$.

For every $d > 0$, let $X_d = \{A \subset \exp^c\ \mathbb{R}^n : \mathrm{diam} A \leqslant d\}$.

In the proof of the following lemma we use the notion of asymptotic dimension; see, e.g., [3] for the necessary definition and properties.

Lemma 1. *The hyperspace* $\exp_2 \mathbb{R}^n$ *does not admit an asymptotically Lipschitz embedding into* X_d *for all* $d > 0$.

Proof. Since the asymptotic dimension $\mathrm{asdim}(\exp_2 \mathbb{R}^n) = 2n$ is strictly larger than $\mathrm{asdim}(X_d) = n$, we see that such an embedding is impossible. □

Lemma 2. *For every* $r > 0,\ C_1 \geqslant 1$*, for every* $W \in \exp_2 \mathbb{R}^n$*, we have*

$$\Phi_{C_1, r}^{\exp_2 \mathbb{R}^n}(R) \leqslant 2^{2^{n+1} \cdot {C_1}^n}.$$

Proof. Since $W \in \exp_2 \mathbb{R}^n$, the neighborhood $O_{C_1 r}(W)$ coincides with the ball $C_1 r$. The cardinality of any r-discrete subset $A \in O_{C_1 r}(W)$ does not exceed $2 \cdot (2 \cdot C_1)^n = 2^{n+1} \cdot {C_1}^n$. In turn, this means that

$$\Phi_{C_1, r}^{\exp_2 \mathbb{R}^n}(R) \leqslant 2^{2^{n+1} \cdot {C_1}^n}.$$

□

Lemma 3. *Let* $C \geqslant 1$*. Then for every* $R < d$ *the inequality*

$$\Phi_{C, r}^{X_d}(R) \geqslant 2^{\frac{R}{r}}$$

holds.

Proof. For every set $W \in X_d$ with $\|W\| < R$ we choose a maximal r-discrete set $B \subset \partial \overline{O_r(W)}$. We see that $|B| > \frac{R}{r}$. Then the points of the set A are all possible unions of the form $W \cup B_k$, where B_k are the shortest segments connecting the points of B to W. Therefore, $A \in O_{C\, r}(W)$ is an r-discrete set. The cardinality of A is at least $2^{\frac{R}{r}}$. Thus,

$$\Phi_{C, r}^{X_d}(R) \geqslant 2^{\frac{R}{r}}.$$

□

Theorem 2. *The spaces* $\exp \mathbb{R}^n$ *and* $\exp^c \mathbb{R}^n$ *are not coarsely equivalent.*

Proof. Suppose that the spaces $\exp \mathbb{R}^n$ and $\exp^c \mathbb{R}^n$ are coarsely equivalent. Then there exist coarse embeddings $f\colon \exp \mathbb{R}^n \to \exp^c \mathbb{R}^n$ and $g\colon \exp^c \mathbb{R}^n \to \exp \mathbb{R}^n$ such that $f \circ g$ and $g \circ f$ are of finite distance from the identity maps $1_{\exp \mathbb{R}^n}$ and $1_{\exp^c \mathbb{R}^n}$ respectively.

By Lemma 1, for every $d > 0$ there exists $x = \{x_1, x_2\} \in \exp_2 \mathbb{R}^n \subset \exp \mathbb{R}^n$ such that

$$\mathrm{diam} f(x) > d.$$

Since f is a coarse embedding, there exist nondecreasing functions $\varphi_1, \varphi_2 \colon [0, \infty) \to [0, \infty)$ such that

$$\varphi_1(d(x, y)) \leqslant \rho(f(x), f(y)) \leqslant \varphi_2(d(x, y)), \quad x, y \in \exp \mathbb{R}^n.$$

Therefore, the image of an r-discrete set from $\exp \mathbb{R}^n$ is a $\varphi_1(r)$-discrete subset in $\exp^c \mathbb{R}^n$.

Using Lemmas 2, 3 and taking into account that for every $C_1 > 1$, $r > 0$ there exists R_0 such that $2^{2^{n+1} \cdot {C_1}^n} < 2^{\frac{R}{r}}$ for all $R > R_0$ we obtain a contradiction. □

A similar result to Theorem 2 can be proved also for the hyperbolic space $\mathbb{H}^n$.

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Стаття: надійшла до редколегії 25.04.2018
прийнята до друку 24.05.2018

ПРО ГРУБУ ЕКВІВАЛЕНТНІСТЬ ГІПЕРПРОСТОРІВ ЕВКЛІДОВИХ ПРОСТОРІВ

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Доведено, що гіперпростори континуумів і компактних опуклих підмножин евклідового простору $\mathbb{R}^n$ не є грубо еквівалентними. Також доведено, що гіперпростір континуумів в $\mathbb{R}^n$, $n \geq 2$, не є геодезійним простором.

Ключові слова: груба еквівалентність, гіперпростір, геодезійний простір.