ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2017. Випуск 84. С. 62–66
Visnyk of the Lviv Univ. Series Mech. Math. 2017. Issue 84. P. 62–66

УДК 512.624.5

PERIODIC WORDS CONNECTED WITH THE LUCAS NUMBERS

Galyna BARABASH, Yaroslav KHOLYAVKA, Iryna TYTAR

Ivan Franko National University of Lviv,
Universytetska Str., 1, Lviv, 79000
e-mails: galynabarabash71@gmail.com,
ya_khol@franko.lviv.ua,
iratytar1217@gmail.com

We introduce periodic words that are connected with the Lucas numbers and investigated their properties.

Key words: Lucas numbers, Lucas words, Fibonacci numbers, Fibonacci words.

1. Introduction. The *Fibonacci numbers* F_n are defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, for any integer $n > 1$, and with initial values $F_0 = 0$ and $F_1 = 1$. Different kinds of the Fibonacci sequence and their properties have been presented in the literature, see, e.g., [1, 4, 7]. Similarly to the Fibonacci numbers, the *Lucas numbers* L_n are defined by the recurrence relation $L_n = L_{n-1} + L_{n-2}$, for any integer $n > 1$, and with initial values $L_0 = 2$ and $L_1 = 1$.

The sequence $L_n \pmod m$ is periodic and repeats by returning to its starting values because there are only a finite number m^2 of pairs of terms possible, and the recurrence of a pair results in recurrence of all following terms.

In analogy to the definition of the infinite Fibonacci word [2, 6], one defines the Lucas words as the contatenation of the two previous terms $l_n = l_{n-1}l_{n-2}$, $n > 1$, with initial values $l_0 = 10$ and $l_1 = 1$ and defines the infinite Lucas word l, $l = \lim l_n$.

Using Lucas words, in the present article we shall introduce some new kinds of infinite words, namely LLP-words, and investigate some of their properties.

For any notations not explicitly defined in this article we refer to [3, 4, 5].

2. Lucas sequence modulo m**.** The letter p, $p > 2$, is reserved to denote a prime, m may be arbitrary integer, $m > 2$.

Let for any integer $n \geqslant 0$, $L_n(m)$ denote the n-th member of the sequence of integers $L_n \pmod m$. We reduce L_n modulo m by taking the least nonnegative residues, and let $k(m)$ denote the length of the period of the repeating sequence $L_n(m)$.

2010 Mathematics Subject Classification: 08A50, 11B39, 11B83

The problem of determining the length of the period of the recurring sequence arose in connection with a method for generating random numbers. A few properties of the function $k(m)$ are in the following theorem [9].

Theorem 1. *For all m the following hold:*
1) Any sequence $L_n(m)$ is periodic.
2) If m has prime factorization $m = \prod_{i=1}^{n} p_i^{e_i}$, then $k(m) = \operatorname{lcm}(k(p_1^{e_1}), \ldots, k(p_n^{e_n}))$.

Theorem 2. *If $m > 2$, then $k(m)$ is an even number.*

Proof. We find:
$$\begin{aligned} L_{k(m)}(m) &= L_0(m) = 2,\\ L_{k(m)-1}(m) &= L_{-1}(m) = m-1 = -L_1(m),\\ L_{k(m)-2}(m) &= L_{k(m)}(m) - L_{k(m)-1}(m) = L_0(m) + L_1(m) = L_2(m). \end{aligned}$$
Let for each t, t_0, $0 \leqslant t \leqslant t_0 - 1 \leqslant k(m) - 1$, we have $L_{k(m)-t}(m) = (-1)^t L_t(m)$. By using the fact that
$$L_{t+1}(m) = L_t(m) + L_{t-1}(m) \pmod{m}$$
for each $t \in \mathbb{N}$, the identity above can be verified by direct calculation for $t = t_0$:
$$\begin{aligned} L_{t_0}(m) &= L_{k(m)-t_0+2}(m) - L_{k(m)-t_0+1}(m) =\\ &= L_{k(m)-(t_0-2)}(m) - L_{k(m)-(t_0-1)}(m) =\\ &= (-1)^{t_0-2} L_{t_0-2}(m) - (-1)^{t_0-1} L_{t_0-1}(m) =\\ &= (-1)^{t_0}(L_{t_0-2}(m) + L_{t_0-1}(m)) =\\ &= (-1)^{t_0} L_{t_0}(m). \end{aligned}$$
If $t = k(m)$, then
$$L_0(m) = (-1)^{k(m)} L_{k(m)}(m), \qquad 2 = (-1)^{k(m)} 2.$$
Suppose that $k(m)$ is odd, then $m = 2$, $k(2) = 3$, or $m = 4$, $k(4) = 6$. For $m > 2$ $k(m)$ is even. □

3. Lucas words.

Let $l_0 = 10$ and $l_1 = 1$. Now $l_n = l_{n-1} l_{n-2}$, $n > 1$, the contatenation of the two previous terms. The successive initial finite Lucas words are:

(1) $\quad l_0 = 10, \quad l_1 = 1, \quad l_2 = 110, \quad l_3 = 1101, \quad l_4 = 1101110 \quad l_5 = 11011101101, \ldots$

The infinite Lucas word l is the limit $l = \lim l_n$. It is referenced A230603 in the On-line Encyclopedia of Integer Sequences [8]. The combinatorial properties of the Fibonacci (A003849 [8]) and Lucas infinite words are of great interest in some aspects of mathematics and physics, such as number theory, fractal geometry, cryptography, formal language, computational complexity, quasicrystals etc. See [5].

As usual we denote by $|l_n|$ the length (the number of symbols) of l_n (see [5]). The following proposition summarizes basic properties of Lucas words [5, 6].

Theorem 3. *The infinite Lucas word and the finite Lucas words satisfy the following properties:*
1) The words 1111 and 00 are not subwords of the infinite Lucas word.

2) For all $n > 1$ let ab be the last two symbols of $l_n, n > 1$, then we have $ab = 10$ if n is even and $ab = 01$ if n is odd.
3) For all n $|l_n| = L_n$.

4. Periodic LLP-words. Let us start with the classical definition of periodicity on words over arbitrary alphabet $\{a_0, a_1, a_2, \dots\}$ (see [3]).

Definition 1. Let $w = a_0a_1a_2\dots$ be an infinite word. We say that w is

1) a *periodic word* if there exists a positive integer t such that $a_i = a_{i+t}$ for all $i \geqslant 0$. The smallest t satisfying previous conditions is called the period of w;
2) an *eventually periodic word* if there exist two positive integers k, p such that $a_i = a_{i+p}$, for all $i > k$;
3) an *aperiodic word* if it is not eventually periodic.

Hypothesis. The infinite Lucas word is aperiodic.

We consider finite Lucas words l_n (1) as numbers written in the binary system and denote them by b_n. Denote by d_n the value of the number b_n in usual decimal numeration system. We write $b_n = d_n$ meaning that b_n and d_n are writings of the same number in different numeration systems.

Example 1.

$$b_0 = 10,\ b_1 = 1,\ b_2 = 110,\ b_3 = 1101,\ b_4 = 1101110,\ b_5 = 11011101101,\ \dots, \tag{2}$$

$$d_0 = 2,\ d_1 = 1,\ d_2 = 6,\ d_3 = 13,\ d_4 = 110,\ d_5 = 1773,\ \dots. \tag{3}$$

Theorem 4. *For any integer n, $n > 1$, we have*

$$d_n = d_{n-1}2^{L_{n-2}} + d_{n-2} \tag{4}$$

with $d_0 = 2$ and $d_1 = 1$.

Proof. One can easily verify (4) for the first few n:

$$\begin{aligned} d_2 &= 6 = 1 \cdot 2^2 + 2 = d_1 2^{L_0} + d_0, \\ d_3 &= 13 = 6 \cdot 2^1 + 1 = d_2 2^{L_1} + d_1, \\ d_4 &= 110 = 13 \cdot 2^3 + 6 = d_3 2^{L_2} + d_2. \end{aligned}$$

Statement (4) follows from Theorem 3 (statement 3) and the equality

$$d_n = b_n = b_{n-1}\underbrace{0\dots0}_{L_{n-2}}+b_{n-2} = d_{n-1}2^{L_{n-2}} + d_{n-2}.$$

□

Let $d_0(m) = 2$, $l_0(m) = 10$ and for arbitrary n, $n \geqslant 1$, $d_n(m) = d_n \pmod m$, $b_n(m) = d_n(m)$ in binary numeration system and $l_n(m) = l_{n-1}(m)b_n(m)$. Denote by $l(m)$ the limit $l(m) = \lim_{n\to\infty} l_n(m)$.

Example 2.

$$m=3;\quad d_0=2,\ d_1=1,\ d_2=6,\ d_3=13,\ d_4=110,\ d_5=1773,\ \dots;$$

$$d_0(3)=2,\ d_1(3)=1,\ d_2(3)=0,\ d_3(3)=1,\ d_4(3)=2,\ d_5(3)=0,\ \dots;$$

$$b_0(3)=10,\ b_1(3)=1,\ b_2(3)=0,\ b_3(3)=1,\ b_4(3)=10,\ b_5(3)=0,\ \dots;$$

$$l_0(3){=}10, l_1(3){=}101, l_2(3){=}1010, l_3(3){=}10101, l_4(3){=}1010110, l_5(3){=}10101100, \dots .$$

Definition 2. We say that
1) $l_n(m)$ is a *finite LLP-word type* 1 modulo m;
2) $l(m)$ is a *infinite LLP-word type* 1 modulo m.

Theorem 5. *The word $l(p)$ is periodic.*

Proof. The statement follows from (4) and Theorem 1 because there are only a finite number of $d_n \pmod p$ and $2^{L_{n-2}} \pmod p$ possible, and the recurrence of the first few terms sequence $d_n \pmod p$ gives recurrence of all subsequent terms. □

Using Lucas words (1) we define a periodic LLP-word $l^*(m)$ (infinite LLP-word type 2 by modulo m). As usual we denote by ϵ the empty word [5].

First we define words $w_n^*(m)$. Let $w_n^*(m)$ be the last $L_n(m)$ symbols of the word l_n. If $L_n(m)=0$ for some n, then $w_n^*(m)=\epsilon$. The word length $|w_n^*(m)|$ coincides with $L_n(m)$. Since $L_n(m)$ is a periodic sequence with period $k(m)$, the sequence $|w_n^*(m)|$ is periodic with the same period.

Theorem 6. *The word $w_n^*(m)$ coincides with the word $w_{n+k(m)}^*(m)$.*

Proof. Since $l_n = l_{n-1}l_{n-2}$, $n>1$, the last L_{n-2} symbols of the word l_n coincide with the word l_{n-2}, and therefore the last L_n elements of the word l_{n+2r} coincide with the word l_{n-2} for any natural number r. The period $k(m)$ is an even number (Theorem 2), so the last $L_n^*(m)$ elements of the word l_n coincide with the last $L_n^*(m)$ elements of the word $l_{n+k(m)}$. □

Let $l_0^*(m)=10$ and for arbitrary integer n, $n\geqslant 1$, $l_n^*(m)=l_{n-1}^*(m)w_n^*(m)$. Denote by $l^*(m)$ the limit $l^*(m)=\lim_{n\to\infty} l_n^*(m)$.

Example 3.

$$l_0=10,\quad l_1=1,\quad l_2=110,\quad l_3=1101,\quad l_4=1101110\quad l_5=11011101101,\dots$$

$$m=3;\quad L_0(3)=2,\ L_1(3)=1,\ L_2(3)=0,\ L_3(3)=1,\ L_4(3)=1,\ L_5(3)=2,\ \dots;$$

$$w_0^*(3)=10,\ w_1^*(3)=1,\ w_2^*(3)=\epsilon,\ w_3^*(3)=1,\ w_4^*(3)=0,\ w_5^*(3)=01,\ \dots;$$

$$l_0^*(3)=10, l_1^*(3)=101, l_2^*(3)=101, l_3^*(3)=1011, l_4^*(3)=10110, l_5^*(3)=1011001, \dots .$$

Definition 3. We say that
1) $l_n^*(m)$ is a *finite LLP-word of type* 2 modulo m;
2) $l^*(m)$ is an *infinite LLP-word of type* 2 by modulo m.

Theorem 7. *The word $l^*(m)$ is a periodic word and has period $L_0(m)+\ldots+L_{k(m)-1}$.*

Proof. The proof is a directly corollary of Theorem 6. □

References

1. K. T. Atanassov, V. Atanassova, A. G. Shannon, and J. C. Turner, *New visual perspectives on Fibonacci numbers*, World Scientific, London, 2002.
2. J. Berstel, *Fibonacci words – a survey*, In: *The book of L*, G. Rosenberg, A. Salomaa (Eds.), Springer, Berlin, 1986, pp. 11–26.
3. J. P. Duval, F. Mignosi, and A. Restivo, *Recurrence and periodicity in infinite words from local periods*, Theor. Comput. Sci. **262** (2001), no. 1–2, 269–284.
4. T. Koshy, *Fibonacci and Lucas numbers with applications*, Wiley-Interscience, New York, 2001.
5. M. Lothaire, *Algebraic combinatorics on words*, Cambridge Univ. Press, Cambridge, 2002.
6. G. Pirillo, *Fibonacci numbers and words*, Discrete Math. **173** (1997), no. 1–3, 197–207.
7. J. L. Rami'rez, G. N. Rubiano, and R. de Castro, *A generalization of the Fibonacci word fractal and the Fibonacci snowflake*, Theor. Comput. Sci. **528** (2014), 40–56.
8. N. J. A. Sloane, *The online encyclopedia of integer sequences*, Published electronically at https://oeis.org
9. D. D. Wall, *Fibonacci series modulo m*, Am. Math. Mon. **67** (1960), no. 6, 525–532.

Стаття: надійшла до редколегії 10.04.2018
прийнята до друку 15.05.2018

ПЕРІОДИЧНІ СЛОВА, ПОВ'ЯЗАНІ З ЧИСЛАМИ ЛЮКА

Галина БАРАБАШ, Ярослав ХОЛЯВКА, Ірина ТИТАР

Львівський національний університет імені Івана Франка,
вул. Університетська, 1, Львів, 79000
e-mails: galynabarabash71@gmail.com,
ya_khol@franko.lviv.ua,
iratytar1217@gmail.com

Означено періодичні слова, які пов'язані з числами Люка. Досліджуємо їхні властивості.

Ключові слова: числа Люка, слова Люка, числа Фібоначчі, слова Фібоначчі.