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# FREE ABELIAN DIBANDS

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We prove that varieties of abelian dibands and $(ln, rn)$-dibands coincide, and consider some properties of free abelian dibands.

*Key words:* dimonoid, abelian diband, free abelian diband, semigroup

**1. Introduction.** As is well-known the notion of a dimonoid was introduced by J.-L. Loday in [1]. Recall that a nonempty set $D$ with two binary associative operations $\dashv$ and $\vdash$ is called a *dimonoid* if for all $x, y, z \in D$ the following conditions hold:

$(D_1)$ $\quad (x\dashv y)\dashv z = x\dashv(y\vdash z)$,

$(D_2)$ $\quad (x\vdash y)\dashv z = x\vdash(y\dashv z)$,

$(D_3)$ $\quad (x\dashv y)\vdash z = x\vdash(y\vdash z)$.

It is not hard to see that a dimonoid becomes a semigroup if the two dimonoid operations coincide. Dimonoids and in particular dialgebras play a prominent role in the theory of Leibniz algebras, these structures and related algebras have been studied by many authors (see e.g., [2, 3, 4, 5]).

J.-L. Loday [1] constructed a free associative dialgebra and a free dimonoid. Later on, free dimonoids and free commutative dimonoids were investigated in detail in [6] and [7], respectively. Free abelian dimonoids (this class does not coincide with the class of commutative dimonoids) were described in [8]. The structure of free normal dibands and other relatively free dimonoids was considered in [9, 10]. In this paper we study free abelian idempotent dimonoids.

The paper is organized as follows. In Section 2 we present necessary definitions and examples of abelian idempotent dimonoids. In Section 3 we give necessary and sufficient conditions under which a dimonoid is an abelian diband, and find a free abelian idempotent dimonoid. In addition, we consider some properties of free abelian dibands.

**2. Examples of abelian dibands.** A nonempty class $H$ of algebraic systems is a *variety* if the Cartesian product of any sequence of $H$-systems is a $H$-system, every subsystem of an arbitrary $H$-system is a $H$-system and any homomorphic image of an arbitrary $H$-system is a $H$-system [11].

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A dimonoid $(D, \dashv, \vdash)$ is called *abelian* [8] if for all $x, y \in D$,

$$x \dashv y = y \vdash x.$$

Recall that a *band* is a semigroup whose elements are idempotents. If for a dimonoid $(D, \dashv, \vdash)$ the semigroups $(D, \dashv)$ and $(D, \vdash)$ are bands, then this dimonoid is called *idempotent* (or simply a *diband*).

The class of all abelian idempotent dimonoids satisfies the conditions of Birkhoff's theorem and therefore it is a variety. A dimonoid which is free in the variety of abelian dibands will be called a *free abelian idempotent dimonoid*.

Consider some examples of abelian dibands.

(i) It is obvious that a non-singleton left zero and right zero dimonoid $(D, \dashv, \vdash)$ i.e., $(D, \dashv)$ is a left zero semigroup and $(D, \vdash)$ is a right zero semigroup, is an abelian diband but not commutative [7].

(ii) Let $(S, \circ)$ be an arbitrary semigroup. A semigroup $(S, *)$, where $x * y = y \circ x$ for all $x, y \in S$, is called a *dual semigroup* to $(S, \circ)$.

A semigroup $(S, \circ)$ is called *left commutative* (respectively, *right commutative*) if it satisfies the identity $x \circ y \circ a = y \circ x \circ a$ (respectively, $a \circ x \circ y = a \circ y \circ x$).

**Proposition 1.** *Let $(S, \circ)$ be an arbitrary right commutative band and $(S, *)$ a dual semigroup to $(S, \circ)$. Then the algebra $(S, \circ, *)$ is an abelian diband.*

*Proof.* The proof follows from Proposition 3 of [8]. □

**Proposition 2.** *Let $(S, *)$ be an arbitrary left commutative band and $(S, \circ)$ a dual semigroup to $(S, *)$. Then the algebra $(S, \circ, *)$ is an abelian diband.*

*Proof.* The proof follows from Proposition 4 of [8]. □

(iii) An idempotent semigroup $S$ is called a *left regular band* if $aba = ab$ for all $a, b \in S$. If instead of the last identity, $aba = ba$ holds, then $S$ is a *right regular band*.

A dimonoid $(D, \dashv, \vdash)$ is called an $(lr, rr)$*-diband* [10] if $(D, \dashv)$ is a left regular band and $(D, \vdash)$ is a right regular band.

Let $X$ be a nonempty set and $FS(X)$ the free semilattice of all nonempty finite subsets of $X$ with respect to the operation of the set theoretical union. Define two binary operations $\dashv$ and $\vdash$ on the set $B_{lz,rz}(X) = \{(a, A) \in X \times FS(X) | a \in A\}$ as follows:

$$(x, A) \dashv (y, B) = (x, A \cup B),$$
$$(x, A) \vdash (y, B) = (y, A \cup B).$$

**Proposition 3.** [10] *The algebra $(B_{lz,rz}(X), \dashv, \vdash)$ is a free $(lr, rr)$-diband.*

It is obvious that the diband $(B_{lz,rz}(X), \dashv, \vdash)$ is abelian. We will denote this diband simply by $B_{lz,rz}(X)$.

Further we show that there are examples of abelian dimonoids which are not idempotent and, conversely, there are idempotent dimonoids which are not abelian.

(iv) For a nonempty set $X$, define two binary operations $\dashv$ and $\vdash$ on the direct product of $X$ and $FS(X)$ (see example (iii) above) by the rule:

$$(x, A) \dashv (y, B) = (x, A \cup \{y\} \cup B),$$
$$(x, A) \vdash (y, B) = (y, A \cup \{x\} \cup B).$$

**Proposition 4.** *The algebra* $(X \times FS(X), \dashv, \vdash)$*, where* $|X| \neq 1$*, is an abelian dimonoid but not idempotent one.*

*Proof.* The fact that $(X \times FS(X), \dashv, \vdash)$ is an abelian dimonoid is proved analogously as Proposition 2 of [8]. This dimonoid is not idempotent since

$$(x, A) \dashv (x, A) = (x, A \cup \{x\}) \neq (x, A),$$

when $x \notin A$. □

(v) Let $X_1, X_2, \ldots, X_n$ $(n \geqslant 3)$ be nonempty sets. Fix a natural number $\alpha$ such that $[0,5n] < \alpha < n$, where $[0,5n]$ is the integer part of 0,5$n$, and take arbitrary $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \prod_{i=1}^{n} X_i$. Define two binary operations $\dashv_\alpha$ and $\vdash_\alpha$ on $\prod_{i=1}^{n} X_i$ by

$$x \dashv_\alpha y = (x_1, \ldots, x_\alpha, y_{\alpha+1}, \ldots, y_n),$$

$$x \vdash_\alpha y = (x_1, \ldots, x_{n-\alpha}, y_{n-\alpha+1}, \ldots, y_n).$$

**Proposition 5.** *The algebraic system* $(\prod_{i=1}^{n} X_i, \dashv_\alpha, \vdash_\alpha)$ *is a diband.*

*Proof.* For all $x, y, z \in \prod_{i=1}^{n} X_i$ we have

$$\begin{aligned}(x \dashv_\alpha y) \dashv_\alpha z &= (x_1, \ldots, x_\alpha, y_{\alpha+1}, \ldots, y_n) \dashv_\alpha (z_1, z_2, \ldots, z_n) = \\ &= (x_1, \ldots, x_\alpha, z_{\alpha+1}, \ldots, z_n) = \\ &= (x_1, x_2, \ldots, x_n) \dashv_\alpha (y_1, \ldots, y_\alpha, z_{\alpha+1}, \ldots, z_n) = \\ &= x \dashv_\alpha (y \dashv_\alpha z),\end{aligned}$$

$$\begin{aligned}(x \vdash_\alpha y) \vdash_\alpha z &= (x_1, \ldots, x_{n-\alpha}, y_{n-\alpha+1}, \ldots, y_n) \vdash_\alpha (z_1, z_2, \ldots, z_n) = \\ &= (x_1, \ldots, x_{n-\alpha}, z_{n-\alpha+1}, \ldots, z_n) = \\ &= (x_1, x_2, \ldots, x_n) \vdash_\alpha (y_1, \ldots, y_{n-\alpha}, z_{n-\alpha+1}, \ldots, z_n) = \\ &= x \vdash_\alpha (y \vdash_\alpha z).\end{aligned}$$

Therefore, operations $\dashv_\alpha$ and $\vdash_\alpha$ are associative. Show that axioms $(D_1)$–$(D_3)$ hold:

$$\begin{aligned}(x \dashv_\alpha y) \dashv_\alpha z &= (x_1, \ldots, x_\alpha, z_{\alpha+1}, \ldots, z_n) = \\ &= (x_1, x_2, \ldots, x_n) \dashv_\alpha (y_1, \ldots, y_{n-\alpha}, z_{n-\alpha+1}, \ldots, z_n) = \\ &= x \dashv_\alpha (y \vdash_\alpha z),\end{aligned}$$

$$\begin{aligned}(x \vdash_\alpha y) \dashv_\alpha z &= (x_1, \ldots, x_{n-\alpha}, y_{n-\alpha+1}, \ldots, y_n) \dashv_\alpha (z_1, z_2, \ldots, z_n) = \\ &= (x_1, \ldots, x_{n-\alpha}, y_{n-\alpha+1}, \ldots, y_\alpha, z_{\alpha+1}, \ldots, z_n) = \\ &= (x_1, x_2, \ldots, x_n) \vdash_\alpha (y_1, \ldots, y_\alpha, z_{\alpha+1}, \ldots, z_n) = \\ &= x \vdash_\alpha (y \dashv_\alpha z),\end{aligned}$$

$$\begin{aligned}(x \dashv_\alpha y) \vdash_\alpha z &= (x_1, \ldots, x_\alpha, y_{\alpha+1}, \ldots, y_n) \vdash_\alpha (z_1, z_2, \ldots, z_n) = \\ &= (x_1, \ldots, x_{n-\alpha}, z_{n-\alpha+1}, \ldots, z_n) = \\ &= x \vdash_\alpha (y \vdash_\alpha z).\end{aligned}$$

So, $(\prod_{i=1}^{n} X_i, \dashv_\alpha, \vdash_\alpha)$ is a dimonoid, in addition, it is clear that this dimonoid is idempotent. □

We note that if for example $|X_1| > 1$, then $(\prod_{i=1}^{n} X_i, \dashv_\alpha, \vdash_\alpha)$ is not abelian. Indeed, for $x, y \in \prod_{i=1}^{n} X_i$ with distinct $x_1$ and $y_1$ we obtain

$$x \dashv_\alpha y = (x_1, ..., x_\alpha, y_{\alpha+1}, ..., y_n) \neq (y_1, ..., y_{n-\alpha}, x_{n-\alpha+1}, ..., x_n) = y \vdash_\alpha x.$$

**3. The free abelian diband.**

An idempotent semigroup $S$ is called a *left* (respectively, *right*) *normal band* if it is right (respectively, left) commutative (see Section 2).

A dimonoid $(D, \dashv, \vdash)$ is called an $(ln, rn)$-*diband* [10] if $(D, \dashv)$ is a left normal band and $(D, \vdash)$ is a right normal band.

It is well-known that every left (right) normal band is left (right) regular. The converse statement is not true in general.

As is known (see Corollary 1 of [10]), the variety of $(ln, rn)$-dibands and the variety of $(lr, rr)$-dibands coincide.

Now we give necessary and sufficient conditions under which an arbitrary dimonoid is an abelian diband.

**Theorem 1.** *A dimonoid $(D, \dashv, \vdash)$ is abelian idempotent if and only if $(D, \dashv, \vdash)$ is an $(ln, rn)$-diband.*

*Proof.* Let $(D, \dashv, \vdash)$ be an abelian idempotent dimonoid. Using dimonoid axioms of $(D_1)$ and $(D_3)$, the property of abelianity for $(D, \dashv, \vdash)$ and associativity of $\dashv, \vdash$, for all $a, b \in D$ we have

$$(a \dashv b) \dashv a = a \dashv (b \vdash a) = a \dashv (a \dashv b) = (a \dashv a) \dashv b = a \dashv b,$$

$$a \vdash (b \vdash a) = (a \dashv b) \vdash a = (b \vdash a) \vdash a = b \vdash (a \vdash a) = b \vdash a.$$

It means that $(D, \dashv)$ is a left regular band and $(D, \vdash)$ is a right regular band, therefore $(D, \dashv, \vdash)$ is an $(lr, rr)$-diband.

By Corollary 1 of [10], $(D, \dashv, \vdash)$ is an $(ln, rn)$-diband.

Now let $(D, \dashv, \vdash)$ be an $(ln, rn)$-diband, then for all $x, y, a \in D$,

$$a \dashv x \dashv y = a \dashv y \dashv x, \qquad x \vdash y \vdash a = y \vdash x \vdash a.$$

Using dimonid axioms, idempotency of $\dashv, \vdash$, and the fact that $(D, \dashv)$ (respectively, $(D, \vdash)$) is a left (respectively, right) normal band, we obtain

$$\begin{aligned} x \dashv y &= (x \dashv y) \vdash (x \dashv y) = \\ &= x \vdash (y \vdash (x \dashv y)) = \\ &= x \vdash ((y \vdash x) \dashv y) = \\ &= (x \vdash (y \vdash x)) \dashv y = \\ &= (y \vdash x) \dashv y = \\ &= y \vdash (x \dashv y) = \\ &= y \vdash ((x \dashv y) \dashv x) = \\ &= (y \vdash (x \dashv y)) \dashv x = \\ &= ((y \vdash x) \dashv y) \dashv x = \\ &= (y \vdash x) \dashv (y \vdash x) = \\ &= y \vdash x. \end{aligned}$$

Thus, $(D, \dashv, \vdash)$ is an abelian diband. □

It should be noted that the sufficiency of this theorem follows also from Corollary 1 of [10] and the necessity of Theorem 1 of [10].

From Theorem 1 we immediately obtain

**Corollary 1.** *The variety of abelian dibands and the variety of* $(ln, rn)$*-dibands coincide.*

Let $B_{lz,rz}(X)$ be a dimonoid from Proposition 3 (see Section 2).

**Corollary 2.** $B_{lz,rz}(X)$ *is the free abelian diband.*

*Proof.* According to Proposition 3, $B_{lz,rz}(X)$ is the an $(lr, rr)$-diband. By Corollary 1 and Corollary 1 of [10], $B_{lz,rz}(X)$ is a free abelian diband. □

Observe that the cardinality of $X$ is the *rank* of the free abelian diband $B_{lz,rz}(X)$ and this diband is uniquely determined up to an isomorphism by $|X|$.

It is clear that operations of the free abelian diband $B_{lz,rz}(X)$ coincide if and only if the rank of this diband is equal to 1.

The following two statements are obvious.

**Proposition 6.** *The semigroups* $(B_{lz,rz}(X), \dashv)$ *and* $(B_{lz,rz}(X), \vdash)$ *are anti-isomorphic.*

We denote the automorphism group of an algebra $\mathfrak{A}$ by $\mathbf{Aut}(\mathfrak{A})$. The symmetric group on $X$ is denoted by $S(X)$.

**Proposition 7.** $\mathbf{Aut}(B_{lz,rz}(X)) \cong S(X)$.

Let $(Fd(X), \prec, \succ)$ be a *free dimonoid* on $X$ (see, e.g., [3]). For every $w \in Fd(X)$, where $w = (w_1, \ldots, \widetilde{w}_l, \ldots, w_k)$, we assume $c(w) = \bigcup_{i=1}^{k} \{w_i\}$.

Define a binary relation $\sigma$ on $Fd(X)$ as follows: $u = (u_1, \ldots, \widetilde{u}_i, \ldots, u_n)$ and $v = (v_1, \ldots, \widetilde{v}_j, \ldots, v_m)$ are $\sigma$*-equivalent* if

$$c(u) = c(v) \qquad \textit{and} \qquad u_i = v_j.$$

A congruence $\rho$ on a dimonoid $(D, \dashv, \vdash)$ is called *abelian idempotent* if $(D, \dashv, \vdash)/\rho$ is an abelian diband. The notion of an $(lr, rr)$-congruence is defined analogously.

By Theorem 6 of [12], $\sigma$ is the least $(lr, rr)$-congruence on $(Fd(X), \prec, \succ)$. From Corollary 1 and Corollary 1 of [10] we obtain

**Proposition 8.** *The binary relation* $\sigma$ *is the least abelian idempotent congruence on the free dimonoid* $(Fd(X), \prec, \succ)$.

Finally we count the cardinality of the free abelian idempotent dimonoid for a finite case.

As usual, we denote the number of all $k$-element subsets of an $n$-element set by $C_n^k$.

**Proposition 9.** *Let* $X$ *be an arbitrary nonempty finite set with* $|X| = n$. *Then*

$$|B_{lz,rz}(X)| = n \cdot 2^{n-1}.$$

*Proof.* Let $A$ be an arbitrary nonempty subset of $X$. Obviously, we can choose $A$ exactly by $2^n - 1$ ways, in addition, for every set $A$ there exist $|A|$ elements of $B_{lz,rz}(X)$ of the form $(a, A), a \in A$. Therefore,

$$\begin{aligned}
|B_{lz,rz}(X)| &= \sum_{A \subseteq X,\, A \neq \emptyset} |A| = \\
&= \sum_{i=1}^{n} C_n^i \cdot i = \\
&= \sum_{i=1}^{n} n \cdot \frac{(n-1)!}{((i-1)! \cdot (n-1) - (i-1))!} = \\
&= n \cdot \sum_{i=1}^{n} C_{n-1}^{i-1} = \\
&= n \cdot \sum_{j=0}^{n-1} C_{n-1}^{j} = \\
&= n \cdot 2^{n-1}.
\end{aligned}$$

□

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## ВІЛЬНІ АБЕЛЕВІ ДІСПОЛУКИ

**Юрій ЖУЧОК**

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Доведено, що многовиди абелевих дісполук і $(ln, rn)$-дісполук збігаються. Розглянуто деякі властивості вільних абелевих дісполук.

*Ключові слова:* дімоноїд, абелева дісполука, вільна абелева дісполука, напівгрупа.