ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2017. Випуск 83. С. 179–188
Visnyk of the Lviv Univ. Series Mech. Math. 2017. Issue 83. P. 179–188

УДК 517.956.4; 517.977.5

SPECTRAL ANALYSIS OF BOUNDARY VALUE PROBLEMS WITH RETARDED ARGUMENT

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In this work we find asymptotic formulas for eigenvalues and eigenfunctions of a Sturm-Liouville type problem with retarded argument which contains a spectral parameter in the boundary conditions and with discontinuous weight function and also we obtain bounds for the distance between eigenvalues. We extend and generalize some approaches and results of the [S. B. Norkin, Differential equations of the second order with retarded argument, Translations of Mathematical Monographs, Vol. 31, AMS, Providence, RI (1972)].

Key words: differential equation with retarded argument, eigenparameter, transmission conditions, asymptotics of eigenvalues, bounds for the distance between eigenvalues.

1. Introduction.

Some discontinuous boundary value problems with retarded argument and some classic boundary value problems have been investigated in [1-18]. Norkin in [2] considered the equation

$$x''(t) + \lambda x(t) + M(t)x(t - \Delta(t)) = 0$$

with boundary conditions

$$x(0) = x(\pi) = 0,$$

obtained asymptotic formulas for eigenvalues and eigenfunctions and found bounds for the distance between eigenvalues of this problem. In this paper we investigate the eigenvalues and eigenfunctions of a discontinuous boundary value problem with retarded argument with discontinuous weight function. Namely, we consider the boundary value problem for the differential equation

$$u''(x) + q(x)u(x - \Delta(x)) + \lambda r(x)\, u(x) = 0 \qquad (1)$$

2010 Mathematics Subject Classification: 34L20, 35R10 on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ with spectral and physical parameter dependent boundary conditions

$$\sqrt{\lambda} r_+ u(0) + u'(0) = 0, \tag{2}$$

$$m\lambda u(\pi) + u'(\pi) = 0, \tag{3}$$

and with transmission conditions

$$\gamma^+ u(\tfrac{\pi}{2} - 0) - \delta^+ u(\tfrac{\pi}{2} + 0) = 0, \tag{4}$$

$$\gamma^- u'(\tfrac{\pi}{2} - 0) - \delta^- u'(\tfrac{\pi}{2} + 0) = 0 \tag{5}$$

where the real-valued function $q(x)$ is continuous in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and has finite limits

$$q(\tfrac{\pi}{2} \pm 0) = \lim_{x \to \frac{\pi}{2} \pm 0} q(x),$$

the real valued function $\Delta(x) \geqslant 0$ is continuous in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and has finite limits

$$\Delta(\tfrac{\pi}{2} \pm 0) = \lim_{x \to \frac{\pi}{2} \pm 0} \Delta(x),$$

$x - \Delta(x) \geqslant 0$ if $x \in \left[0, \frac{\pi}{2}\right)$; $x - \Delta(x) \geqslant \frac{\pi}{2}$, if $x \in \left(\frac{\pi}{2}, \pi\right]$; $r(x) = r_+^2$ if $x \in \left[0, \frac{\pi}{2}\right)$ and $r(x) = r_-^2$ if $x \in \left(\frac{\pi}{2}, \pi\right]$; λ is a real positive spectral parameter; m is a positive physical parameter; $r_+, r_-, d, \delta^+, \delta^-, \gamma^+, \gamma^- \neq 0$ are arbitrary real numbers.

We want to note that differential equations with retarded argument are of importance in the theory of automatic control and in the theory of self-oscillatory systems. For instance, in automatic control systems retardation is the time interval which the system requires to react to an input impulse ([2]).

Let $w_1(x, \lambda)$ be a solution of Eq. (1) on $\left[0, \frac{\pi}{2}\right]$ satisfying the initial conditions

$$w_1(0, \lambda) = r_+^{-1} \quad \text{and} \quad w_1'(0, \lambda) = -\sqrt{\lambda}. \tag{6}$$

Conditions (6) determine a unique solution of Eq. (1) on $\left[0, \frac{\pi}{2}\right]$ ([2], p. 12).

After determining the above solution, we shall determine the solution $y_2(x, \lambda)$ of Eq. (1) on $\left[\frac{\pi}{2}, \pi\right]$ by means of the solution $y_1(x, \lambda)$ using the initial conditions

$$w_2\left(\tfrac{\pi}{2}, \lambda\right) = \tfrac{\gamma^+}{\delta^+} w_1(\tfrac{\pi}{2}, \lambda) \qquad \text{and} \qquad w_2'\left(\tfrac{\pi}{2}, \lambda\right) = \tfrac{\gamma^-}{\delta^-} w_1'(\tfrac{\pi}{2}, \lambda). \tag{7}$$

The conditions (7) define a unique solution of Eq. (1) on $\left[\frac{\pi}{2}, \pi\right]$.

Consequently, the function $w(x, \lambda)$ defined on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ by the equality

$$w(x, \lambda) = \begin{cases} w_1(x, \lambda), & x \in \left[0, \frac{\pi}{2}\right), \\ w_2(x, \lambda), & x \in \left(\frac{\pi}{2}, \pi\right], \end{cases}$$

is a solution of the Eq. (1) on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ which satisfies one of the boundary conditions and transmission conditions.

2. Eigenvalues and Eigenfunctions of the Problem (1)–(5)**.**

Lemma 1. *Let* $w(x,\lambda)$ *be a solution of Eq.* (1)*. Then the following integral equations hold:*

$$w_1(x,\lambda) = \tfrac{\sqrt{2}}{r_+}\cos(r_+\sqrt{\lambda}x + \tfrac{\pi}{4}) - \tfrac{r_+}{\sqrt{\lambda}}\int\limits_0^x q(\tau)\sin r_+\sqrt{\lambda}(x-\tau)\, w_1(\tau-\Delta(\tau),\lambda)\,d\tau, \tag{8}$$

$$\begin{aligned} w_2(x,\lambda) = {} & \tfrac{\gamma^+}{\delta^+} w_1(\tfrac{\pi}{2},\lambda)\cos r_-\sqrt{\lambda}\left(x-\tfrac{\pi}{2}\right) + \frac{\gamma^- w_1'(\tfrac{\pi}{2},\lambda)}{\sqrt{\lambda}r_-\delta^-}\sin r_-\sqrt{\lambda}\left(x-\tfrac{\pi}{2}\right) - \\ & - \frac{r_-}{\sqrt{\lambda}}\int\limits_{\frac{\pi}{2}}^x q(\tau)\sin r_-\sqrt{\lambda}(x-\tau)\, w_2(\tau-\Delta(\tau),\lambda)\,d\tau. \end{aligned} \tag{9}$$

Proof. To prove this lemma, it is enough to substitute $-\lambda^2 w_1(\tau,\lambda) - w_1''(\tau,\lambda)$ and $-\lambda^2 w_2(\tau,\lambda) - w_2''(\tau,\lambda)$ instead of $-q(\tau)w_1(\tau-\Delta(\tau),\lambda)$ and $-q(\tau)w_2(\tau-\Delta(\tau),\lambda)$ in the integrals in (9), (10) respectively and integrate by parts twice. □

Theorem 1. *Problem* (1)–(5) *can have only simple eigenvalues.*

Proof. The proof is similar to the proof of Theorem 1 in [8]. □

The function $w(x,\lambda)$ defined in Section 1 is a nontrivial solution of Eq. (1) satisfying conditions (2) and (4)-(5). Putting $w(x,\ \lambda)$ into (3), we get the characteristic equation

$$Z(\lambda) \equiv w'(\pi,\ \lambda) + m\lambda w(\pi,\ \lambda) = 0. \tag{10}$$

By Theorem 1 the set of eigenvalues of boundary-value problem (1)-(5) coincides with the set of real roots of Eq. (10). Let

$$Q_1 = r_+\int\limits_0^{\frac{\pi}{2}} |q(\tau)|d\tau,\ Q_2 = r_-\int\limits_{\frac{\pi}{2}}^{\pi} |q(\tau)|\,d\tau.$$

Lemma 2. *Let* $\lambda \geqslant \max\{2Q_1, 2Q_2\}$*. Then for the solutions* $w_1(x,\lambda)$ *and* $w_2(x,\lambda)$*of Eq.* (8) *and Eq.* (9) *the following inequalities hold*:

$$|w_1(x,\lambda)| \leqslant \text{const}., \quad x \in \left[0, \tfrac{\pi}{2}\right], \tag{11}$$

$$|w_2(x,\lambda)| \leqslant \text{const}., \quad x \in \left[\tfrac{\pi}{2}, \pi\right]. \tag{12}$$

Proof. The proof is similar to the proof of Theorem 1 in [7]. □

From (8)-(10)

$$-\frac{\gamma^+ r_-\sqrt{\lambda}}{\delta^+}\left[\frac{\sqrt{2}}{r_+}\cos\left(\frac{r^+\pi\sqrt{\lambda}}{2}+\frac{\pi}{4}\right)-\frac{r_+}{\sqrt{\lambda}}\int_0^{\pi/2} q(\tau)\sin\left(r_+\sqrt{\lambda}\left(\frac{\pi}{2}-\tau\right)\right)w_1(\tau-\Delta(\tau),\lambda)d\tau\right]\sin\frac{r_-\pi\sqrt{\lambda}}{2}$$

$$+\frac{\gamma^-}{\delta^-}\left[-\sqrt{2\lambda}\sin\left(\frac{r^+\pi\sqrt{\lambda}}{2}+\frac{\pi}{4}\right)-r_+^2\int_0^{\pi/2} q(\tau)\cos\left(r_+\sqrt{\lambda}\left(\frac{\pi}{2}-\tau\right)\right)w_1(\tau-\Delta(\tau),\lambda)d\tau\right]\cos\frac{r_-\pi\sqrt{\lambda}}{2}$$

$$-r_-^2\int_{\pi/2}^{\pi} q(\tau)\cos\left(r_-\sqrt{\lambda}(\pi-\tau)\right)w_2(\tau-\Delta(\tau),\lambda)d\tau-$$

$$+m\lambda\left\{\frac{\gamma^+}{\delta^+}\left[\frac{\sqrt{2}}{r_+}\cos\left(\frac{r^+\pi\sqrt{\lambda}}{2}+\frac{\pi}{4}\right)-\frac{r_+}{\sqrt{\lambda}}\int_0^{\pi/2} q(\tau)\sin\left(r_+\sqrt{\lambda}\left(\frac{\pi}{2}-\tau\right)\right)w_1(\tau-\Delta(\tau),\lambda)d\tau\right]\cos\frac{r_-\pi\sqrt{\lambda}}{2}\right.$$

$$+\frac{\gamma^-}{\delta^-\sqrt{\lambda}r_-}\left[-\sqrt{2\lambda}\sin\left(\frac{r^+\pi\sqrt{\lambda}}{2}+\frac{\pi}{4}\right)-r_+^2\int_0^{\pi/2} q(\tau)\cos\left(r_+\sqrt{\lambda}\left(\frac{\pi}{2}-\tau\right)\right)w_1(\tau-\Delta(\tau),\lambda)d\tau\right]$$

(13)

$$\left.\times\sin\frac{r_-\pi\sqrt{\lambda}}{2}-\frac{r_-}{\sqrt{\lambda}}\int_{\pi/2}^{\pi} q(\tau)\sin\left(r_-\sqrt{\lambda}(\pi-\tau)\right)w_2(\tau-\Delta(\tau),\lambda)d\tau\right\}=0.$$

Let λ be sufficiently large and $\gamma^+\delta^- r_- = r_+\delta^+\gamma^-$. With the helps of (8), (9), (11) and (12), we have

$$\sqrt{\lambda}\cos\left(\frac{\sqrt{\lambda}\pi}{2}\left[r_+ + r_-\right]+\frac{\pi}{4}\right)+O(1)=0.$$

So we have the following formula for the eigenvalues:

$$\sqrt{\lambda_n}=\frac{4n-3}{2\left[r_+ + r_-\right]}+O\left(\frac{1}{n}\right).$$

Using the same techniques in [2] we find the next asymptotic formulas for the eigenfunctions of problem (1)–(5):

$$u_{1n}=r_+^{-1}\left\{\cos\left(\frac{[4n-3]r_-^2 r_+ x}{2[r_+ + r_-]}\right)-\sin\left(\frac{[4n-3]r_-^2 r_+ x}{2[r_+ + r_-]}\right)\right\}+O\left(\frac{1}{n}\right),\quad x\in\left[0,\frac{\pi}{2}\right)$$

and

$$u_{2n}=\frac{r_+^{-1}\gamma^+}{\delta^+}\left\{\cos\left(\frac{[4n-3]r_- r_+^2 x}{2[r_+ + r_-]}+\frac{[4n-3][r_+ + r_-]\pi}{16[r_+ + r_-]}\right)\right.$$

$$\left.-\sin\left(\frac{[4n-3]r_- r_+^2 x}{2[r_+ + r_-]}+\frac{[4n-3][r_+ + r_-]\pi}{16[r_+ + r_-]}\right)\right\}+O\left(\frac{1}{n}\right),\quad x\in\left(\frac{\pi}{2},\pi\right].$$

Now let us assume that the following conditions hold: The derivatives $q'(x)$ and $\Delta''(x)$ exist and are bounded in $[0,\frac{\pi}{2})\bigcup(\frac{\pi}{2},\pi]$ and have finite limits $q'(\frac{\pi}{2}\pm 0)=\lim\limits_{x\to\frac{\pi}{2}\pm 0} q'(x)$ and $\Delta''(\frac{\pi}{2}\pm 0)=\lim\limits_{x\to\frac{\pi}{2}\pm 0}\Delta''(x)$, respectively; $\Delta'(x)\leqslant 1$ in $[0,\frac{\pi}{2})\bigcup(\frac{\pi}{2},\pi]$, $\Delta(0)=0$ and $\lim\limits_{x\to\frac{\pi}{2}+0}\Delta(x)=0$.

Under these additional conditions we have

$$w_1\left(\tau-\Delta\left(\tau\right),\lambda\right)=\tfrac{\sqrt{2}}{r_+}\cos\left(\tfrac{\pi+r_+4\sqrt{\lambda}(\tau-\Delta(\tau))}{4}\right)+O\left(\tfrac{1}{\sqrt{\lambda}}\right), \tag{14}$$

$$w_2\left(\tau-\Delta\left(\tau\right),\lambda\right)=\tfrac{\sqrt{2}\gamma^+}{r_+\delta^+}\cos\left(\tfrac{\pi+\sqrt{\lambda}2\pi(r_+-r_-)+r_-4\sqrt{\lambda}(\tau-\Delta(\tau))}{4}\right)+O\left(\tfrac{1}{\sqrt{\lambda}}\right). \tag{15}$$

Let

$$R_1\left(x,\lambda,\Delta(\tau)\right)=\int\limits_0^x\tfrac{q(\tau)}{\sqrt{2}}\sin\left(\tfrac{r_+4\sqrt{\lambda}\Delta(\tau)-\pi}{4}\right)d\tau,$$

$$R_2(x,\lambda,\Delta(\tau))=\int\limits_0^x\tfrac{q(\tau)}{\sqrt{2}}\cos\left(\tfrac{r_+4\sqrt{\lambda}\Delta(\tau)-\pi}{4}\right)d\tau,$$

$$R_3\left(x,\lambda,\Delta(\tau)\right)=\int\limits_{\pi/2}^x\tfrac{q(\tau)}{\sqrt{2}}\sin\left(\tfrac{r_-4\sqrt{\lambda}\Delta(\tau)-\pi}{4}\right)d\tau,$$

$$R_4\left(x,\lambda,\Delta(\tau)\right)=\int\limits_{\pi/2}^x\tfrac{q(\tau)}{\sqrt{2}}\cos\left(\tfrac{r_-4\sqrt{\lambda}\Delta(\tau)-\pi}{4}\right)d\tau.$$

The following formulas

$$\left\{\begin{array}{l}\int\limits_0^x\tfrac{q(\tau)}{\sqrt{2}}\sin\left(\tfrac{r_+4\sqrt{\lambda}(2\tau-\Delta(\tau))+\pi}{4}\right)d\tau\\ \int\limits_0^x\tfrac{q(\tau)}{\sqrt{2}}\cos\left(\tfrac{r_+4\sqrt{\lambda}(2\tau-\Delta(\tau))+\pi}{4}\right)d\tau\\ \int\limits_{\pi/2}^x\tfrac{q(\tau)}{\sqrt{2}}\sin\left(\tfrac{r_-4\sqrt{\lambda}(2\tau-\Delta(\tau))+\pi}{4}\right)d\tau\\ \int\limits_{\pi/2}^x\tfrac{q(\tau)}{\sqrt{2}}\cos\left(\tfrac{r_-4\sqrt{\lambda}(2\tau-\Delta(\tau))+\pi}{4}\right)d\tau\end{array}\right.=O\left(\tfrac{1}{\sqrt{\lambda}}\right) \tag{16}$$

can be proved by the similar method as in Lemma 3.3.3 in [2]. Putting formulas (14) and (15) in (13) and using (16) we obtain following equality:

$$\cos\left(\tfrac{\sqrt{\lambda}\pi(r_++r_-)}{2}+\frac{\pi}{4}\right)$$

$$=\frac{\sin\left(\frac{\sqrt{\lambda}\pi(r_++r_-)}{2}+\frac{\pi}{4}\right)}{\sqrt{\lambda}}\left(\tfrac{\delta^+\gamma^-+m\delta^-\gamma^+[r_+R_2(x,\lambda,\Delta(\tau))+r_-R_4(x,\lambda,\Delta(\tau))]}{\delta^+\delta^-}\right)+O\left(\tfrac{1}{\lambda}\right).$$

Now replacing $\sqrt{\lambda}$ by $\sqrt{\lambda_n}=\dfrac{4n-3}{2\left[r_++r_-\right]}+\delta_n$ we get

$$\delta_n=\frac{4\left[\delta^+\gamma^-+m\delta^-\gamma^+\right]\Big[r_+R_2\big(\frac{\pi}{2},\frac{4n-3}{2[r_++r_-]},\Delta\left(\tau\right)\big)+r_-R_4\big(\pi,\frac{4n-3}{2[r_++r_-]},\Delta\left(\tau\right)\big)\Big]}{\left(4n-3\right)\pi\delta^+\delta^-}+O\left(\tfrac{1}{n^2}\right).$$

Thus, we now may obtain a sharper asymptotic formula for the eigenfunctions. Putting (14) in (8) and replacing $\sqrt{\lambda}$ by $\sqrt{\lambda_n}$ for $x \in \left[0, \frac{\pi}{2}\right)$ we have

$$
\begin{aligned}
u_{1n}(x) &= \cos\left(\frac{r_-^2(8n-6)x+\pi(r_++r_-)}{4(r_++r_-)}\right)\left[\frac{(8n-6)+2\sqrt{2}r_+^2(r_++r_-)R_1\left(x,\frac{4n-3}{2[r_++r_-]},\Delta(\tau)\right)}{r_+\sqrt{2}(4n-3)}\right]\\
&-4\sqrt{2}\sin\left(\frac{r_-^2(8n-6)x+\pi(r_++r_-)}{4(r_++r_-)}\right)\\
&\times\left(\frac{\delta^+\gamma^-+r_+\delta^- m\gamma^+R_2\left(\frac{\pi}{2},\frac{4n-3}{2[r_++r_-]},\Delta(\tau)\right)+mr_-\gamma^+\delta^-R_4\left(\pi,\frac{4n-3}{2[r_++r_-]},\Delta(\tau)\right)}{(4n-3)\pi r_+\delta^+\delta^-}\right)+O\left(\frac{1}{n^2}\right).
\end{aligned}
$$

Putting (15) in (9) and replacing $\sqrt{\lambda}$ by $\sqrt{\lambda_n}$ for $x \in \left(\frac{\pi}{2}, \pi\right]$ we have

$$
\begin{aligned}
u_{2n}(x) &= \frac{\gamma^+}{2\delta^+}\Bigg\{\Bigg[(-1)^{n+1}\sin\left(\frac{r_-(4n-3)x}{2(r_++r_-)}\right)+\cos\left(\frac{\pi[(r_+-r_-)(4n-3)+(r_++r_-)]+(8n-6)r_-x}{4(r_++r_-)}\right)\Bigg]\\
&\times\left[\frac{(8n-6)+2\sqrt{2}r_+^2(r_++r_-)R_1\left(\frac{\pi}{2},\frac{4n-3}{2[r_++r_-]},\Delta(\tau)\right)}{r_+\sqrt{2}(4n-3)}\right]+\Bigg[(-1)^n\cos\left(\frac{r_-(4n-3)x}{2(r_++r_-)}\right)\\
&-\sin\left(\frac{\pi[(r_+-r_-)(4n-3)+(r_++r_-)]+(8n-6)r_-x}{4(r_++r_-)}\right)\Bigg]\times\left[\frac{2r_+(r_++r_-)R_2\left(\frac{\pi}{2},\frac{4n-3}{2[r_++r_-]},\Delta(\tau)\right)}{(4n-3)}\right]\Bigg\}\\
&-\frac{\sqrt{2}\gamma^-}{2r_-\delta^-}\Bigg\{(-1)^{n+1}\sin\left(\frac{(4n-3)r_-^2r_+x}{2(r_++r_-)}\right)+\frac{4}{(4n-3)\pi}\Bigg[(-1)^n\cos\left(\frac{(4n-3)r_-^2r_+x}{2(r_++r_-)}\right)+\\
&\times\sin\left(\frac{\pi[(r_+-r_-)(4n-3)+(r_++r_-)]+(8n-6)r_-x}{4(r_++r_-)}\right)\Bigg]\\
&\times\left[\frac{\gamma^-\delta^++\gamma^+\delta^- m\left[r_+R_2\left(\frac{\pi}{2},\frac{4n-3}{2[r_++r_-]},\Delta(\tau)\right)\right)+r_-R_4\left(\pi,\frac{4n-3}{2[r_++r_-]},\Delta(\tau)\right)\right)\right]}{\delta^+\delta^-}\right]\\
&+\cos\left(\frac{\pi[(r_+-r_-)(4n-3)+(r_++r_-)]+(8n-6)r_-x}{4(r_++r_-)}\right)\Bigg\}\\
&+\frac{\gamma^-r_+^2(r_++r_-)}{r_-\delta^-(4n-3)}R_2\left(\frac{\pi}{2},\frac{4n-3}{2[r_++r_-]},\Delta(\tau)\right)\Bigg\{(-1)^n\cos\left(\frac{r_-(4n-3)x}{2(r_++r_-)}\right)\\
&-\sin\left(\frac{\pi[(r_+-r_-)(4n-3)+(r_++r_-)]+(8n-6)r_-x}{4(r_++r_-)}\right)\Bigg\}\\
&+\frac{2\gamma^-r_+^3(r_++r_-)R_1\left(\frac{\pi}{2},\frac{4n-3}{2[r_++r_-]},\Delta(\tau)\right)}{\delta^-(4n-3)}\\
&\times\Bigg\{(-1)^n\sin\left(\frac{r_-(4n-3)x}{2(r_++r_-)}\right)+\cos\left(\frac{\pi[(r_+-r_-)(4n-3)+(r_++r_-)]+(8n-6)r_-x}{4(r_++r_-)}\right)\Bigg\}-\\
&\frac{2\gamma^+r_-(r_++r_-)}{(4n-3)r_+\delta^+}\Bigg\{\sin\left(\frac{\pi[(r_+-r_-)(4n-3)+(r_++r_-)]+(8n-6)r_-x}{4(r_++r_-)}\right)\\
&\times R_4\left(x,\frac{4n-3}{2[r_++r_-]},\Delta(\tau)\right)-R_3\left(x,\frac{4n-3}{2[r_++r_-]},\Delta(\tau)\right)\\
&\times\cos\left(\frac{\pi[(r_+-r_-)(4n-3)+(r_++r_-)]+(8n-6)r_-x}{4(r_++r_-)}\right)\Bigg\}+O\left(\frac{1}{n^2}\right).
\end{aligned}
$$

3. Bounds for the Distance Between Eigenvalues.
Let us define

$$\chi_0 = \begin{cases} \min\left\{\beta_0^2 Q_\pi^2, \gamma_0^2 Q_0^2 \pi^2\right\}, \\ \gamma_0^2 Q_\pi^2, \text{ if } x\left|q\left(x\right)\right| \geqslant \displaystyle\int_0^x q(t)dt, \ 0 \leqslant x \leqslant \pi. \end{cases}$$

where β_0 is the unique real root of the equation $\beta = \left(\sqrt{2+\sqrt{2}}+\sqrt{2}\right)e^{1/\beta}$; γ_0 is the unique real root of the equation $\gamma = \frac{\sqrt{2}}{4}\left(\sqrt{9+4\sqrt{2}}+3\right)e^{1/\gamma}$; $Q_\pi = \displaystyle\int_0^\pi |q(x)|\,dx$ and $Q_0 = \max_{[0,\pi]}|q\left(x\right)|$. Assume that $\lambda \geqq \chi_0$ and let $\lambda_N, \lambda_{N+1}, \ldots, \lambda_{N+p}, \ldots$ be the eigenvalues of problem (4)–(5) listed in the increasing order, N is the number of zeros on the set $(0,\pi/2)\cup(\pi/2,\pi)$ of the eigenfunctions corresponding to the eigenvalue λ_N. In what follows the eigenvalues with odd index will be called odd, and those with even index will be called even.

Now, we will state the following theorem which can be proven easily using the same method as in [2].

Theorem 2 (**Asymptotic Oscillation Theorem**). *The eigenvalues of problem* (1)–(5) *form an unbounded increasing sequence* $\lambda_N, \lambda_{N+1}, \ldots, \lambda_{N+p}, \ldots,$ *in the region* $\lambda \geqq \chi_0$. *Moreover, the eigenfunction corresponding to the eigenvalue* λ_{N+p} *has exactly* $N+p$ *zeros on the set* $(0,\pi/2)\cup(\pi/2,\pi)$, *where* N *is the number of zeros of the eigenfunction corresponding to the first eigenvalue* λ_N *of the sequence.*

Lemma 3. *Suppose that* $\lambda \geqq \chi_0$ *in (1) and that* λ' *is an eigenvalue of problem* (1)–(5). *Then* $\sqrt{\lambda'} = \mu' = \dfrac{4n'-3}{2\left[r_+ + r_-\right]} + \delta_{n'}$, *where* n' *is an integer, and* $|\delta_{n'}| \leqslant \dfrac{1}{2\left[r_+ + r_-\right]}$. *Moreover, if* λ' *is an odd eigenvalue, then* n' *is even; for an even eigenvalue,* n' *is odd.*

Proof. Suppose that λ' is an odd eigenvalue of the problem (1)–(5) and that

$$\sqrt{\lambda'} = \mu' = \frac{4n'-3}{2\left[r_+ + r_-\right]} + \delta_{n'} \tag{17}$$

where n' is an integer, and

$$|\delta_{n'}| \leqslant \frac{1}{2\left[r_+ + r_-\right]}. \tag{18}$$

Differentiating (9) with respect to x and evaluating its value at $x = \pi$ we obtain

$$\left|\sin\left(\tfrac{\mu'\pi[r_+ + r_-]}{2} + \frac{\pi}{4}\right)\right| > \tfrac{\sqrt{2}}{2}. \tag{19}$$

However, if $\lambda' \geqq \chi_0$, from (6) and Lemma 2.3.6 in [2] it follows that

$$\frac{1}{\mu'}\left|\int_{\pi/2}^{\pi} q\left(\tau\right)\cos\left(\mu' r_-\left(\pi-\tau\right)\right)w_2\left(\tau-\Delta(\tau),\lambda'\right)d\tau\right| < \tfrac{\sqrt{2}}{2} \tag{20}$$

and it follows from the (19) and (20) that the sign of the derivative coincides with the sign of $\sin\left(\frac{\mu'\pi[r_+ + r_-]}{2} + \frac{\pi}{4}\right)$. From Theorem 3.1 and Lemma 2.3.3 in [2] we obtain that

$w'_x(\pi,\lambda')>0$. Therefore we get

$$\sin\left(\frac{\mu'\pi[r_++r_-]}{2}+\frac{\pi}{4}\right)>0. \tag{21}$$

From (17) it now follows that

$$\begin{aligned}\sin\left(\frac{\mu'\pi[r_++r_-]}{2}+\frac{\pi}{4}\right) &= \sin\left(\left(\frac{4n'-3}{2[r_++r_-]}+\delta_{n'}\right)\frac{\pi[r_++r_-]}{2}+\frac{\pi}{4}\right)\\ &= \sin\frac{(4n'-3)\pi}{2}\cos\left(\frac{\delta_{n'}\pi[r_++r_-]}{2}+\frac{\pi}{4}\right).\end{aligned}$$

If the equality holds in (18), then $\cos\left(\frac{\delta_{n'}\pi[r_++r_-]}{2}+\frac{\pi}{4}\right)=0$ and therefore $\sin\left(\frac{\mu'\pi[r_++r_-]}{2}+\frac{\pi}{4}\right)=0$, which contradicts (21), the integer n' is defined uniquely and $\left|\frac{\delta_{n'}\pi[r_++r_-]}{2}+\frac{\pi}{4}\right|<\frac{\pi}{2}$. Then $\cos\left(\frac{\delta_{n'}\pi[r_++r_-]}{2}+\frac{\pi}{4}\right)>0$ and, from (21), $\sin\frac{(4n'-3)\pi}{4}>0$. Thus the proof is completed. □

Theorem 3. *Let $\lambda'=\mu_1^2$, $\lambda''=\mu_2^2$, $\lambda'''=\mu_3^2$ $(\lambda'''>\lambda''>\lambda'\geqq\chi_0)$ be three successive eigenvalues of problem (1)–(5). Then*

$$\frac{3}{r_++r_-}<\mu_3-\mu_1<\frac{6}{r_++r_-} \tag{22}$$

$$\mu_3-\mu_2<\frac{4}{r_++r_-},\ \mu_2-\mu_1<\frac{4}{r_++r_-}. \tag{23}$$

Proof. By Lemma 3.2, $\mu_3=\dfrac{4n_3-3}{2\left[r_++r_-\right]}+\delta_{n_3}$ and $\mu_1=\dfrac{4n_1-3}{2\left[r_++r_-\right]}+\delta_{n_1}$, with $n_3-n_1\geqq 2$ and $|\delta_{n_1}|<\dfrac{1}{2\left[r_++r_-\right]}$, $|\delta_{n_3}|<\dfrac{1}{2\left[r_++r_-\right]}$. Therefore

$$\mu_3-\mu_1\geqq\frac{2\left(n_3-n_1\right)}{r_++r_-}-\left(|\delta_{n_3}|+|\delta_{n_1}|\right)>\frac{3}{r_++r_-}.$$

The inequalities in (23) and the second inequality in (22) may be proved using the same method in the proof of Theorem 3.6.1 in [2]. □

Acknowledgements. The author acknowledges the referee for useful important comments and suggestions.

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Стаття: надійшла до редколегії 25.09.2016
доопрацьована 11.03.2017
прийнята до друку 13.11.2017

СПЕКТРАЛЬНИЙ АНАЛІЗ ЗАДАЧІ ШТУРМА-ЛІУВІЛЛЯ ІЗ ЗАПІЗНЕННЯМ

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Знайдено асимптотичні формули для власних функцій задачі типу Штурма-Ліувілля з відхиленнями аргументу. У цій задачі спектральний параметр міститься також у крайових умовах, а вагова функція є розривною. Отримано оцінки на відстань між власними значеннями. Ми розширили й узагальнили деякі підходи та результати статті [S. B. Norkin, Differential equations of the second order with retarded argument, Translations of Mathematical Monographs, Vol. 31, AMS, Providence, RI (1972)]

Ключові слова: диференціальне рівняння з відхиленням аргументу, спектральний параметр, умови спряження, асимптотика власних значень, оцінка відстані між власними значеннями.