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RESTORATION OF A SOLUTION'S INITIAL DATA AND A SOURCE OF THE FRACTIONAL DIFFUSION EQUATION IN THE SPACE OF PERIODIC DISTRIBUTIONS

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We prove the correctness of an inverse problem for a time fractional subdiffusion equation. This problem is to find a solution of direct problem, which is classical in time with values in the space of periodic spatial distributions, its initial data and a source term of the equation. We show that the same kind time integral over-determination conditions may be used.

Key words: fractional derivative, inverse problem, periodic distribution, time integral over-determination condition.

Inverse Cauchy and boundary-value problems for a time fractional diffusion equations with different unknown quantities and under different over-determination conditions are actively studied in connection with their applications (see, for instance, [1]–[9]).

We study the inverse problem for a time fractional diffusion equation. This problem is to find a solution for direct problem, classical in time with values in the space of periodic spatial distributions, its initial data and a source term of the equation. We use the time integral over-determination conditions. Such kind of conditions generalise the multi-point conditions. Space integral over-determination conditions have been used, for instance, in [4, 10, 11] for study the inverse problems.

Note that the sufficient conditions of classical solvability of fractional Cauchy and boundary-value problems were obtained, for example, in [12]–[17], the existence and uniqueness theorems to the boundary-value problems for partial differential equations in Sobolev spaces were obtained by Yu. Berezansky, V. I. Gorbachuk and M. L. Gorbachuk, Ya. Roitberg, J.-L. Lions, E. Magenes, V. A. Mikhailets, A. A. Murach and others (see [18] and references therein), and in [19] the existence and uniqueness theorems to the space fractional Cauchy problem in Schwartz spaces were proved. The solvability of

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some nonclassical direct problems for partial differential equations with integral initial conditions, in particular, in the space of periodic spatial variable functions, have been established, for example, in [20, 21], the multi-point non-local problem for parabolic pseudo-differential equations with non-smooth symbols has been investigated in [22]. The inverse problem on determination only the initial data of the solution (classical in time with values in the space of periodic spatial distributions) of a time fractional diffusion equation, or only a source term of a such type equation, were studied in [8] and [9], respectively.

1. Auxiliary definitions. Assume that $\mathbb{N}$ is a set of natural numbers, $\mathbb{Z}_+ = \mathbb{N}\cup\{0\}$, $\mathcal{D}(\mathbb{R})$ is the space of infinitely differentiable functions with compact supports, $\mathcal{S}(\mathbb{R})$ is the space of rapidly decreasing infinitely differentiable functions [23, p. 90], while $\mathcal{D}'(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ are the spaces of linear continuous functionals (distributions) over $\mathcal{D}(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$, respectively, and the symbol (f, φ) stands for the value of the distribution f on the test function φ. Note that $\mathcal{S}'(\mathbb{R})$ is the space of slowly increasing distributions.

Recall that the Caputo derivative (or the Caputo-Djrbashian derivative) of order $\alpha \in (0, 1)$ is defined by

$$^cD_t^\alpha v(x,t) = \frac{1}{\Gamma(1-\alpha)}\Big[\frac{\partial}{\partial t}\int_0^t \frac{v(x,\tau)}{(t-\tau)^\alpha}d\tau - \frac{v(x,0)}{t^\alpha}\Big].$$

Let $X_k(x) = \sin kx$, $k \in \mathbb{N}$. Similarly to [23, p. 120], we denote by $\mathcal{D}'_{2\pi}(\mathbb{R})$ the space of periodic distributions, i.e., the space of $v \in \mathcal{D}'(\mathbb{R})$ such that

$$v(x + 2\pi) = v(x) = -v(-x) \qquad \forall x \in \mathbb{R}.$$

The formal series

$$\sum_{k=1}^{\infty} v_k X_k(x), \quad x \in \mathbb{R} \tag{1}$$

is the Fourier series of the distribution $v \in \mathcal{D}'_{2\pi}(\mathbb{R})$, and numbers

$$v_k = \frac{2}{\pi}(v, X_k)_{2\pi} = \frac{2}{\pi}(v, hX_k)$$

are its Fourier coefficients. Here $h(x)$ is an even function from $\mathcal{D}(\mathbb{R})$ possessing the properties:

$$h(x) = \begin{cases} 1, & x \in (-\pi+\varepsilon, \pi-\varepsilon) \\ 0, & x \in \mathbb{R}\setminus(-\pi,\pi) \end{cases}, \quad 0 \le h(x) \le 1.$$

Note that

$$v_k = \tfrac{2}{\pi}\int_0^\pi v(x)X_k(x)dx \quad \text{for} \quad v \in \mathcal{D}'_{2\pi}(\mathbb{R}) \cap L^1_{loc}(\mathbb{R}),$$

and then the series (1) is the classical Fourier series of v by the system X_k, $k \in \mathbb{N}$.

As it is known (see [23, p. 123]) $\mathcal{D}'_{2\pi}(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$, the series (1) of $v \in \mathcal{D}'_{2\pi}(\mathbb{R})$ converges in $\mathcal{S}'(\mathbb{R})$ to v, and for the Fourier coefficients the estimates hold

$$(1+k)^{-m}|v_k| \le C(v,m) \quad \forall k \in \mathbb{N}$$

with some $m \in \mathbb{Z}_+$ where $C(v,m)$ is the positive constant, the same for all $k \in \mathbb{N}$.

We use the following: for $\gamma \in \mathbb{R}$

$$H^\gamma(\mathbb{R}) = \Big\{ v \in \mathcal{D}'_{2\pi}(\mathbb{R}) : \|v\|_{H^\gamma(\mathbb{R})} = \sup_{k\in\mathbb{N}} |v_k|(1+k)^\gamma < +\infty \Big\}$$

(note that $H^{\gamma+\varepsilon}(\mathbb{R}) \subset H^\gamma(\mathbb{R})$ for all $\varepsilon > 0$, $\gamma \in \mathbb{R}$),

$C\big([0,T]; H^\gamma(\mathbb{R})\big)$ is the space of continuous in $t \in [0,T]$ functions $v(x,t)$ with values $v(\cdot,t) \in H^\gamma(\mathbb{R})$ endowed with the norm $\|v\|_{C\left([0,T];H^\gamma(\mathbb{R})\right)} = \max\limits_{t\in[0,T]} \|v(\cdot,t)\|_{H^\gamma(\mathbb{R})}$,

$C_b\big((0,T]; H^\gamma(\mathbb{R})\big)$ is the space of continuous in $t \in (0,T]$ functions $v(x,t)$ with values $v(\cdot,t) \in H^\gamma(\mathbb{R})$ endowed with the norm $\|v\|_{C_b\left((0,T];H^\gamma(\mathbb{R})\right)} = \sup\limits_{t\in(0,T]} \|v(\cdot,t)\|_{H^\gamma(\mathbb{R})}$,

$C_{2,\alpha}\big([0,T]; H^\gamma(\mathbb{R})\big) = \Big\{ v \in C\big([0,T]; H^{2+\gamma}(\mathbb{R})\big) : {}^c D^\alpha v \in C_b\big((0,T]; H^\gamma(\mathbb{R})\big) \Big\}$ is its subspace endowed with the norm

$$\|v\|_{C_{2,\alpha}\left([0,T];H^\gamma(\mathbb{R})\right)} = \max\Big\{ \|v\|_{C\left([0,T];H^{2+\gamma}(\mathbb{R})\right)}, \|{}^c D^\alpha v\|_{C_b\left((0,T];H^\gamma(\mathbb{R})\right)} \Big\}.$$

2. The inverse problem. We study the inverse problem

$$ {}^c D_t^\alpha u - u_{xx} = F_0(x), \quad (x,t) \in Q_T := \mathbb{R} \times (0,T], \tag{2}$$

$$ u(x,0) = F_1(x), \quad x \in \mathbb{R}, \tag{3}$$

$$ \int_0^{t_0} u(x,t)dt = \Phi_0(x), \quad \int_0^{t_1} u(x,t)dt = \Phi_1(x), \quad x \in \mathbb{R},\ t_0, t_1 \in (0,T] \tag{4}$$

where $\alpha \in (0,1)$, Φ_0, Φ_1 are given functions, T is a given positive number, u, F_0, F_1 are unknown functions.

Let the following assumption holds:

(A) $\gamma \in \mathbb{R}$, $\Phi_0, \Phi_1 \in H^{\gamma+4}(\mathbb{R})$, $t_0, t_1 \in (0,T]$, $t_0 \neq t_1$.

Expand the functions $F_j(x)$, $\Phi_j(x)$, $j \in \{0,1\}$, in the formal Fourier series by the system $X_k(x)$, $k \in \mathbb{N}$:

$$ F_j(x) = \sum_{k=1}^{\infty} F_{jk} X_k(x), \quad x \in \mathbb{R}, \tag{5}$$

$$ \Phi_j(x) = \sum_{k=1}^{\infty} \Phi_{jk} X_k(x), \quad x \in \mathbb{R}, \ j = 0,1. \tag{6}$$

Definition 1. The vector-function

$$(u, F_0, F_1) \in \mathcal{M}_{\alpha,\gamma} := C_{2,\alpha}\big([0,T]; H^\gamma(\mathbb{R})\big) \times H^\gamma(\mathbb{R}) \times H^{\gamma+2}(\mathbb{R})$$

given by the series

$$ u(x,t) = \sum_{k=1}^{\infty} u_k(t) X_k(x), \quad (x,t) \in Q_T \tag{7}$$

and (5), satisfying the equation (2) in $\mathcal{S}'(\mathbb{R})$ for every $t \in (0,T]$ and the conditions (3), (4), is called a *solution of the problem* (2)–(4).

Substituting the function (7) in the equation (2) and the conditions (3), (4), we obtain the problems

$$^{c}D_t^{\alpha}u_k + k^2u_k = F_{0k},\ t \in (0,T],\ \ u_k(0) = F_{1k}, \tag{8}$$

$$\int\limits_0^{t_0} u_k(t)dt = \Phi_{0k},\ \int\limits_0^{t_1} u_k(t)dt = \Phi_{1k}, \quad k \in \mathbb{N}, \tag{9}$$

for the unknown $u_k(t)$, $t \in [0,T]$ and F_{jk}, $j = 0,1$, $k \in \mathbb{N}$.

So, the vector-functions $(u_k(t), F_{0k}, F_{1k})$ $(k \in \mathbb{N})$ of the Fourier coefficients of the solution satisfy the inverse problems (8), (9).

We use the Mittag–Leffler function $E_{\alpha,\mu}(x) = \sum_{p=0}^{\infty} \frac{x^p}{\Gamma(p\alpha+\mu)}$.

The function $E_{\alpha,\mu}(-x)$ $(x > 0)$ is infinitely differentiable for $\alpha \in (0,1]$, $\mu > 0$ and compactly monotonic. We have $0 < E_{\alpha,\mu}(-k^2t^{\alpha}) < 1$ for all $t > 0$, $\mu \geq \alpha$,

$$E_{\alpha,\mu}(-x) \leq \frac{r_{\alpha,\mu}}{1+x}, \quad x > 0, \quad \text{where } r_{\alpha,\mu} \text{ is a positive constant,}$$

and the asymptotic behavior [12]

$$E_{\alpha,\mu}(-x) = O\Big(\frac{1}{x}\Big), \quad x \to +\infty.$$

Theorem 1. *Assume that $\gamma \in \mathbb{R}$, $F_0 \in H^{\gamma}(\mathbb{R})$, $F_1 \in H^{\gamma+2}(\mathbb{R})$. Then there exists a unique solution $u \in C_{2,\alpha}\big([0,T]; H^{\gamma}(\mathbb{R})\big)$ to the direct problem* (2), (3). *It is given by* (7) *where*

$$u_k(t) = F_{0k}k^{-2}\big[1 - E_{\alpha,1}(-k^2t^{\alpha})\big] + F_{1k}E_{\alpha,1}(-k^2t^{\alpha}), \quad t \in [0,T], \quad k \in \mathbb{N}. \tag{10}$$

The solution depends continuously on the data (F_0, F_1), and the following inequality holds:

$$||u||_{C_{2,\alpha}\big([0,T];H^{\gamma}(\mathbb{R})\big)} \leq a_0||F_0||_{H^{\gamma}(\mathbb{R})} + a_1||F_1||_{H^{\gamma+2}(\mathbb{R})}, \tag{11}$$

where a_j, $j \in \{0,1\}$ are positive constants independent of data.

Proof. It follows from the theorem 1 in [8] that there exists the unique solution $u \in C_{2,\alpha}\big([0,T]; H^{\gamma}(\mathbb{R})\big)$ to the problem (2), (3) under the theorem's conditions, that it is given by (7) where

$$u_k(t) = F_{0k}\int\limits_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-k^2\tau^{\alpha})d\tau + F_{1k}E_{\alpha,1}(-k^2t^{\alpha}), \quad t \in [0,T], \quad k \in \mathbb{N}.$$

By the link

$$\lambda\int\limits_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda\tau^{\alpha})d\tau = 1 - E_{\alpha,1}(-\lambda t^{\alpha}),$$

we obtain the formulas (10) and, using [8, th.1], we obtain (11). This inequality implies that a solution of the problem is unique and depends continuously on the data. □

Theorem 2. *Assume that (A) holds. Then there exists a unique solution* $(u, F_0, F_1) \in \mathcal{M}_{\alpha,\gamma}$ *of the inverse problem* (2)–(4). *It is given by the Fourier series* (7) *and* (6) *where* $u_k(t)$ *are defined by* (10),

$$F_{0k} = \Big[\frac{\Phi_{0k}}{t_0} E_{\alpha,2}(-k^2 t_1^\alpha) - \frac{\Phi_{1k}}{t_1} E_{\alpha,2}(-k^2 t_0^\alpha)\Big] k^2\, G_k^{-1}, \tag{12}$$

$$F_{1k} = \Big[\frac{\Phi_{1k}}{t_1}\big(1 - E_{\alpha,2}(-k^2 t_0^\alpha)\big) - \frac{\Phi_{0k}}{t_0}\big(1 - E_{\alpha,2}(-k^2 t_1^\alpha)\big)\Big]\, G_k^{-1},$$

$$G_k = E_{\alpha,2}(-k^2 t_1^\alpha) - E_{\alpha,2}(-k^2 t_0^\alpha), \quad k \in \mathbb{N}.$$

The solution depends continuously on the data Φ_0, Φ_1 *and the following inequality holds:*

$$||u||_{C_{2,\alpha}\big([0,T];H^\gamma(\mathbb{R})\big)} + ||F_0||_{H^\gamma(\mathbb{R})} + ||F_1||_{H^{\gamma+2}(\mathbb{R})}$$

$$\le b_0||\Phi_0||_{H^{\gamma+4}(\mathbb{R})} + b_1||\Phi_1||_{H^{\gamma+4}(\mathbb{R})}, \tag{13}$$

where b_j, $j \in \{0, 1\}$ *are positive constants independent of data.*

Proof. Using (10), we write the conditions (9) as follows

$$F_{0k}k^{-2}\int\limits_0^{t_0}\Big[1 - E_{\alpha,1}(-k^2 t^\alpha)\Big]dt + F_{1k}\int\limits_0^{t_0} E_{\alpha,1}(-k^2 t^\alpha)dt = \Phi_{0k},$$

$$F_{0k}k^{-2}\int\limits_0^{t_1}\Big[1 - E_{\alpha,1}(-k^2 t^\alpha)\Big]dt + F_{1k}\int\limits_0^{t_1} E_{\alpha,1}(-k^2 t^\alpha)dt = \Phi_{1k},$$

$k \in \mathbb{N}$. Note that [8]

$$\int\limits_0^{t_j} E_{\alpha,1}(-k^2 t^\alpha)dt = t_j E_{\alpha,2}(-k^2 {t_j}^\alpha), \quad j = 0, 1, \quad k \in \mathbb{N}.$$

From here, according to the assumption (A), we find the expressions (12) for the unknown Fourier coefficients F_{jk}, $k \in \mathbb{N}$, $j = 0, 1$. The numbers $G_k \neq 0$ for all $k \in \mathbb{N}$ by the mentioned monotonic property of the Mittag–Leffler function.

Let us show that the founded solution belongs to $\mathcal{M}_{\alpha,\gamma}$.

Taking the behavior of the Mittag–Leffler function for large k and the formulas (12) into account, one obtains

$$(1+k)^\gamma |F_{0k}| \le c_0\Big[|\Phi_{0k}|(1+k)^{\gamma+2} + |\Phi_{1k}|(1+k)^{\gamma+2}\Big],$$

$$(1+k)^{\gamma+2}|F_{1k}| \le c_0\Big[|\Phi_{0k}|(1+k)^{\gamma+4} + |\Phi_{1k}|(1+k)^{\gamma+4}\Big], \ \ k \in \mathbb{N},$$

where c_0 is a positive constant, and therefore,

$$||F_0||_{H^\gamma(\mathbb{R})} \le c_0\Big[||\Phi_0||_{H^{\gamma+2}(\mathbb{R})} + ||\Phi_1||_{H^{\gamma+2}(\mathbb{R})}\Big],$$

$$||F_1||_{H^{\gamma+2}(\mathbb{R})} \le c_0\Big[||\Phi_0||_{H^{\gamma+4}(\mathbb{R})} + ||\Phi_1||_{H^{\gamma+4}(\mathbb{R})}\Big].$$

So, under the theorem's assumptions, $F_0 \in H^{\gamma}(\mathbb{R})$, $F_1 \in H^{\gamma+2}(\mathbb{R})$. Then, using (11), we obtain the inequality (13). The inequality (13) implies that a solution of the problem is unique and depends continuously on the problem's data. □

3. Remarks. 1. The obtained result can be transferred to the case of the boundary value problem for a time fractional diffusion equation

$$^cD_t^{\alpha}u - A(x,D)u = F_0(x),$$

where $A(x,D)$ is an elliptic differential expression of the second order with infinitely differentiable coefficients and when the corresponding Sturm–Liouville problem has positive eigenvalues.

2. In the case $\alpha \in (1,2)$

$$^cD_t^{\alpha}v(x,t) = \frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{v_{\tau\tau}(x,\tau)}{(t-\tau)^{\alpha-1}}d\tau = \frac{1}{\Gamma(2-\alpha)}\Big[\frac{\partial}{\partial t}\int_0^t \frac{v_{\tau}(x,\tau)}{(t-\tau)^{\alpha-1}}d\tau - \frac{v_t(x,0)}{t^{\alpha-1}}\Big]$$

and we may study the inverse problem

$$^cD_t^{\alpha}u - u_{xx} = F_0(x), \quad (x,t) \in Q_T := \mathbb{R}\times(0,T], \tag{14}$$

$$u(x,0) = F_1(x), \quad u_t(x,0) = F_2(x), \quad x \in \mathbb{R}, \tag{15}$$

$$\int_0^{t_0} u(x,t)dt = \Phi_0(x), \quad \int_0^{t_1} u(x,t)dt = \Phi_1(x), \quad x \in \mathbb{R}, \tag{16}$$

where Φ_0, Φ_1, F_2 are the given functions, T is a given positive number, u, F_0, F_1 are unknown functions, $t_0, t_1 \in (0,T]$, $t_0 \neq t_1$.

By [8, th.1], assuming $\gamma \in \mathbb{R}$, $\theta \in (0,1)$, $F_0 \in H^{\gamma+2\theta}(\mathbb{R})$, $F_j \in H^{\gamma+2}(\mathbb{R})$, $j = 1,2$, we obtain the existence of the unique solution $u \in C_{2,\alpha}\big([0,T]; H^{\gamma}(\mathbb{R})\big)$ to the direct problem (14), (15). It is given by (7) where

$$u_k(t) = F_{0k}k^{-2}\big[1-E_{\alpha,1}(-k^2t^{\alpha})\big]+F_{1k}E_{\alpha,1}(-k^2t^{\alpha})+F_{2k}tE_{\alpha,2}(-k^2t^{\alpha}), \ t \in [0,T], \ k \in \mathbb{N}. \tag{17}$$

The solution depends continuously on the data (F_0, F_1, F_2), and the following inequality holds:

$$||u||_{C_{2,\alpha}\big([0,T];H^{\gamma}(\mathbb{R})\big)} \leq a_0||F_0||_{H^{\gamma+2\theta}(\mathbb{R})} + \sum_{j=1}^{2} a_j||F_j||_{H^{\gamma+2}(\mathbb{R})}, \tag{18}$$

where a_j, $j \in \{0,1,2\}$ are positive constants independent of data.

Using (17), we write the conditions (16) as follows

$$F_{0k}k^{-2}\int_0^{t_0}\Big[1-E_{\alpha,1}(-k^2t^{\alpha})\Big]dt + F_{1k}\int_0^{t_0}E_{\alpha,1}(-k^2t^{\alpha})dt + F_{2k}\int_0^{t_0}tE_{\alpha,2}(-k^2t^{\alpha})dt = \Phi_{0k},$$

$$F_{0k}k^{-2}\int_0^{t_1}\Big[1-E_{\alpha,1}(-k^2t^{\alpha})\Big]dt + F_{1k}\int_0^{t_1}E_{\alpha,1}(-k^2t^{\alpha})dt + F_{2k}\int_0^{t_1}tE_{\alpha,2}(-k^2t^{\alpha})dt = \Phi_{1k},$$

$k \in \mathbb{N}$. Note that $\int\limits_0^{t_j} tE_{\alpha,2}(-k^2t^\alpha)dt = \dfrac{1}{\alpha k^{4/\alpha}} \int\limits_0^{k^2 {t_j}^\alpha} E_{\alpha,2}(-z)z^{\frac{2}{\alpha}-1}dz = t_j^2 E_{\alpha,3}(-k^2{t_j}^\alpha)$, $k \in \mathbb{N}$ and we have

$$F_{0k}k^{-2}t_0\Big[1 - E_{\alpha,2}(-k^2t_0^\alpha)\Big] + F_{1k}t_0E_{\alpha,2}(-k^2t_0^\alpha)dt = \Phi_{0k} - F_{2k}t_0^2E_{\alpha,3}(-k^2t_0^\alpha),$$

$$F_{0k}k^{-2}t_1\Big[1 - E_{\alpha,2}(-k^2t_1^\alpha)\Big] + F_{1k}t_1E_{\alpha,2}(-k^2t_1^\alpha) = \Phi_{1k} - F_{2k}t_1^2E_{\alpha,3}(-k^2t_1^\alpha),$$

$k \in \mathbb{N}$. From here if

$$G_k := E_{\alpha,2}(-k^2t_1^\alpha) - E_{\alpha,2}(-k^2t_0^\alpha) \neq 0 \ \forall k \in \mathbb{N}$$

we may find the expressions

$$\begin{aligned} F_{0k} &= \Big[\Big(\frac{\Phi_{0k}}{t_0} - F_{2k}t_0E_{\alpha,3}(-k^2t_0^\alpha)\Big)E_{\alpha,2}(-k^2t_1^\alpha) - \\ &-\Big(\frac{\Phi_{1k}}{t_1} - F_{2k}t_1E_{\alpha,3}(-k^2t_1^\alpha)\Big)E_{\alpha,2}(-k^2t_0^\alpha)\Big]k^2\ G_k^{-1}, \\ F_{1k} &= \Big[\Big(\frac{\Phi_{1k}}{t_1} - F_{2k}t_1E_{\alpha,3}(-k^2t_1^\alpha)\Big)\big(1 - E_{\alpha,2}(-k^2t_0^\alpha)\big) - \\ &-\Big(\frac{\Phi_{0k}}{t_0} - F_{2k}t_0E_{\alpha,3}(-k^2t_0^\alpha)\Big)\big(1 - E_{\alpha,2}(-k^2t_1^\alpha)\big)\Big]\ G_k^{-1} \end{aligned} \tag{19}$$

which imply the inequalities

$$||F_0||_{H^{\gamma+2\theta}(\mathbb{R})} \leq \sum_{j=0}^{1} c_j||\Phi_j||_{H^{\gamma+2+2\theta}(\mathbb{R})} + c_2||F_2||_{H^{\gamma+2\theta}(\mathbb{R})},$$

$$||F_1||_{H^{\gamma+2}(\mathbb{R})} \leq \sum_{j=0}^{1} c_j||\Phi_j||_{H^{\gamma+4}(\mathbb{R})} + c_2||F_2||_{H^{\gamma+2}(\mathbb{R})},$$

and, by using (18),

$$||u||_{C_{2,\alpha}\big([0,T];H^\gamma(\mathbb{R})\big)} \leq \sum_{j=0}^{1} c_j||\Phi_j||_{H^{\gamma+4}(\mathbb{R})} + c_2||F_2||_{H^{\gamma+2}(\mathbb{R})},$$

where c_j, $j \in \{0,1,2\}$ are positive constants independent of problem's data.

In the case $\alpha \in (1,2)$, the function $E_{\alpha,2}(-z)$ does not monotonic. But it has a finite number of real positive zeroes. So, assuming $\Phi_0, \Phi_1 \in H^{\gamma+4}(\mathbb{R})$, $F_2 \in H^{\gamma+2}(\mathbb{R})$, under severe constraints of existing $t_0, t_1 \in (0,T]$ such that $t_0 \neq t_1$ and $G_k \neq 0$ for all $k \in \mathbb{N}$ we obtain that there exists the unique solution

$$(u, F_0, F_1) \in \mathcal{M}_{\alpha,\gamma,\theta} := C_{2,\alpha}\big([0,T];H^\gamma(\mathbb{R})\big) \times H^{\gamma+2\theta}(\mathbb{R}) \times H^{\gamma+2}(\mathbb{R})$$

of the inverse problem (14)—(16) for all $\theta \in (0,1)$, that it is given by the Fourier series (7) and (5) with u_k defined by (17), F_{0k}, F_{1k} defined by (19) and depends continuously on the data Φ_0, Φ_1, F_2.

4. Conclusion. For a time fractional sub-diffusion equation we prove the correctness of the inverse problem which is to find a classical in time with values in the space of periodic spatial distributions solution for the direct problem, its initial data and a source term of the equation under the same kind time integral over-determination conditions. More complicated situation is for a time fractional super-diffusion equation. The numbers t_1, t_2 can not be arbitrarily taken from $(0, T]$ in such type over-determination conditions.

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ВІДНОВЛЕННЯ ПОЧАТКОВИХ ДАНИХ РОЗВ'ЯЗКУ ТА ПРАВОЇ ЧАСТИНИ РІВНЯННЯ ДРОБОВОЇ ДИФУЗІЇ В ПРОСТОРІ ПЕРІОДИЧНИХ УЗАГАЛЬНЕНИХ ФУНКЦІЙ

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Доводимо коректність оберненої задачі для рівняння суб-дифузії з регуляризованою похідною дробового порядку за часом, що полягає у знаходженні розв'язку прямої задачі, класичного за часом зі значеннями в просторі періодичних узагальнених функцій, його невідомих початкових даних і правої частини рівняння. З'ясовуємо, що можна використовувати однакового вигляду інтегральні за часом умови перевизначення.

Ключові слова: похідна дробового порядку, обернена задача, періодична узагальнена функція, інтегральна за часом умова перевизначення.