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ON DECREASE OF NONPOSITIVE $\mathcal{M}$-SUBHARMONIC FUNCTIONS IN THE UNIT BALL

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We describe asymptotic behavior of nonpositive $\mathcal{M}$-subharmonic functions in the unit ball in terms of smoothness properties of the measure which is determined by the Riesz measure and boundary values on the unit sphere. We generalize recent results of the second author using more general growth (decrease) scale.

Key words: $\mathcal{M}$-subharmonic function, Green potential, unit ball, Riesz measure

Let us introduce definitions and notations which will be used in this paper. Let $\mathbb{C}^n$, $n \in \mathbb{N}$ denote the n-dimensional complex space with the inner product $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w}_j$, and the corresponding norm $|z| = \sqrt{\langle z, z\rangle}$, $z, w \in \mathbb{C}^n$. We denote by B the unit ball $\{z \in \mathbb{C}^n : |z| < 1\}$ with the boundary $S = \{z \in \mathbb{C}^n : |z| = 1\}$.

Let u be a measurable function locally integrable on B. For $0 < p < \infty$ we define

$$m_p(r, u) = \left(\int_S |u(r\xi)|^p \, d\sigma(\xi)\right)^{\frac{1}{p}}, \ 0 < r < 1,$$

where $d\sigma$ is the Lebesgue measure on the unit sphere S normalized so that $\sigma(S) = 1$.

We are interested in description of asymptotic behavior of $m_p(r, u)$, where u belongs to a class of subharmonic functions in B. Such problems were considered in [4], [5], [8], [1], [2], [10]. The aim of this paper is to generalize results [10] in two directions. We start with definition of $\mathcal{M}$-subharmonic functions which are the main object of our research.

For $z, w \in B$, define the *involutive automorphism* φ_w of the unit ball B given by

$$\varphi_w(z) = \frac{w - P_w z - (1 - |w|^2)^{1/2} Q_w z}{1 - \langle z, w\rangle}$$

where $P_0 z = 0$, $P_w z = \frac{\langle z, w\rangle}{|w|^2} w$, $w \neq 0$, is the orthogonal projection of $\mathbb{C}^n$ onto the subspace generated by w and $Q_w = I - P_w$ ([6, 7]).

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The *invariant Laplacian* $\tilde{\Delta}$ on B is defined by

$$\tilde{\Delta}f(a) = \Delta(f \circ \varphi_a)(0),$$

where $f \in C^2(B)$, Δ is the ordinary Laplacian. We note that $u \in C^2$ is an $\mathcal{M}$-subharmonic function if and only if $(\tilde{\Delta}u)(a) \geqslant 0$ for all $a \in B$.

The *Green function* for the invariant Laplacian ([3], [9], [7, Chap. 6.2]) is defined by $G(z, w) = g(\varphi_w(z))$, where $g(z) = \frac{n+1}{2n}\int_{|z|}^{1}(1-t^2)^{n-1}t^{-2n+1}dt$.

If μ is a nonnegative Borel measure on B, the function G_μ defined by

$$G_\mu(z) = \int_B G(z, w)d\mu(w)$$

is called the *(invariant) Green potential* of μ, provided $G_\mu \not\equiv +\infty$. It is known that ([7, Chap. 6.4]) the last condition is equivalent to

$$\int_B (1-|w|^2)^n d\mu(w) < \infty. \tag{1}$$

An upper semicontinuous function $u : B \to [-\infty, \infty)$, with $u \not\equiv -\infty$, is $\mathcal{M}$-*subharmonic* on B if

$$u(a) \leqslant \int_S u(\varphi_a(r\xi))d\sigma(\xi) \tag{2}$$

for all $a \in B$ and all r sufficiently small. A continuous function u for which the equality holds in (2) is said to be $\mathcal{M}$-*harmonic* on B.

Let $C_0^2(B)$ denote the class of twice continuously differentiable functions with compact support in B. For the $\mathcal{M}$-subharmonic functions the following theorem holds.

Theorem A ([7]). *If u is $\mathcal{M}$-subharmonic on B, then there exists a unique Borel measure μ_u on B such that*

$$\int_B \psi d\mu_u = \int_B u\tilde{\Delta}\psi d\tau \tag{3}$$

for all $\psi \in C_0^2(B)$, where $d\tau(z) = \frac{dA(z)}{(1-|z|^2)^{n+1}}$ is the invariant volume measure on B, A is the volume measure, i.e. $d\mu_u = \tilde{\Delta}u d\tau$ in the sense of distributions.

If u is $\mathcal{M}$-subharmonic on B, the unique Borel measure μ_u satisfying (3) is called the *Riesz measure* of u.

For $z \in B$, $\xi \in S$

$$\mathcal{P}(z, \xi) = \left\{\frac{1-|z|^2}{|1-\langle z, \xi\rangle|^2}\right\}^n, \quad \mathcal{P}[\mu](z) = \int_S \mathcal{P}(z, \xi)d\mu(\xi) \tag{4}$$

are called the *Poisson kernel* of B and the *Poisson integral*, respectively, where μ is a complex Borel measure on S.

An $\mathcal{M}$-subharmonic function u on B has an $\mathcal{M}$-*harmonic majorant* on B if there exists an $\mathcal{M}$-harmonic function h on B such that $u(z) \leqslant h(z)$ for all $z \in B$. Furthermore, if there exists an $\mathcal{M}$-harmonic function H satisfying $u(z) \leqslant H(z)$, for all $z \in B$, and $H(z) \leqslant h(z)$ for any $\mathcal{M}$-harmonic majorant h of u, then H is called the *least $\mathcal{M}$-harmonic majorant* of u, and will be denoted by H_u.

Theorem B (Riesz Decomposition Theorem, [9, Th.2.16]**).** *Suppose that $u \not\equiv -\infty$ is $\mathcal{M}$-subharmonic on B and has an $\mathcal{M}$-harmonic majorant on B. Then*

$$u(z) = H_u(z) - \int_B G(z,w)d\mu_u(w), \tag{5}$$

where μ_u is the Riesz measure of u and H_u is the least $\mathcal{M}$-harmonic majorant u.

Note that, if $u \leqslant 0$, $u \not\equiv -\infty$, is $\mathcal{M}$-subharmonic on B, then $v \equiv 0$ is an $\mathcal{M}$-harmonic majorant. Therefore, for H_u in representation (5), we have $H_u(z) \leqslant 0$, $z \in B$. Since, every (nonnegative) $\mathcal{M}$-harmonic function on B can be represented by the Poisson integral ([7, Prop. 5.10]), there exists a nonnegative Borel measure ν on S such that

$$H_u(z) = -\mathcal{P}[\nu](z),\ z \in B. \tag{6}$$

Define for $a, b \in \bar{B}$ the *nonisotropic metric* on S by $d(a,b) = |1 - \langle a,b\rangle|^{1/2}$ ([6, Chap. 5.1]). For $\xi \in S$ and $\delta > 0$ we set $C(\xi,\delta) = \{z \in B : d(z,\xi) < \delta^{1/2}\}$.

Given an $\mathcal{M}$-subharmonic function u, let us define

$$d\lambda(w) = \frac{4n^2}{n+1}d\nu(w) + (1-|w|^2)^n d\mu_u(w) \tag{7}$$

for $w \in \bar{B}$, where ν is the nonnegative Borel measure on S defined by (6), μ_u is its Riesz measure.

In [10] the second author described the growth of pth means, $1 < p < \frac{2n-1}{2(n-1)}$, of nonpositive $\mathcal{M}$-subharmonic functions in the unit ball in $\mathbb{C}^n$ in terms of smoothness properties of the measure λ.

Theorem C. *Let u be a nonpositive $\mathcal{M}$-subharmonic function in B, $u \not\equiv -\infty$, $1 < p < \frac{2n-1}{2(n-1)}$, $0 \leqslant \gamma < 2n$, $n \in \mathbb{N}$. Then*

$$m_p(r,u) = O\left((1-r)^{\gamma-n}\right),\ r \uparrow 1 \tag{8}$$

holds if and only if

$$\left(\int_S \lambda^p\left(C(\xi,\delta)\right)d\sigma(\xi)\right)^{\frac{1}{p}} = O\left(\delta^\gamma\right),\ 0 < \delta < 1. \tag{9}$$

As a corollary we have got criteria of the growth of pth means, $1 < p < \frac{2n-1}{2(n-1)}$, of the invariant Green potential in the unit ball in $\mathbb{C}^n$ in terms of smoothness properties of the Riesz measure ([2]) that generalize a result of M. Stoll [8].

Note that in Theorem C the growth of $\mathcal{M}$-subharmonic functions is described by using power functions, and the case $\gamma = 2n$ is not covered. An example from [1] shows that (9) does not imply (8) in the case $n = 1$ and $\gamma = 2$. More precisely, it is proved that for the measure $d\mu(w) = \frac{dA(w)}{1-|w|}$ one has

$$m_p(r,u) = (1+o(1))2\pi(1-r)\log\frac{1}{1-r}, \quad r \uparrow 1, \tag{10}$$

while $\left(\int_0^{2\pi}\lambda^p(\mathcal{C}(\varphi,\delta))\,d\varphi\right)^{\frac{1}{p}} \asymp \delta^2,\ \delta \to 0+$.

The purpose of this paper is to describe the asymptotic behavior of the pth means of $\mathcal{M}$-subharmonic functions by using a wider class than that of power functions and to investigate the case $\gamma = 2n$.

The following theorem is a generalization of Theorem C and the main result of this paper.

Theorem 1. *Let u be a nonpositive $\mathcal{M}$-subharmonic function in B, $n \in \mathbb{N}$, $u \not\equiv -\infty$, $1 \leqslant p < \frac{2n-1}{2(n-1)}$. Let $\Phi\colon [0,2] \to [0,\infty)$ be a function such that for all $t > 1$ and $0 < t\delta < 2$ we have*

$$\Phi(t\delta) = O\left(\frac{t^{2n}}{\psi(\log(e+t))}\Phi(\delta)\right) \tag{11}$$

for some positive increasing function ψ satisfying $\int_1^\infty \frac{dt}{\psi(t)} < \infty$ and $\psi(ct) \asymp \psi(t)$, $c > 1$. Then

$$m_p(r,u) = O\left(\frac{\Phi(1-r)}{(1-r)^n}\right), \ r \uparrow 1 \tag{12}$$

holds if and only if

$$\left(\int_S \lambda^p\left(C(\xi,\delta)\right) d\sigma(\xi)\right)^{\frac{1}{p}} = O\left(\Phi(\delta)\right), \ 0 < \delta < 1. \tag{13}$$

Specifying ψ in Theorem 1 we can get the following corollary.

Corollary 1. *Let u be a nonpositive $\mathcal{M}$-subharmonic function in B, $u \not\equiv -\infty$, $1 \leq p < \frac{2n-1}{2(n-1)}$. Let $\varepsilon > 0$, $\Phi\colon [0,2] \to [0,\infty)$ be an increasing function such that for all $t > 1$ and $0 < t\delta \leqslant 2$ we have $\Phi(t\delta) = O\left(\frac{t^{2n}}{\log^{1+\varepsilon}(e+t)}\Phi(\delta)\right)$. Then*

$$m_p(r,u) = O\left(\frac{\Phi(1-r)}{(1-r)^n}\right), \ r \uparrow 1$$

holds if and only if

$$\left(\int_S \lambda^p\left(C(\xi,\delta)\right) d\sigma(\xi)\right)^{\frac{1}{p}} = O\left(\Phi(\delta)\right), \ 0 < \delta < 1.$$

Example 1. The function $\Phi(t) = \frac{t^\alpha}{\log^\beta \frac{e}{t}}$, $t \in [0,2]$, where $\alpha \in (0,2n)$, $\beta \in \mathbb{R}$ satisfies the assumptions of Theorem 1.

In fact, let firstly, $\beta \leqslant 0$, $0 < \alpha < 2n$. Since $(\log \frac{e}{\delta})^\beta < (\log \frac{e}{t\delta})^\beta$, $t > 1$ we get

$$\Phi(t\delta) = \frac{t^\alpha \delta^\alpha}{\log^\beta \frac{e}{t\delta}} \leqslant \frac{c_0 t^{2n}}{\log^k(e+t)} \frac{\delta^\alpha}{\log^\beta \frac{e}{\delta}} = \frac{c_0 t^{2n}}{\log^k(e+t)}\Phi(\delta),$$

where $k > 1$, $c_0 = \max_{t>1} \frac{\log^k(1+t)}{t^{2n-\alpha}}$.

Now, let $\beta > 0$, $0 < \alpha < 2n$. Since

$$\frac{(\log \frac{e}{\delta})^\beta}{(\log \frac{e}{t\delta})^\beta} = \left(1 + \frac{\log t}{\log \frac{e}{t\delta}}\right)^\beta \leq \left(1 + \frac{\log t}{\log \frac{e}{2}}\right)^\beta, \ 0 < t\delta \leqslant 2$$

we deduce

$$\Phi(t\delta) = \frac{t^\alpha \delta^\alpha}{\log^\beta \frac{e}{t\delta}} \leqslant \frac{\tilde{c_0} t^{2n}}{\log^k(e+t)} \Phi(\delta),$$

where $k > 1$, $\tilde{c_0} = \max\limits_{t>1} \dfrac{(1+\frac{\log t}{\log \frac{e}{2}})^\beta \log^k(e+t)}{t^{2n-\alpha}}$. Thus, the function $\Phi(t)$ satisfies inequality (11) with $\psi(x) = x^k,\ k > 1$.

One can check that the function $\Phi(t) = \frac{t^{2n}}{\log^\beta \frac{e}{t}}$, $t \in (0, 2]$, $\Phi(0) = 0$ does not satisfy (11) for any $\beta \in \mathbb{R}$. In this limit case we have the following statement.

Theorem 2. *Let u be a nonpositive $\mathcal{M}$-subharmonic function in B, $n \in \mathbb{N}$, $u \not\equiv -\infty$, $1 \le p < \frac{2n-1}{2(n-1)}$. Let $\beta > 1$ and $\varkappa > 1$. If*

$$\left(\int_S \lambda^p\left(C(\xi,\delta)\right) d\sigma(\xi)\right)^{\frac{1}{p}} = O\left(\delta^{2n} \log^\beta \frac{e}{\delta}\right),\ 0 < \delta < 2, \tag{14}$$

then

$$m_p(r,u) = O\left((1-r)^n \log^{\beta+\varkappa} \frac{e}{1-r}\right),\ r \uparrow 1. \tag{15}$$

Proof of Theorem 1. Sufficiency. Let us define the kernel

$$K(z,w) = \begin{cases} \dfrac{G(z,w)}{(1-|w|^2)^n}, & \text{if } w \in B,\ z \in B; \\ \dfrac{n+1}{4n^2} \mathcal{P}(z,\xi), & \text{if } w \in S,\ z \in B. \end{cases}$$

The following properties of $K(z,w)$ are described in [10] and will be used later.

Proposition A. *For $z, w = \rho\xi \in \bar{B}$ the following hold:*

a) *For $w \in \{w :\ |\varphi_w(z)| \geqslant \frac{1}{4}\}$ the inequality*

$$0 \leqslant K(z,w) \leqslant c \frac{(1-|z|^2)^n}{|1-\langle z,w\rangle|^{2n}} \tag{16}$$

holds for some $c > 0$.

b) $\lim\limits_{\rho\to 1-} \frac{G(z,\rho\xi)}{(1-\rho^2)^n} = \frac{n+1}{4n^2}\mathcal{P}(z,\xi)$ *uniformly in $\xi \in S$.*

c)

$$|K(z,w)| \geqslant \frac{n+1}{4n^2} \frac{(1-|z|^2)^n}{|1-\langle z,w\rangle|^{2n}},\ z \in B,\ w \in \bar{B}. \tag{17}$$

Denote

$$B^*\Big(z, \frac{1}{4}\Big) = \Big\{w \in B : |\varphi_w(z)| < \frac{1}{4}\Big\}.$$

Since representation for $\mathcal{M}$-subharmonic functions (5) can be rewritten as

$$u(z) = -\int_{\bar{B}} K(z,w) d\lambda(w),$$

let us estimate the absolute values of

$$u_1(z) := \int\limits_{B^*\left(z,\frac14\right)} K(z,w)d\lambda(w) \quad \text{and} \quad u_2(z) := \int\limits_{B\setminus B^*\left(z,\frac14\right)} K(z,w)d\lambda(w).$$

We start with u_1. In this case

$$d\lambda(w) = (1-|w|^2)^n d\mu_u(w) \quad \text{and} \quad u_1(z) = \int\limits_{B^*\left(z,\frac14\right)} G(z,w)d\mu_u(w).$$

Thus we can use the same method of proof as that in the proof of Theorem 1.5 ([2]).

By definition

$$0 \leqslant u_1(z) = \int_{B^*\left(z,\frac14\right)} G(z,w)d\mu(w) = \int_{B^*\left(z,\frac14\right)} g(\varphi_w(z))d\mu(w).$$

Now we need the following lemma.

Lemma A ([7]). *Let $0<\delta<\frac12$ be fixed. Then g satisfies the following two inequalities:*

$$g(z) \geqslant \frac{n+1}{4n^2}(1-|z|^2)^n, \ z\in B,$$

$$g(z) \leqslant c(\delta)(1-|z|^2)^n, \ z \in B, |z| \geqslant \delta, \tag{18}$$

where $c(\delta)$ is a positive constant. Furthermore, if $n>1$ then

$$g(z) \asymp |z|^{-2n+2}, \qquad |z| \leqslant \delta. \tag{19}$$

By (19) we have $g(z) \leqslant c|z|^{-2n+2}$ for $|z| \leqslant \frac14$ and some positive constant c. Thus,

$$|u_1(z)| \leqslant c\int_{B^*\left(z,\frac14\right)} |\varphi_w(z)|^{-2n+2}\, d\mu(w).$$

Denote $z = r\xi$, where $r=|z|$, $\frac12<r<1$ and $w=|w|\eta$, $\xi,\eta\in S$. Let

$$K(z,\sigma_1,\sigma_2) = \{w \in B : |r-|w|| \leqslant \sigma_1, d(\xi,\eta) \leqslant \sigma_2\}.$$

The following inclusion is proved in [2]

$$B^*\left(z,\frac14\right) \subset K\left(z, \frac23(1-r), 4\sqrt2(1-r)^{\frac12}\right). \tag{20}$$

We denote

$$K(z) := K\left(z, \frac23(1-r), 4\sqrt2(1-r)^{\frac12}\right),$$
$$\tilde K(z) := K\left(z, \frac23(1-r), 8\sqrt2(1-r)^{\frac12}\right).$$

In [2] it is proved that

$$I_1 := \int_S |u_1(r\xi)|^p d\sigma(\xi)$$
$$\leqslant c_1(1-r)^n \int\limits_{||w|-r|<\frac{2}{3}(1-r)} \mu^{p-1}\left(\tilde{K}\,(r\eta)\right) d\mu(|w|\eta).$$

To obtain the final estimate of I_1, for a fixed $r \in (\frac{1}{2}, 1)$, we define the measure ν_1 on the balls $\{D(\eta, t) : \eta \in S, t > 0\}$ by

$$\nu_1(D(\eta,t)) = \lambda\Big(\Big\{\rho\zeta \in B : |\rho - r| < \frac{2}{3}(1-r), d(\zeta,\eta) < t\Big\}\Big).$$

It can be expanded to the family of all Borel sets on B in the standard way. It is clear that

$$\nu_1(D(\eta,t)) \asymp (1-r)^n \mu\Big(\Big\{\rho\zeta \in B : |\rho - r| < \frac{2}{3}(1-r), d(\zeta,\eta) < t\Big\}\Big).$$

Lemma B ([2]). *Let ν be a finite positive Borel measure on S, $0 < \delta < \frac{1}{2}$, and $p \geq 1$. Then*

$$\int_S \nu^{p-1}(D(\xi,\delta))d\nu(\xi) \leqslant \frac{N^p}{\delta^{2n}} \int_S \nu^p(D(\xi,\delta))\,d\sigma(\xi),$$

where N is a positive constant independent of p and δ.

By using Lemma B we get

$$I_1 \leq \frac{c_2}{(1-r)^{n(p-1)}} \int\limits_{||w|-r|<\frac{2}{3}(1-r)} \lambda^{p-1}\left(\tilde{K}\,(r\eta)\right) d\lambda(|w|\eta)$$
$$= \frac{c_2}{(1-r)^{n(p-1)}} \int_S \nu_1^{p-1}\left(D(\eta, 8\sqrt{2}(1-r)^{\frac{1}{2}})\right) d\nu_1(\eta)$$
$$\leqslant \frac{c_2 N^p}{(128)^n (1-r)^{np}} \int_S \nu_1^p\left(D(\eta, 8\sqrt{2}(1-r)^{\frac{1}{2}})\right) d\sigma(\eta)$$
$$= \frac{c_3(n,p)}{(1-r)^{np}} \int_S \lambda^p\left(\tilde{K}(r\eta)\right) d\sigma(\eta).$$

Note that if $\rho\zeta \in \tilde{K}(r\eta)$ then

$$|1 - \langle\rho\zeta,\eta\rangle| \leqslant |1 - \langle\zeta,\eta\rangle| + (1-\rho)\,|\langle\zeta,\eta\rangle| \leqslant (4c_4^2 + c_5 + 1)(1-r). \tag{21}$$

Hence, by the assumption of the theorem

$$I_1 \leqslant c_3(1-r)^{-np} \int_S \lambda^p\left(C(\eta, (4c_2^2 + c_1 + 1)(1-r))\right) d\sigma(\eta)$$
$$\leqslant c_6(1-r)^{-np}\Phi^p((4c_2^2 + c_1 + 1)(1-r)). \tag{22}$$

Since for all $t > 1$ and $0 < t\delta < 2$ the inequality $\Phi(t\delta) \leq \frac{t^{2n}}{\psi(\log(e+t))}\Phi(\delta)$ holds we get

$$I_1 \leqslant c_6(1-r)^{-np}\frac{(4c_2^2 + c_1 + 1)^{2np}}{\psi^p(\log(4c_2^2 + c_1 + 2))}\Phi^p(1-r) = \tilde{c_6}\frac{\Phi^p(1-r)}{(1-r)^{np}}$$

where ψ is a positive increasing function satisfying $\int_1^\infty \frac{dt}{\psi(t)} < \infty$.

$$\int_S |u_1(r\xi)|^p d\sigma(\xi) \leqslant \tilde{c_6}\left(\frac{\Phi(1-r)}{(1-r)^n}\right)^p. \tag{23}$$

Let us estimate

$$u_2(z) = -\int_B K(z,w)d\tilde{\lambda}(w)$$

where $d\tilde{\lambda}(w) = \frac{4n^2}{n+1}d\nu(w)+(1-|w|)^n\chi_{B\setminus B^*\left(z,\frac{1}{4}\right)}(w)d\mu(w)$, χ_E is the characteristic function of a set E. We may assume that $|z| \geq \frac{1}{2}$.

We denote

$$E_k = E_k(z) = \left\{w\in B : \left|1-\left\langle\frac{z}{|z|}, w\right\rangle\right| < 2^{k+1}(1-|z|)\right\}, \quad k\in\mathbb{Z}_+.$$

One has that for $w \in E_{k+1}\setminus E_k$, $|1-\langle z,w\rangle| \geqslant 2^{k-1}(1-|z|)$.

By (16) we get that

$$|u_2(z)| \leqslant c\int_B \frac{(1-|z|^2)^n}{|1-\langle z,w\rangle|^{2n}}d\tilde{\lambda}(w) \leqslant c\int_B \frac{(1+|w|)^n(1-|z|^2)^n}{|1-\langle z,w\rangle|^{2n}}d\tilde{\lambda}(w).$$

Arguments from [2] give

$$|u_2(z)| \leqslant \frac{4^n c_7}{(1-|z|)^n}\sum_{k=1}^{[\log_2 \frac{1}{1-r}]}\frac{\tilde{\lambda}(E_k)}{2^{2n(k-2)}}.$$

By Hölder's inequality ($\frac{1}{p}+\frac{1}{q}=1$)

$$\left(\sum_{k=1}^{[\log_2 \frac{1}{1-r}]}\frac{\tilde{\lambda}(E_k)}{2^{2nk}}\right)^p = \left(\sum_{k=1}^{[\log_2 \frac{1}{1-r}]}\frac{\tilde{\lambda}(E_k)(\psi(k))^{\frac{1}{q}}}{2^{2nk}(\psi(k))^{\frac{1}{q}}}\right)^p$$

$$\leqslant \sum_{k=1}^{[\log_2 \frac{1}{1-r}]}\frac{\tilde{\lambda}^p(E_k)(\psi(k))^{\frac{p}{q}}}{2^{2npk}}\left(\sum_{k=1}^{\infty}\frac{1}{\psi(k)}\right)^{p/q} \leq C\sum_{k=1}^{[\log_2 \frac{1}{1-r}]}\frac{\tilde{\lambda}^p(E_k)(\psi(k))^{p-1}}{2^{2npk}}. \tag{24}$$

Therefore

$$\begin{aligned}\int_S |u_2(r\xi)|^p d\sigma(\xi) \leqslant& \frac{c_8}{(1-r)^{np}}\sum_{k=1}^{[\log_2 \frac{1}{1-r}]}\int_S \frac{\tilde{\lambda}^p(E_k(r\xi))(\psi(k))^{p-1}}{2^{2npk}}d\sigma(\xi)\\ =& \frac{c_8}{(1-r)^{np}}\sum_{k=1}^{[\log_2 \frac{1}{1-r}]}\frac{(\psi(k))^{p-1}}{2^{2npk}}\int_S \tilde{\lambda}^p\left(C\left(\xi, 2^{k+1}(1-r)\right)\right)d\sigma(\xi)\\ \leq& \frac{c_8}{(1-r)^{np}}\sum_{k=1}^{[\log_2 \frac{1}{1-r}]}\frac{(\psi(k))^{p-1}}{2^{2npk}}\Phi^p(2^{k+1}(1-r))\\ \leq& \frac{c_8\Phi^p(1-r)}{(1-r)^{np}}\sum_{k=1}^{[\log_2 \frac{1}{1-r}]}\left(\frac{2^{(k+1)2n}}{\psi(\log(e+2^{k+1}))}\right)^p\frac{(\psi(k))^{p-1}}{2^{2npk}}\end{aligned}$$

$$\leq \frac{c_8\Phi^p(1-r)}{(1-r)^{np}}\sum_{k=1}^{\infty}\frac{\psi^{p-1}(k)}{\psi^p(k\log 2)} \leq \frac{c_9\Phi^p(1-r)}{(1-r)^{np}}\sum_{k=1}^{\infty}\frac{1}{\psi(k)}$$
$$\leq \frac{c_{10}\Phi^p(1-r)}{(1-r)^{np}}.$$

Thus

$$\int_S |u_2(r\xi)|^p d\sigma(\xi) \leqslant \frac{\tilde{c}_{10}(n,p,\gamma)\Phi^p(1-r)}{(1-r)^{np}}.$$

The latter inequality together with (23) completes the proof of the sufficiency.

Necessity. By (17)

$$|u(z)| \geqslant \int_B K(z,w)d\lambda(w) \geqslant \frac{n+1}{4n^2}\int_B \frac{(1-|z|^2)^n}{|1-\langle z,w\rangle|^{2n}}d\lambda(w)$$
$$\geqslant \frac{n+1}{4n^2}\int_{C(\xi,1-r)} \frac{(1-|z|^2)^n}{|1-\langle z,w\rangle|^{2n}}d\lambda(w).$$

Further we argue as in the proof of Theorem 1.5 ([2]). Since, for $w \in C(\xi, 1-r)$, $|1-\langle z,w\rangle| \leqslant 2(1-|z|)$, we have

$$|u(z)| \geqslant \frac{n+1}{4^{n+1}n^2}\frac{\lambda(C(\xi,1-r))}{(1-r)^n}.$$

From the assumption of the theorem it follows that

$$\left(\frac{n+1}{2^{2(n+1)}n^2}\right)^p \int_S \frac{\lambda^p(C(\xi,1-r))}{(1-r)^{np}}d\sigma(\xi) \leqslant \int_S |u(r\xi)|^p d\sigma(\xi) \leqslant c_{11}^p\frac{\Phi^p(1-r)}{(1-r)^{np}}.$$

Thus

$$\int_S \lambda^p(C(\xi,1-r))d\sigma(\xi) \leqslant c_{11}^p\Phi^p(1-r), \quad 0<r<1.$$

□

Proof of Theorem 2. We choose $\psi(x) = x^{\varkappa}$. Repeating arguments from the proof of the sufficiency of Theorem 1 with $\Phi(\delta) = \delta^{2n}\log^{\beta}\frac{e}{\delta}$ we arrive to the estimate

$$\int_S |u_2(r\xi)|^p d\sigma(\xi) \leqslant \frac{c_8}{(1-r)^{np}}\sum_{k=1}^{[\log_2\frac{1}{1-r}]}\frac{k^{(p-1)\varkappa}}{2^{2npk}}(2^{k+1}(1-r))^{2np}\log^{\beta p}\frac{e}{2^{k+1}(1-r)}$$
$$= c_8 4^{np}(1-r)^{np}\log^{p(\beta+\varkappa)}\frac{e}{1-r}\sum_{k=1}^{[\log_2\frac{1}{1-r}]}k^{(p-1)\varkappa}\frac{\log^{\beta p}\frac{e}{2^{k+1}(1-r)}}{(\log\frac{e}{2^{k+1}(1-r)}+\log 2^{k+1})^{p(\beta+\varkappa)}}$$
$$\leq c_8 4^{np}(1-r)^{np}\log^{p(\beta+\varkappa)}\frac{e}{1-r}\sum_{k=1}^{[\log_2\frac{1}{1-r}]}k^{(p-1)\varkappa}\frac{1}{(\log\frac{e}{2}+\log 2^{k+1})^{p\varkappa}}$$
$$\leq c_{12}(1-r)^{np}\log^{p(\beta+\varkappa)}\frac{e}{1-r}\sum_{k=1}^{\infty}\frac{1}{k^{\varkappa}}$$
$$\leq c_{13}(1-r)^{np}\log^{p(\beta+\varkappa)}\frac{e}{1-r}.$$

This yields (15). Theorem 2 is proved. □

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ПРО СПАДАННЯ НЕДОДАТНОЇ $\mathcal{M}$-СУБГАРМОНІЙНОЇ ФУНКЦІЇ В ОДИНИЧНІЙ КУЛІ

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Описано асимптотичну поведінку недодатних $\mathcal{M}$-субгармонічних функцій в одиничній кулі в термінах гладкості міри, яка визначається її мірою Рісса та межовими значеннями на одиничній сфері. Узагальнено недавні результати другого автора, використовуючи загальнішу шкалу зростання (спадання).

Ключові слова: $\mathcal{M}$-субгармонійна функція, потенціал Гріна, одинична куля, міра Рісса.