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ON FEEBLY COMPACT SEMITOPOLOGICAL SYMMETRIC INVERSE SEMIGROUPS OF A BOUNDED FINITE RANK

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We study feebly compact shift-continuous T_1-topologies on the symmetric inverse semigroup $\mathscr{I}_\lambda^n$ of finite transformations of the rank $\leqslant n$. For any positive integer $n \geqslant 2$ and any infinite cardinal λ a Hausdorff countably pracompact non-compact shift-continuous topology on $\mathscr{I}_\lambda^n$ is constructed. We show that for an arbitrary positive integer n and an arbitrary infinite cardinal λ for a T_1-topology τ on $\mathscr{I}_\lambda^n$ the following conditions are equivalent: (i) τ is countably pracompact; (ii) τ is feebly compact; (iii) τ is d-feebly compact; (iv) $(\mathscr{I}_\lambda^n, \tau)$ is H-closed; (v) $(\mathscr{I}_\lambda^n, \tau)$ is $\mathbb{N}_\mathfrak{d}$-compact for the discrete countable space $\mathbb{N}_\mathfrak{d}$; (vi) $(\mathscr{I}_\lambda^n, \tau)$ is $\mathbb{R}$-compact; (vii) $(\mathscr{I}_\lambda^n, \tau)$ is infra H-closed. Also we prove that for an arbitrary positive integer n and an arbitrary infinite cardinal λ every shift-continuous semiregular feebly compact T_1-topology τ on $\mathscr{I}_\lambda^n$ is compact.

Key words: semigroup, inverse semigroup, semitopological semigroup, compact, countably compact, countably pracompact, feebly compact, H-closed, infra H-closed, X-compact, semiregular space.

We follow the terminology of [6, 7, 8, 26, 27]. If X is a topological space and $A \subseteq X$, then by $\operatorname{cl}_X(A)$ and $\operatorname{int}_X(A)$ we denote the topological closure and interior of A in X, respectively. By $|A|$ we denote the cardinality of a set A, by $A\triangle B$ the symmetric difference of sets A and B, by $\mathbb{N}$ the set of positive integers, and by ω the first infinite cardinal.

A semigroup S is called *inverse* if every a in S possesses an unique inverse a^{-1}, i.e. if there exists an unique element a^{-1} in S such that

$$aa^{-1}a = a \qquad \text{and} \qquad a^{-1}aa^{-1} = a^{-1}.$$

A map which associates to any element of an inverse semigroup its inverse is called the *inversion*.

A *topological* (*inverse*) *semigroup* is a topological space together with a continuous semigroup operation (and an inversion, respectively). Obviously, the inversion defined on a topological inverse semigroup is a homeomorphism. If S is a semigroup (an inverse

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semigroup) and τ is a topology on S such that (S,τ) is a topological (inverse) semigroup, then we shall call τ a *semigroup* (*inverse*) *topology* on S. A *semitopological semigroup* is a topological space together with a separately continuous semigroup operation. If S is a semigroup (an inverse semigroup) and τ is a topology on S such that (S,τ) is a semitopological semigroup (with continuous inversion), then we shall call τ a *shift-continuous* (*inverse*) *topology* on S.

If S is a semigroup, then by $E(S)$ we denote the subset of all idempotents of S. On the set of idempotents $E(S)$ there exists a natural partial order: $e \leqslant f$ *if and only if* $ef = fe = e$. A *semilattice* is a commutative semigroup of idempotents. A *topological* (*semitopological*) *semilattice* is a topological space together with a continuous (separately continuous) semilattice operation. If S is a semilattice and τ is a topology on S such that (S,τ) is a topological semilattice, then we shall call τ a *semilattice topology* on S.

Every inverse semigroup S admits a partial order:

$$a \preccurlyeq b \qquad \text{if and only if there exists} \qquad e\in E(S) \quad \text{such that} \quad a = eb.$$

We shall say that $\preccurlyeq$ is the *natural partial order* on S.

Let λ be an arbitrary non-zero cardinal. A map α from a subset D of λ into λ is called a *partial transformation* of λ. In this case the set D is called the *domain* of α and is denoted by $\operatorname{dom}\alpha$. The image of an element $x \in \operatorname{dom}\alpha$ under α is denoted by $x\alpha$ Also, the set $\{x\in\lambda\colon y\alpha = x \text{ for some } y\in Y\}$ is called the *range* of α and is denoted by $\operatorname{ran}\alpha$. The cardinality of $\operatorname{ran}\alpha$ is called the *rank* of α and is denoted by $\operatorname{rank}\alpha$. For convenience we denote by $\varnothing$ the empty transformation, a partial mapping with $\operatorname{dom}\varnothing = \operatorname{ran}\varnothing = \varnothing$.

Let $\mathscr{I}_\lambda$ denote the set of all partial one-to-one transformations of λ together with the following semigroup operation:

$$x(\alpha\beta) = (x\alpha)\beta \quad \text{if} \quad x\in\operatorname{dom}(\alpha\beta) = \{y\in\operatorname{dom}\alpha\colon y\alpha\in\operatorname{dom}\beta\}, \qquad \text{for} \quad \alpha,\beta\in\mathscr{I}_\lambda.$$

The semigroup $\mathscr{I}_\lambda$ is called the *symmetric inverse semigroup* over the cardinal λ (see [7]). The symmetric inverse semigroup was introduced by V. V. Wagner [29] and it plays a major role in the theory of semigroups.

Put $\mathscr{I}_\lambda^n = \{\alpha\in\mathscr{I}_\lambda\colon \operatorname{rank}\alpha\leqslant n\}$, for $n = 1,2,3,\ldots$. Obviously, $\mathscr{I}_\lambda^n$ $(n=1,2,3,\ldots)$ are inverse semigroups, $\mathscr{I}_\lambda^n$ is an ideal of $\mathscr{I}_\lambda$, for each $n=1,2,3,\ldots$. The semigroup $\mathscr{I}_\lambda^n$ is called the *symmetric inverse semigroup of finite transformations of the rank* $\leqslant n$. By

$$\left(\begin{array}{cccc} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{array}\right)$$

we denote a partial one-to-one transformation which maps x_1 onto y_1, x_2 onto y_2, …, and x_n onto y_n. Obviously, in such case we have $x_i\neq x_j$ and $y_i\neq y_j$ for $i\neq j$ $(i,j = 1,2,3,\ldots,n)$. The empty partial map $\varnothing\colon\lambda\rightharpoonup\lambda$ is denoted by $\mathbf{0}$. It is obvious that $\mathbf{0}$ is zero of the semigroup $\mathscr{I}_\lambda^n$.

Let λ be a non-zero cardinal. On the set $B_\lambda = (\lambda\times\lambda)\cup\{0\}$, where $0\notin\lambda\times\lambda$, we define the semigroup operation "$\cdot$" as follows

$$(a,b)\cdot(c,d) = \begin{cases} (a,d), & \text{if } b = c;\\ 0, & \text{if } b\neq c,\end{cases}$$

and $(a,b)\cdot 0 = 0\cdot(a,b) = 0\cdot 0 = 0$ for $a,b,c,d\in\lambda$. The semigroup B_λ is called the *semigroup of* $\lambda\times\lambda$*-matrix units* (see [7]). Obviously, for any cardinal $\lambda > 0$, the semigroup of $\lambda\times\lambda$-matrix units B_λ is isomorphic to $\mathscr{I}_\lambda^1$.

A subset A of a topological space X is called *regular open* if $\operatorname{int}_X(\operatorname{cl}_X(A)) = A$.

We recall that a topological space X is said to be

- *functionally Hausdorff* if for every pair of distinct points $x_1, x_2 \in X$ there exists a continuous function $f\colon X \to [0,1]$ such that $f(x_1) = 0$ and $f(x_2) = 1$;
- *semiregular* if X has a base consisting of regular open subsets;
- *quasiregular* if for any non-empty open set $U \subset X$ there exists a non-empty open set $V \subset U$ such that $\operatorname{cl}_X(V) \subseteq U$;
- *compact* if each open cover of X has a finite subcover;
- *sequentially compact* if each sequence $\{x_i\}_{i\in\mathbb{N}}$ of X has a convergent subsequence in X;
- *countably compact* if each open countable cover of X has a finite subcover;
- *H-closed* if X is a closed subspace of every Hausdorff topological space in which it is contained;
- *infra H-closed* provided that any continuous image of X into any first countable Hausdorff space is closed (see [20]);
- *countably compact at a subset* $A \subseteq X$ if every infinite subset $B \subseteq A$ has an accumulation point x in X;
- *countably pracompact* if there exists a dense subset A in X such that X is countably compact at A;
- *feebly compact* if each locally finite open cover of X is finite;
- *d-feebly compact* (or DFCC) if every discrete family of open subsets in X is finite (see [24]);
- *pseudocompact* if X is Tychonoff and each continuous real-valued function on X is bounded;
- *Y-compact* for some topological space Y, if $f(X)$ is compact, for any continuous map $f\colon X \to Y$.

According to Theorem 3.10.22 of [8], a Tychonoff topological space X is feebly compact if and only if X is pseudocompact. Also, a Hausdorff topological space X is feebly compact if and only if every locally finite family of non-empty open subsets of X is finite. Every compact space and every sequentially compact space are countably compact, every countably compact space is countably pracompact, every countably pracompact space is feebly compact (see [3]), every H-closed space is feebly compact too (see [15]). Also, every space feebly compact is infra H-closed by Proposition 2 and Theorem 3 of [20].

Topological properties of an infinite (semi)topological semigroup $\lambda\times\lambda$-matrix units were studied in [12, 13, 14]. In [13] it was shown that on the infinite semitopological semigroup $\lambda\times\lambda$-matrix units B_λ there exists a unique Hausdorff topology τ_c such that (B_λ, τ_c) is a compact semitopological semigroup and it was also shown that every pseudocompact Hausdorff shift-continuous topology τ on B_λ is compact. Also, in [13] it was proved that every non-zero element of a Hausdorff semitopological semigroup $\lambda\times\lambda$-matrix units B_λ is an isolated point in the topological space B_λ. In [12] it was shown that the infinite semigroup $\lambda\times\lambda$-matrix units B_λ cannot be embedded into a compact Hausdorff topological semigroup, every Hausdorff topological inverse semigroup S that contains B_λ as a subsemigroup, contains B_λ as a closed subsemigroup, i.e., B_λ is *algebraically complete* in the class of Hausdorff topological inverse semigroups. This result in [11] was extended onto so called inverse semigroups with *tight ideal series* and, as a corollary, onto the semigroup $\mathscr{I}_\lambda^n$. Also, in [16] it was proved that for every positive integer n the semigroup $\mathscr{I}_\lambda^n$ is *algebraically h-complete* in the class of Hausdorff topological inverse semigroups, i.e., every homomorphic image of $\mathscr{I}_\lambda^n$ is algebraically complete in the class of Hausdorff topological inverse semigroups. In the paper [17] this result was extended onto the class of Hausdorff semitopological inverse semigroups and it was shown therein that for an infinite cardinal λ the semigroup $\mathscr{I}_\lambda^n$ admits a unique Hausdorff topology τ_c such that $(\mathscr{I}_\lambda^n,\tau_c)$ is a compact semitopological semigroup. Also, it was proved in [17] that every countably compact Hausdorff shift-continuous topology τ on B_λ is compact. In [14] it was shown that a topological semigroup of finite partial bijections $\mathscr{I}_\lambda^n$ with a compact subsemigroup of idempotents is absolutely H-closed (i.e., every homomorphic image of $\mathscr{I}_\lambda^n$ is algebraically complete in the class of Hausdorff topological semigroups) and any countably compact topological semigroup does not contain $\mathscr{I}_\lambda^n$ as a subsemigroup for infinite cardinal λ. In [14] there were given sufficient conditions onto a topological semigroup $\mathscr{I}_\lambda^1$ to be non-H-closed. Also in [10] it was proved that an infinite semitopological semigroup of $\lambda\times\lambda$-matrix units B_λ is H-closed in the class of semitopological semigroups if and only if the space B_λ is compact.

For an arbitrary positive integer n and an arbitrary non-zero cardinal λ we put

$$\exp_n\lambda=\{A\subseteq\lambda\colon |A|\leqslant n\}.$$

It is obvious that for any positive integer n and any non-zero cardinal λ the set $\exp_n\lambda$ with the binary operation $\cap$ is a semilattice. Later in this paper by $\exp_n\lambda$ we shall denote the semilattice $(\exp_n\lambda,\cap)$. It is easy to see that $\exp_n\lambda$ is isomorphic to the subsemigroup of idempotents (the band) of the semigroup $\mathscr{I}_\lambda^n$ for any positive integer n. We observe that for every positive integer n the band of the semigroup $\mathscr{I}_\lambda^n$ is isomorphic to the semilattice $\exp_n\lambda$ by the mapping $E(\mathscr{I}_\lambda^n)\ni\varepsilon\mapsto\operatorname{dom}\varepsilon$.

In the paper [18] feebly compact shift-continuous topologies τ on the semilattice $\exp_n\lambda$ were studied, and all compact semilattice topologies on $\exp_n\lambda$ were described. In [18] it was whown that for an arbitrary positive integer n and an arbitrary infinite cardinal λ for a T_1-topology τ on $\exp_n\lambda$ the following conditions are equivalent: (i) $(\exp_n\lambda,\tau)$ is a compact topological semilattice; (ii) $(\exp_n\lambda,\tau)$ is a countably compact topological semilattice; (iii) $(\exp_n\lambda,\tau)$ is a feebly compact topological semilattice; (iv) $(\exp_n\lambda,\tau)$ is a compact semitopological semilattice; (v) $(\exp_n\lambda,\tau)$ is a countably compact semitopological semilattice. Also, in [18] there was constructed a countably pracompact H-closed quasiregular non-semiregular topology τ_{fc}^2 such that $\left(\exp_2\lambda,\tau_{\mathsf{fc}}^2\right)$ is a semitopological semilattice with the discontinuous semilattice operation and it was proved that for an arbitrary positive integer n and an arbitrary infinite cardinal λ a semiregular feebly compact semitopological semilattice $\exp_n\lambda$ is a compact topological semilattice. In [19] it was shown that for an arbitrary positive integer n and an arbitrary infinite cardinal λ for a T_1-topology τ on $\exp_n\lambda$ the following conditions are equivalent: (i) τ is countably pracompact; (ii) τ is feebly compact; (iii) τ is d-feebly compact; (iv) $(\exp_n\lambda,\tau)$ is an H-closed space.

This paper is a continuation of [11, 13, 16, 17]. We study feebly compact shift-continuous T_1-topologies on the semigroup $\mathscr{I}_\lambda^n$. For any positive integer $n\geqslant 2$ and any infinite cardinal λ a Hausdorff countably pracompact non-compact shift-continuous topology on $\mathscr{I}_\lambda^n$ is constructed. We show that for an arbitrary positive integer n and an arbitrary infinite cardinal λ for a T_1-topology τ on $\mathscr{I}_\lambda^n$ the following conditions are equivalent: (i) τ is countably pracompact; (ii) τ is feebly compact; (iii) τ is d-feebly compact; (iv) $(\mathscr{I}_\lambda^n,\tau)$ is H-closed; (v) $(\mathscr{I}_\lambda^n,\tau)$ is $\mathbb{N}_{\mathfrak{d}}$-compact for the discrete countable space $\mathbb{N}_{\mathfrak{d}}$; (vi) $(\mathscr{I}_\lambda^n,\tau)$ is $\mathbb{R}$-compact; (vii) $(\mathscr{I}_\lambda^n,\tau)$ is infra H-closed. Also we prove that for an arbitrary positive integer n and an arbitrary infinite cardinal λ every shift-continuous semiregular feebly compact T_1-topology τ on $\mathscr{I}_\lambda^n$ is compact.

Later we shall assume that n is an arbitrary positive integer.

For every element α of the semigroup $\mathscr{I}_\lambda^n$ we put

$$\uparrow_l\alpha=\left\{\beta\in\mathscr{I}_\lambda^n\colon \alpha\alpha^{-1}\beta=\alpha\right\} \qquad \text{and} \qquad \uparrow_r\alpha=\left\{\beta\in\mathscr{I}_\lambda^n\colon \beta\alpha^{-1}\alpha=\alpha\right\}.$$

Then Proposition 5 of [17] implies that $\uparrow_l\alpha=\uparrow_r\alpha$ and by Lemma 6 of [23, Section 1.4] we have that $\alpha\preccurlyeq\beta$ if and only if $\beta\in\uparrow_l\alpha$ for $\alpha,\beta\in\mathscr{I}_\lambda^n$. Hence we put $\uparrow_{\preccurlyeq}\alpha=\uparrow_l\alpha=\uparrow_r\alpha$ for any $\alpha\in\mathscr{I}_\lambda^n$.

The definition of the semigroup operation of $\mathscr{I}_\lambda^n$ implies the following trivial lemma.

Lemma 1. *Let n be an arbitrary positive integer and λ be any cardinal. Then for any elements α and β of the semigroup $\mathscr{I}_\lambda^n$ the sets $\alpha\mathscr{I}_\lambda^n\beta$ and*

$$\downarrow_{\preccurlyeq}\alpha=\{\gamma\in\mathscr{I}_\lambda^n\colon \gamma\preccurlyeq\alpha\}$$

are finite.

Proof. For any elements α and β of $\mathscr{I}_\lambda^n$ we have that

$$\alpha\mathscr{I}_\lambda^n\beta=\alpha\mathscr{I}_\lambda^n\cap\mathscr{I}_\lambda^n\beta=\{\gamma\in\mathscr{I}_\lambda^n\colon\ \operatorname{dom}\gamma\subseteq\operatorname{dom}\alpha \ \text{ and } \ \operatorname{ran}\gamma\subseteq\operatorname{ran}\beta\}.$$

Since the sets $\operatorname{dom}\alpha$ and $\operatorname{ran}\beta$ are finite, $\alpha\mathscr{I}_\lambda^n\beta$ is finite, as well.

For every $\gamma\in\downarrow_{\preccurlyeq}\alpha$ the definition of the natural partial order $\preccurlyeq$ on the semigroup $\mathscr{I}_\lambda^n$ (see [23, Chapter 1]) implies that the finite partial map γ is a restriction of the finite partial map α onto the set $A=\operatorname{dom}\alpha\cap\operatorname{dom}\varepsilon$, where ε is an idempotent of $\mathscr{I}_\lambda^n$ such that $\gamma=\varepsilon\alpha$. This implies that the set $\downarrow_{\preccurlyeq}\alpha$ is finite. $\square$

Lemma 2. *Let n be an arbitrary positive integer, λ be any infinite cardinal and τ be a shift-continuous T_1-topology on semigroup $\mathscr{I}_\lambda^n$. Then for every element α of the semigroup $\mathscr{I}_\lambda^n$ the set $\uparrow_{\preccurlyeq}\alpha$ is open-and-closed in $(\mathscr{I}_\lambda^n,\tau)$, the space $(\mathscr{I}_\lambda^n,\tau)$ is functionally Hausdorff and hence it is quasi-regular.*

Proof. Fix an arbitrary $\alpha\in\mathscr{I}_\lambda^n$. Then $\alpha\in\alpha\mathscr{I}_\lambda^n\alpha$ and

$$\alpha\mathscr{I}_\lambda^n\alpha=\alpha\mathscr{I}_\lambda^n\cap\mathscr{I}_\lambda^n\alpha=\alpha\alpha^{-1}\mathscr{I}_\lambda^n\cap\mathscr{I}_\lambda^n\alpha^{-1}\alpha=\alpha\alpha^{-1}\mathscr{I}_\lambda^n\alpha^{-1}\alpha,$$

because $\mathscr{I}_\lambda^n$ is an inverse semigroup. Since the topology τ is T_1, Lemma 1 implies that the set $(\alpha\mathscr{I}_\lambda^n\alpha)\setminus\{\alpha\}$ is closed in $(\mathscr{I}_\lambda^n,\tau)$. By the separate continuity of the semigroup operation in $(\mathscr{I}_\lambda^n,\tau)$ we have that there exists an open neighbourhood $U(\alpha)$ of the point α in $(\mathscr{I}_\lambda^n,\tau)$ such that

$$\alpha\alpha^{-1}\cdot U(\alpha)\cdot\alpha^{-1}\alpha\subseteq\mathscr{I}_\lambda^n\setminus\left((\alpha\mathscr{I}_\lambda^n\cup\mathscr{I}_\lambda^n\alpha)\setminus\{\alpha\}\right).$$

The last inclusion implies that $U(\alpha)\subseteq\uparrow\alpha$. Again, since the semigroup operation in $(\mathscr{I}_\lambda^n,\tau)$ is separately continuous the set $\uparrow_{\preccurlyeq}\alpha$ is open in $(\mathscr{I}_\lambda^n,\tau)$ as a full preimage of $U(\alpha)$ and the set $\uparrow_{\preccurlyeq}\alpha$ is closed in $(\mathscr{I}_\lambda^n,\tau)$ as a full preimage of the singleton set $\{\alpha\}$.

Fix arbitrary distinct elements α and β of the semigroup $\mathscr{I}_\lambda^n$. Then either α and β are comparable or not with respect to the natural partial order on $\mathscr{I}_\lambda^n$. If $\alpha \preccurlyeq \beta$ or α and β are incomparable in $(\mathscr{I}_\lambda^n, \preccurlyeq)$ then it is obvious that the map $g\colon \mathscr{I}_\lambda^n \to [0,1]$ defined by the formula

$$(\gamma)f = \begin{cases} 1, & \text{if } \gamma \in {\uparrow_{\preccurlyeq}}\beta; \\ 0, & \text{if } \gamma \notin {\uparrow_{\preccurlyeq}}\beta \end{cases}$$

is continuous. We observe that quasi-regularity of $(\mathscr{I}_\lambda^n, \tau)$ follows from the fact that every non-empty open subset U of $(\mathscr{I}_\lambda^n, \tau)$ contains a maximal element δ with respect to the natural partial order $\preccurlyeq$ on $\mathscr{I}_\lambda^n$ such that ${\uparrow_{\preccurlyeq}}\alpha$ is an open-and-closed subset of $(\mathscr{I}_\lambda^n, \tau)$ and hence, since τ is a T_1-topology, $\{\alpha\} \subseteq U$ is an open-and-closed subset of $(\mathscr{I}_\lambda^n, \tau)$. $\square$

A topological space X is called

- *totally disconnected* if the connected components in X are singleton sets;
- *scattered* if X does not contain non-empty dense in itself subset, which is equivalent that every non-empty subset of X has an isolated point in itself.

Lemma 2 implies the following corollary:

Corollary 1. *Let n be an arbitrary positive integer, λ be any infinite cardinal and τ be a shift-continuous T_1-topology on the semigroup $\mathscr{I}_\lambda^n$. Then $(\mathscr{I}_\lambda^n, \tau)$ is a totally disconnected scattered space.*

A partial order $\leqslant$ on a topological space X is called closed if the relation $\leqslant$ is a closed subset of $X \times X$ in the product topology. In this case $(X, \leqslant)$ is called a *pospace* [9].

Lemma 2 and Proposition VI-1.4 from [9] imply the following corollary:

Corollary 2. *Let n be an arbitrary positive integer, λ be any infinite cardinal and τ be a shift-continuous T_1-topology on semigroup $\mathscr{I}_\lambda^n$. Then $(\mathscr{I}_\lambda^n, \tau, \preccurlyeq)$ is a pospace*

The following example shows that the statement of Lemma 2 does not hold in the case when $(\mathscr{I}_\lambda^n, \tau)$ is a T_0-space.

Example 1. For an arbitrary positive integer n and an arbitrary infinite cardinal λ we define a topology τ_0 on $\mathscr{I}_\lambda^n$ in the following way:

(i) all non-zero elements of the semigroup $\mathscr{I}_\lambda^n$ are isolated points in $(\mathscr{I}_\lambda^n, \tau_0)$; and
(ii) $\mathscr{I}_\lambda^n$ is the unique open neighbourhood of zero in $(\mathscr{I}_\lambda^n, \tau_0)$.

Simple verifications show that the semigroup operation and inversion on $(\mathscr{I}_\lambda^n, \tau_0)$ are continuous.

We need the following example from [17].

Example 2 ([17])**.** Fix an arbitrary positive integer n. The following family

$$\mathscr{B}_{\mathsf{c}} = \big\{U_\alpha(\alpha_1, \ldots, \alpha_k) = {\uparrow_{\preccurlyeq}}\alpha \setminus ({\uparrow_{\preccurlyeq}}\alpha_1 \cup \cdots \cup {\uparrow_{\preccurlyeq}}\alpha_k)\colon \\ \alpha_i \in {\uparrow_{\preccurlyeq}}\alpha \setminus \{\alpha\}, \alpha, \alpha_i \in \mathscr{I}_\lambda^n, i = 1, \ldots, k\big\}$$

determines a base of the topology τ_{c} on $\mathscr{I}_\lambda^n$. By Proposition 10 from [17], $(\mathscr{I}_\lambda^n, \tau_{\mathsf{c}})$ is a Hausdorff compact semitopological semigroup with continuous inversion.

By Theorem 7 from [17], for an arbitrary infinite cardinal λ and any positive integer n every countably compact Hausdorff semitopological semigroup $\mathscr{I}_\lambda^n$ is topologically isomorphic to $(\mathscr{I}_\lambda^n, \tau_{\mathsf{c}})$. By Corollary 1 the topological space $(\mathscr{I}_\lambda^n, \tau_{\mathsf{c}})$ is scattered. Since every countably compact scattered T_3-space is sequentially compact (see [28, Theorem 5.7]), $(\mathscr{I}_\lambda^n, \tau_{\mathsf{c}})$ is a sequentially compact space.

Next we summarise the above results in the following theorem.

Theorem 1. *Let n be an arbitrary positive integer, λ be any infinite cardinal and τ be a T_1-shift continuous topology on the semigroup $\mathscr{I}_\lambda^n$. Then the following conditions are equivalent:*

(i) *τ is compact;*
(ii) *$\tau = \tau_{\mathsf{c}}$;*
(iii) *τ is countably compact;*
(iv) *τ is sequentially compact.*

Since every feebly compact Hausdorff topology on the semigroup $\mathscr{I}_\lambda^1$ is compact, it is natural to ask: *Does there exist a shift-continuous Hausdorff non-compact feebly compact topology τ on the semigroup $\mathscr{I}_\lambda^n$ for $n \geqslant 2$?*

The following example shows that for any infinite cardinal λ and any positive integer $n \geqslant 2$ there exists a Hausdorff feebly compact topology τ on the semigroup $\mathscr{I}_\lambda^n$ such that $(\mathscr{I}_\lambda^n, \tau)$ is a non-compact semitopological semigroup.

Example 3. Let λ be any infinite cardinal and $\tau_{\mathsf{c}}^2 = \tau_{\mathsf{c}}$ be the topology on the semigroup $\mathscr{I}_\lambda^2$ which is defined in Example 2. We construct a stronger topology τ_{fc}^2 on $\mathscr{I}_\lambda^2$ then τ_{c}^2 in the following way. By $\pi\colon \lambda \to \mathscr{I}_\lambda^2\colon a \mapsto \varepsilon_a$ we denote the map which assigns to any element $a \in \lambda$ the identity partial map $\varepsilon_a\colon \{a\} \to \{a\}$. Fix an arbitrary infinite subset A of λ. For every non-zero element $x \in \mathscr{I}_\lambda^2$ we assume that the base $\mathscr{B}_{\mathsf{fc}}^2(x)$ of the topology τ_{fc}^2 at the point x coincides with the base of the topology τ_{c}^2 at x, and

$$\mathscr{B}_{\mathsf{fc}}^2(0) = \big\{U_B(\mathbf{0}) = U(\mathbf{0}) \setminus ((B)\pi \cup \{\alpha_1, \ldots, \alpha_s\}) : U(0) \in \mathscr{B}_{\mathsf{c}}^2(0), \alpha_1, \ldots, \alpha_s \in \mathscr{I}_\lambda^2 \setminus \{\mathbf{0}\} \text{ and } B \subseteq \lambda \text{ such that } |A \triangle B| < \infty\big\}$$

form a base of the topology τ_{fc}^2 at zero $\mathbf{0}$ of the semigroup $\mathscr{I}_\lambda^2$. Simple verifications show that the family $\big\{\mathscr{B}_{\mathsf{fc}}^2(x)\colon x \in \mathscr{I}_\lambda^2\big\}$ satisfies conditions **(BP1)**–**(BP4)** of [8], and hence τ_{fc}^2 is a Hausdorff topology on $\mathscr{I}_\lambda^2$.

Proposition 1. *Let λ be an arbitrary infinite cardinal. Then $\big(\mathscr{I}_\lambda^2, \tau_{\mathsf{fc}}^2\big)$ is a countably pracompact semitopological semigroup with continuous inversion.*

Proof. It is obvious that the inversion in $\big(\mathscr{I}_\lambda^2, \tau_{\mathsf{fc}}^2\big)$ is continuous and later we shall show that all translations in $\big(\mathscr{I}_\lambda^2, \tau_{\mathsf{fc}}^2\big)$ are continuous maps. We consider the following possible cases.

(1) $\mathbf{0} \cdot \mathbf{0} = \mathbf{0}$. For every basic open neighbourhood $U_B(\mathbf{0})$ of zero in $\big(\mathscr{I}_\lambda^2, \tau_{\mathsf{fc}}^2\big)$ we have that

$$U_B(\mathbf{0}) \cdot \mathbf{0} = \mathbf{0} \cdot U_B(\mathbf{0}) = \{\mathbf{0}\} \subset U_\pi(\mathbf{0}).$$

(2) $\alpha \cdot \mathbf{0} = \mathbf{0}$. For all basic open neighbourhoods $U_B(\mathbf{0})$ and $U_\alpha(\beta_1, \ldots, \beta_k)$ of zero and an element $\alpha \neq \mathbf{0}$ in $\big(\mathscr{I}_\lambda^2, \tau_{\mathsf{fc}}^2\big)$, respectively, we have that

$$U_\alpha(\beta_1, \ldots, \beta_k) \cdot \mathbf{0} = \{\mathbf{0}\} \subset U_B(\mathbf{0}).$$

Let $V_B(\mathbf{0}) = \mathscr{I}_\lambda^2 \setminus (\uparrow_{\preccurlyeq}\alpha_1 \cup \cdots \cup \uparrow_{\preccurlyeq}\alpha_k \cup (B)\pi)$ be an arbitrary basic neighbourhood of zero in $\left(\mathscr{I}_\lambda^2, \tau_{\mathsf{fc}}^2\right)$. Without loss of generality we may assume that

$$\operatorname{rank}\alpha_1 = \ldots = \operatorname{rank}\alpha_k = 1 \leqslant \operatorname{rank}\alpha.$$

Put

$$\mathcal{C}_l = \left\{\gamma \in \mathscr{I}_\lambda^2 : \operatorname{rank}\gamma = 1 \text{ such that } \alpha\gamma = \alpha_i \text{ for some } i = 1, \ldots, k \right. \\ \left. \text{or } \alpha\gamma \in E(\mathscr{I}_\lambda^2) \setminus \{\mathbf{0}\}\right\}.$$

The definition of the semigroup $\mathscr{I}_\lambda^2$ implies that the set $\mathcal{C}_l$ is finite. Then we have that $\alpha \cdot W_B(0) \subseteq V_B(0)$ for $W_B(0) = \mathscr{I}_\lambda^2 \setminus \bigcup\left\{\uparrow_{\preccurlyeq}\gamma \colon \gamma \in \mathcal{C}_l\right\}$.

(3) $\mathbf{0} \cdot \alpha = \mathbf{0}$. For all basic open neighbourhoods $U_B(\mathbf{0})$ and $U_\alpha(\beta_1, \ldots, \beta_k)$ of zero and an element $\alpha \neq \mathbf{0}$ in $\left(\mathscr{I}_\lambda^2, \tau_{\mathsf{fc}}^2\right)$, respectively, we have that

$$\mathbf{0} \cdot U_\alpha(\beta_1, \ldots, \beta_k) = \{\mathbf{0}\} \subset U_B(\mathbf{0}).$$

Let $V_B(\mathbf{0}) = \mathscr{I}_\lambda^2 \setminus (\uparrow_{\preccurlyeq}\alpha_1 \cup \cdots \cup \uparrow_{\preccurlyeq}\alpha_k \cup (B)\pi)$ be an arbitrary basic neighbourhood of zero in $\left(\mathscr{I}_\lambda^2, \tau_{\mathsf{fc}}^2\right)$. Without loss of generality we may assume that

$$\operatorname{rank}\alpha_1 = \ldots = \operatorname{rank}\alpha_k = 1 \leqslant \operatorname{rank}\alpha.$$

Put

$$\mathcal{C}_r = \left\{\gamma \in \mathscr{I}_\lambda^2 \colon \operatorname{rank}\gamma = 1 \text{ such that } \gamma\alpha = \alpha_i \text{ for some } i = 1, \ldots, k \right. \\ \left. \text{or } \gamma\alpha \in E(\mathscr{I}_\lambda^2) \setminus \{\mathbf{0}\}\right\}.$$

The definition of the semigroup $\mathscr{I}_\lambda^2$ implies that the set $\mathcal{C}_r$ is finite. Then we have that $W_B(0) \cdot \alpha \subseteq V_B(0)$ for $W_B(0) = \mathscr{I}_\lambda^2 \setminus \bigcup\left\{\uparrow_{\preccurlyeq}\gamma \colon \gamma \in \mathcal{C}_r\right\}$.

(4) $\alpha \cdot \beta = \gamma \neq \mathbf{0}$ and $\operatorname{rank}\alpha = \operatorname{rank}\beta = \operatorname{rank}\gamma$, i.e., $\operatorname{ran}\alpha = \operatorname{dom}\beta$. Then for any open neighbourhoods $U_\alpha(\alpha_1, \ldots, \alpha_k)$, $U_\beta(\beta_1, \ldots, \beta_n)$, $U_\gamma(\gamma_1, \ldots, \gamma_m)$ of the points α, β and γ in $\left(\mathscr{I}_\lambda^2, \tau_{\mathsf{fc}}^2\right)$, respectively, we have that

$$U_\alpha(\alpha_1, \ldots, \alpha_k) \cdot \beta = \alpha \cdot U_\beta(\beta_1, \ldots, \beta_n) = \{\gamma\} \subseteq U_\gamma(\gamma_1, \ldots, \gamma_m).$$

(5) $\alpha \cdot \beta = \gamma \neq \mathbf{0}$ and $\operatorname{rank}\alpha = \operatorname{rank}\gamma = 1$ and $\operatorname{rank}\beta = 2$, i.e., $\operatorname{ran}\alpha \subsetneqq \operatorname{dom}\beta$. Then for any open neighbourhoods $U_\beta(\beta_1, \ldots, \beta_n)$ and $U_\gamma(\gamma_1, \ldots, \gamma_m)$ of the points β and γ in $\left(\mathscr{I}_\lambda^2, \tau_{\mathsf{fc}}^2\right)$, respectively, we have that

$$\alpha \cdot U_\beta(\beta_1, \ldots, \beta_n) = \{\gamma\} \subseteq U_\gamma(\gamma_1, \ldots, \gamma_m).$$

Let $U_\gamma(\gamma_1, \ldots, \gamma_k)$ be an arbitrary open neighbourhood of the point γ in $\left(\mathscr{I}_\lambda^2, \tau_{\mathsf{fc}}^2\right)$ for some $\gamma_1, \ldots, \gamma_k \in \uparrow_{\preccurlyeq}\gamma$, i.e., $\operatorname{rank}\gamma_1 = \ldots = \operatorname{rank}\gamma_k = 2$. Put

$$\mathcal{Q} = \left\{\delta \in \uparrow_{\preccurlyeq}\alpha \colon \delta\beta \in \{\gamma_1, \ldots, \gamma_k\}\right\}.$$

The definition of the semigroup $\mathscr{I}_\lambda^2$ implies that the set $\mathcal{Q}$ is finite. Then we have that

$$U_\alpha(\mathcal{Q}) \cdot \beta \subseteq U_\gamma(\gamma_1, \ldots, \gamma_k)$$

for $U_\alpha(\mathcal{Q}) = \uparrow_{\preccurlyeq}\alpha \setminus \left\{\delta \in \uparrow_{\preccurlyeq}\alpha \colon \delta \in \mathcal{Q}\right\}$.

(6) $\alpha \cdot \beta = \gamma \neq \mathbf{0}$ and $\operatorname{rank}\beta = \operatorname{rank}\gamma = 1$ and $\operatorname{rank}\alpha = 2$, i.e., $\operatorname{dom}\beta \subsetneqq \operatorname{ran}\alpha$. In this case the proof of separate continuity of the semigroup operation on $\left(\mathscr{I}_\lambda^2, \tau_{\mathsf{fc}}^2\right)$ is dual to case **(5)**.

(7) $\alpha \cdot \beta = \gamma \neq \mathbf{0}$, $\operatorname{rank} \gamma = 1$ and $\operatorname{rank} \alpha = \operatorname{rank} \beta = 2$. Then α and β are isolated points in $\left(\mathscr{I}_\lambda^2, \tau_{\mathsf{fc}}^2\right)$ and hence

$$\alpha \cdot \beta = \gamma \subseteq U_\gamma(\gamma_1, \ldots, \gamma_k),$$

for any basic open neighbourhood $U_\gamma(\gamma_1, \ldots, \gamma_k)$ of γ in $\left(\mathscr{I}_\lambda^2, \tau_{\mathsf{fc}}^2\right)$.

(8) $\alpha \cdot \beta = \mathbf{0}$. Then $\operatorname{dom} \beta \cap \operatorname{ran} \alpha = \varnothing$ and hence

$$U_\alpha(\alpha_1, \ldots, \alpha_k) \cdot \beta = \alpha \cdot U_\beta(\beta_1, \ldots, \beta_n) = \{\mathbf{0}\} \subset U_B(\mathbf{0}),$$

for any basic open neighbourhoods $U_\alpha(\alpha_1, \ldots, \alpha_k)$, $U_\beta(\beta_1, \ldots, \beta_n)$ and $U_B(\mathbf{0})$ of α, β and zero $\mathbf{0}$ in $\left(\mathscr{I}_\lambda^2, \tau_{\mathsf{fc}}^2\right)$, respectivelly.

Thus we have shown that the translations in $\left(\mathscr{I}_\lambda^2, \tau_{\mathsf{fc}}^2\right)$ are continuous maps.

Also, the definition of the topology τ_{fc}^2 on $\mathscr{I}_\lambda^2$ implies that the set $\mathscr{I}_\lambda^2 \setminus \mathscr{I}_\lambda^1$ is dense in $\left(\mathscr{I}_\lambda^2, \tau_{\mathsf{fc}}^2\right)$ and every infinite subset of $\mathscr{I}_\lambda^2 \mathscr{I}_\lambda^1$ has an accumulation point in $\left(\mathscr{I}_\lambda^2, \tau_{\mathsf{fc}}^2\right)$, and hence the space $\left(\mathscr{I}_\lambda^2, \tau_{\mathsf{fc}}^2\right)$ is countably pracompact. □

Proposition 2. *Let n be an arbitrary positive integer and λ be an arbitrary infinite cardinal. Then for every d-feebly compact shift-continuous T_1-topology τ on $\mathscr{I}_\lambda^n$ the subset $\mathscr{I}_\lambda^n \setminus \mathscr{I}_\lambda^{n-1}$ is dense in $(\mathscr{I}_\lambda^n, \tau)$.*

Proof. Since every quasi-regular d-feebly compact space is feebly compact (see [19, Theorem 2]), by Lemma 2 the topology τ is feebly compact.

Suppose to the contrary that there exists a feebly compact shift-continuous T_1-topology τ on $\mathscr{I}_\lambda^n$ such that $\mathscr{I}_\lambda^n \setminus \mathscr{I}_\lambda^{n-1}$ is not dense in $(\mathscr{I}_\lambda^n, \tau)$. Then there exists a point $\alpha \in \mathscr{I}_\lambda^{n-1}$ of the space $(\mathscr{I}_\lambda^n, \tau)$ such that $\alpha \notin \operatorname{cl}_{\mathscr{I}_\lambda^n}\left(\mathscr{I}_\lambda^n \setminus \mathscr{I}_\lambda^{n-1}\right)$. This implies that there exists an open neighbourhood $U(\alpha)$ of α in $(\mathscr{I}_\lambda^n, \tau)$ such that $U(\alpha) \cap \left(\mathscr{I}_\lambda^n \setminus \mathscr{I}_\lambda^{n-1}\right) = \varnothing$. Lemma 2 implies that ${\uparrow_{\preccurlyeq}}\alpha$ is an open-and-closed subset of $(\mathscr{I}_\lambda^n, \tau)$ and hence by Theorem 14 of [4], ${\uparrow_{\preccurlyeq}}\alpha$ is feebly compact. This implies that without loos of generality we may assume that $U(\alpha) \subseteq {\uparrow_{\preccurlyeq}}x \cap \mathscr{I}_\lambda^{n-1}$. By the definition of the semigroup $\mathscr{I}_\lambda^n$ we have that there exists a point $\beta \in U(\alpha)$ such that ${\uparrow_{\preccurlyeq}}\beta \cap U(\alpha) = \{\beta\}$. Again, by Lemma 2 we have that ${\uparrow_{\preccurlyeq}}\beta$ is an open-and-closed subset of $(\mathscr{I}_\lambda^n, \tau)$ and hence by Theorem 14 of [4], ${\uparrow_{\preccurlyeq}}\beta$ is feebly compact. Moreover, our choice implies that β is an isolated point in the subspace ${\uparrow_{\preccurlyeq}}\beta$ of $(\mathscr{I}_\lambda^n, \tau)$.

Suppose that

$$\beta = \begin{pmatrix} x_1 & \cdots & x_k \\ y_1 & \cdots & y_k \end{pmatrix},$$

for some finite subsets $\{x_1, \cdots, x_k\}$ and $\{y_1, \cdots, y_k\}$ of distinct points from λ. Then the above arguments imply that $k < n$. Put $p = n - k$. Next we fix an arbitrary infinite sequence $\{a_i\}_{i \in \mathbb{N}}$ of distinct elements of the set $\lambda \setminus (\{x_1, \cdots, x_k\} \cup \{y_1, \cdots, y_k\})$.

For arbitrary positive integer j we put

$$\beta_j = \begin{pmatrix} x_1 & \cdots & x_k & a_{p(j-1)+1} & \cdots & a_{pj} \\ y_1 & \cdots & y_k & a_{p(j-1)+1} & \cdots & a_{pj} \end{pmatrix}.$$

Then $\beta_j \in \mathscr{I}_\lambda^n$ for any positive integer j. Moreover, we have that $\beta_j \in \mathscr{I}_\lambda^n \setminus \mathscr{I}_\lambda^{n-1}$ and $\beta_j \in {\uparrow_{\preccurlyeq}}\beta$ for any positive integer j.

We claim that the set $\uparrow_{\preccurlyeq}\gamma \cap \{\beta_j : j \in \mathbb{N}\}$ is finite for any $\gamma \in \uparrow_{\preccurlyeq}\beta \setminus \{\beta\}$. Indeed, if the set $\uparrow_{\preccurlyeq}\gamma \cap \{\beta_j : j \in \mathbb{N}\}$ is infinite for some $\gamma \in \uparrow_{\preccurlyeq}\beta \setminus \{\beta\}$ then $\operatorname{dom}\gamma$ contains infinitely many points of the set $\{a_i : i \in \mathbb{N}\}$, which contradicts that $\gamma \in \mathscr{I}_\lambda^n$.

By Lemma 2 for every $\gamma \in \mathscr{I}_\lambda^n$ the set $\uparrow_{\preccurlyeq}\gamma$ is open in $(\mathscr{I}_\lambda^n, \tau)$. Then since β is an isolated point in $\uparrow_{\preccurlyeq}\beta$, our claim implies that the infinite family of isolated points $\mathscr{U} = \{\{b_j\} : j \in \mathbb{N}\}$ is locally finite in $\uparrow_{\preccurlyeq}\beta$, which contradicts that the subspace $\uparrow_{\preccurlyeq}\beta$ of $(\mathscr{I}_\lambda^n, \tau)$ is feebly compact. The obtained contradiction implies the statement of the proposition. □

Remark 1. The following three examples of topological semigroups of matrix units (B_λ, τ_{mv}), (B_λ, τ_{mh}) and (B_λ, τ_{mi}) from [12] imply that the converse to Proposition 2 is not true for any infinite cardinal λ.

Later by $\mathbb{N}_{\mathfrak{d}}$ and $\mathbb{R}$ we denote the sets of positive integers with the discrete topology and the real numbers with the usual topology.

Theorem 2. *Let n be an arbitrary positive integer and λ be an arbitrary infinite cardinal. Then for every shift-continuous T_1-topology τ on the semigroup $\mathscr{I}_\lambda^n$ the following statements are equivalent:*

(i) *τ is countably pracompact;*
(ii) *τ is feebly compact;*
(iii) *τ is d-feebly compact;*
(iv) *$(\mathscr{I}_\lambda^n, \tau)$ is H-closed;*
(v) *$(\mathscr{I}_\lambda^n, \tau)$ is $\mathbb{N}_{\mathfrak{d}}$-compact;*
(vi) *$(\mathscr{I}_\lambda^n, \tau)$ is $\mathbb{R}$-compact;*
(vii) *$(\mathscr{I}_\lambda^n, \tau)$ is infra H-closed.*

Proof. Implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are trivial.

$(iii) \Rightarrow (ii)$ Suppose that a space $(\mathscr{I}_\lambda^n, \tau)$ is d-feebly compact. By Lemma 2 it is quasi-regular. Then by Theorem 1 of [19] every quasiregular d-feebly compact space is feebly compact and hence so is $(\mathscr{I}_\lambda^n, \tau)$.

$(ii) \Rightarrow (i)$ Suppose that a space $(\mathscr{I}_\lambda^n, \tau)$ is feebly compact. By Lemma 2 the topological space $(\mathscr{I}_\lambda^n, \tau)$ is Hausdorff. Then by Lemma 1 of [19] every Hausdorff feebly compact space with a dense discrete subspace is countably pracompact (also see Lemma 4.5 of [5] or Proposition 1 from [2] for Tychonoff spaces) and hence so is $(\mathscr{I}_\lambda^n, \tau)$.

Implication $(iv) \Rightarrow (ii)$ follows from Proposition 4 of [15].

$(ii) \Rightarrow (iv)$ We shall show by induction that if τ is a shift-continuous feebly compact T_1-topology on the semigroup $\mathscr{I}_\lambda^n$ then the subspace $\uparrow_{\preccurlyeq}\alpha$ of $(\mathscr{I}_\lambda^n, \tau)$ is H-closed for any $\alpha \in \mathscr{I}_\lambda^n$.

It is obvious that for any $\alpha \in \mathscr{I}_\lambda^n$ with $\operatorname{rank}\alpha = n$ the set $\uparrow_{\preccurlyeq}\alpha = \{\alpha\}$ is singleton, and since $(\mathscr{I}_\lambda^n, \tau)$ is a T_1-space, $\uparrow_{\preccurlyeq}\alpha$ is H-closed.

Fix an arbitrary $\alpha \in \mathscr{I}_\lambda^n$ with $\operatorname{rank}\alpha = n-1$. By Lemma 2, $\uparrow_{\preccurlyeq}\alpha$ is an open-and-closed subset of $(\mathscr{I}_\lambda^n, \tau)$ and hence by Theorem 14 from [4] the space $\uparrow_{\preccurlyeq}\alpha$ is feebly compact. Since by Lemma 2 every point β of $\uparrow_{\preccurlyeq}\alpha$ with $\operatorname{rank}\alpha = n$ is isolated in $(\mathscr{I}_\lambda^n, \tau)$, the feeble compactness of $\uparrow_{\preccurlyeq}\alpha$ implies that α is a non-isolated point of $(\mathscr{I}_\lambda^n, \tau)$ and the space $\uparrow_{\preccurlyeq}\alpha$ is compact. This implies that $\uparrow_{\preccurlyeq}\alpha$ is H-closed.

Next we shall prove the following statement: *if for some positive integer* $k < n$ *for any* $\alpha \in \mathscr{I}_\lambda^n$ *with* $\operatorname{rank}\alpha \leqslant k$ *the subspace* ${\uparrow_{\preccurlyeq}}\alpha$ *is H-closed then* ${\uparrow_{\preccurlyeq}}\beta$ *is H-closed for any* $\beta \in \mathscr{I}_\lambda^n$ *with* $\operatorname{rank}\beta = k-1$.

Suppose to the contrary that there exists a shift-continuous feebly compact T_1-topology τ on the semigroup $\mathscr{I}_\lambda^n$ such that for some positive integer $k < n$ for any $\alpha \in \mathscr{I}_\lambda^n$ with $\operatorname{rank}\alpha = k$ the subspace ${\uparrow_{\preccurlyeq}}\alpha$ is H-closed and ${\uparrow_{\preccurlyeq}}\beta$ is not an H-closed space for some $\beta \in \mathscr{I}_\lambda^n$ with $\operatorname{rank}\beta = k-1$. Then there exists a Hausdorff topological space X which contains the space ${\uparrow_{\preccurlyeq}}\beta$ as a dense proper subspace. We observe that by Lemma 2 and Theorem 14 of [4] the space ${\uparrow_{\preccurlyeq}}\beta$ is feebly compact.

Fix an arbitrary $x \in X \setminus {\uparrow_{\preccurlyeq}}\beta$. The Hausdorffness of X implies that there exist open neighbourhoods $U_X(x)$ and $U_X(\beta)$ of the points x and β in X, respectively, such that $U_X(x) \cap U_X(\beta) = \varnothing$. Then the assumption of induction implies that without loss of generality we may assume that there do not exist finitely many $\alpha_1, \ldots, \alpha_m \in {\uparrow_{\preccurlyeq}}\beta$ with $\operatorname{rank}\alpha_1 = \ldots = \operatorname{rank}\alpha_m = k$ such that

$$U_X(x) \cap {\uparrow_{\preccurlyeq}}\beta \subseteq {\uparrow_{\preccurlyeq}}\alpha_1 \cup \cdots \cup {\uparrow_{\preccurlyeq}}\alpha_m.$$

Fix an arbitrary $\alpha_1 \in {\uparrow_{\preccurlyeq}}\beta$ such that $\operatorname{rank}\alpha_1 = k$ and ${\uparrow_{\preccurlyeq}}\alpha_1 \cap U_X(x) \neq \varnothing$. Proposition 1.3.1 of [8], Lemma 2 and Proposition 2 imply that there exists $\gamma_1 \in \mathscr{I}_\lambda^n \setminus \mathscr{I}_\lambda^{n-1}$ such that $\gamma_1 \in {\uparrow_{\preccurlyeq}}\alpha_1 \cap U_X(x)$. Next, by induction using Proposition 1.3.1 of [8], Lemma 2 and Proposition 2 we construct sequences $\{\alpha_i\}_{i\in\mathbb{N}}$ and $\{\gamma_i\}_{i\in\mathbb{N}}$ of distinct points of the set ${\uparrow_{\preccurlyeq}}\beta$ such that the following conditions hold:

(a) $\operatorname{rank}\alpha_{i+1} = k$ and ${\uparrow_{\preccurlyeq}}\alpha_{i+1} \setminus \left({\uparrow_{\preccurlyeq}}\alpha_1 \cup \cdots \cup {\uparrow_{\preccurlyeq}}\alpha_i\right) \cap U_X(x) \neq \varnothing$; and

(b) $\gamma_{i+1} \in \mathscr{I}_\lambda^n \setminus \mathscr{I}_\lambda^{n-1}$ and $\gamma_{i+1} \in {\uparrow_{\preccurlyeq}}\alpha_{i+1} \setminus \left({\uparrow_{\preccurlyeq}}\alpha_1 \cup \cdots \cup {\uparrow_{\preccurlyeq}}\alpha_i\right) \cap U_X(x)$,

for all positive integers $i > 1$.

Then Lemma 1 implies that the infinite family of non-empty open subsets $\mathscr{U} = \{\{\gamma_i\} : i \in \mathbb{N}\}$ is locally finite, which contradicts the feeble compactness of ${\uparrow_{\preccurlyeq}}\beta$. The obtained contradiction implies the statement of induction which completes the proof of the statement that the space $(\mathscr{I}_\lambda^n, \tau)$ is H-closed.

$(iv) \Rightarrow (v)$ By Katětov's Theorem every continuous image of an H-closed topological space into a Hausdorff space is H-closed (see [8, 3.15.5 (b)] or [22]). Hence the image $f(\mathscr{I}_\lambda^n)$ is H-closed for every continuous map $f\colon (\mathscr{I}_\lambda^n, \tau) \to \mathbb{N}_{\mathfrak{d}}$, which implies that $f(\mathscr{I}_\lambda^n)$ is compact (see [8, 3.15.5 (a)]).

$(v) \Rightarrow (ii)$ Suppose to the contrary that there exists a Hausdorff shift-continuous $\mathbb{N}_{\mathfrak{d}}$-compact topology τ on $\mathscr{I}_\lambda^n$ which is not feebly compact. Then there exists an infinite locally finite family $\mathscr{U} = \{U_i\}$ of open non-empty subsets of $(\mathscr{I}_\lambda^n, \tau)$. Without loss of generality we may assume that the family $\mathscr{U} = \{U_i\}$ is countable., i.e., $\mathscr{U} = \{U_i \colon i \in \mathbb{N}\}$. Then the definition of the semigroup $\mathscr{I}_\lambda^n$ and Lemma 2 imply that for every $U_i \in \mathscr{U}$ there exists $\alpha_i \in U_i$ such that ${\uparrow_{\preccurlyeq}}\alpha_i \cap U_i = \{\alpha_i\}$ and hence $\mathscr{U}^* = \{\{\alpha_i\} \colon i \in \mathbb{N}\}$ is a family of isolated points of $(\mathscr{I}_\lambda^n, \tau)$. Since the family $\mathscr{U}$ is locally finite, without loss of generality we may assume that $\alpha_i \neq \alpha_j$ for distinct $i, j \in \mathbb{N}$. We claim that the family $\mathscr{U}^*$ is locally finite. Indeed, if we assume the contrary then there exists $\alpha \in \mathscr{I}_\lambda^n$ such that every open neighbourhood of α contains infinitely many elements of the family $\mathscr{U}^*$. This implies that the family $\mathscr{U}$ is not locally finite, a contradiction. Since $(\mathscr{I}_\lambda^n, \tau)$ is a T_1-space and the family $\mathscr{U}^*$ is locally finite, we have that $\bigcup \mathscr{U}^*$ is a closed subset in

$(\mathscr{I}_\lambda^n,\tau)$ and hence the map $f\colon(\mathscr{I}_\lambda^n,\tau)\to\mathbb{N}_{\mathfrak{d}}$ defined by the formula

$$f(\beta)=\begin{cases} 1, & \text{if } \beta\in\mathscr{I}_\lambda^n\setminus\bigcup\mathscr{U}^*;\\ i+1, & \text{if } \beta=\alpha_i \text{ for some } i\in\mathbb{N},\end{cases}$$

is continuous. This contradicts that the space $(\mathscr{I}_\lambda^n,\tau)$ is $\mathbb{N}_{\mathfrak{d}}$-compact.

The proofs of implications $(iv)\Rightarrow(vi)$ and $(vi)\Rightarrow(ii)$ are same as the proofs of $(iv)\Rightarrow(v)$ and $(v)\Rightarrow(ii)$, respectively.

Implication $(ii)\Rightarrow(vii)$ follows from Proposition 2 and Theorem 3 of [20].

$(vii)\Rightarrow(ii)$ Suppose to the contrary that there exists a Hausdorff shift-continuous infra H-closed topology τ on $\mathscr{I}_\lambda^n$ which is not feebly compact. Then similarly as in the proof of implication $(v)\Rightarrow(ii)$ we choice a locally finite family $\mathscr{U}^*=\{\{\alpha_i\}\colon i\in\mathbb{N}\}$ of isolated points of $(\mathscr{I}_\lambda^n,\tau)$. Then the map $f\colon(\mathscr{I}_\lambda^n,\tau)\to\mathbb{R}$ defined by the formula

$$f(\beta)=\begin{cases} 1, & \text{if } \beta\in\mathscr{I}_\lambda^n\setminus\bigcup\mathscr{U}^*;\\ \dfrac{1}{i+1}, & \text{if } \beta=\alpha_i \text{ for some } i\in\mathbb{N},\end{cases}$$

is continuous. This contradicts that the space $(\mathscr{I}_\lambda^n,\tau)$ is infra H-closed. □

Remark 2. By Theorem 5 from [20] conditions (ii) and (vii) of Theorem 2 are equivalent for any Tychonoff space X.

It is not, however, the case that feebly compact and infra H-closed are equivalent in general. In [21] Herrlich, beginning with a T_1-space Y, constructs a regular space X such that the only continuous functions from X into Y are constant. His construction involves the cardinality of Y, but only as the cardinality of collections of open sets whose intersections are singletons. Thus only the most trivial modifications are needed in his argument to produce a regular Hausdorff infra H-closed space. It is also easily shown that the space constructed in this manner is not feebly compact.

Any regular lightly compact space must be a Baire space [25, Lemma 3], and thus it is of interest to note that the space constructed in [21] can be shown to be the countable union of nowhere dense subsets using essentially the same argument as can be used to show it is not feebly compact.

Later we need the following technical lemma.

Lemma 3. *Let n be an arbitrary positive integer and λ be an arbitrary infinite cardinal. Let τ be a feebly compact shift-continuous T_1-topology on the semigroup $\mathscr{I}_\lambda^n$. Then for every $\alpha\in\mathscr{I}_\lambda^n$ and any open neighbourhood $U(\alpha)$ of α in $(\mathscr{I}_\lambda^n,\tau)$ there exist finitely many $\alpha_1,\ldots,\alpha_k\in{\uparrow_{\preccurlyeq}}\alpha\setminus\{\alpha\}$ such that*

$$\mathscr{I}_\lambda^n\setminus\mathscr{I}_\lambda^{n-1}\cap{\uparrow_{\preccurlyeq}}\alpha\subseteq U(\alpha)\cup{\uparrow_{\preccurlyeq}}\alpha_1\cup\cdots\cup{\uparrow_{\preccurlyeq}}\alpha_k.$$

Proof. Suppose to the contrary that there exists a feebly compact shift-continuous T_1-topology on the semigroup $\mathscr{I}_\lambda^n$ which satisfies the following property: some element α of the semigroup $\mathscr{I}_\lambda^n$ has an open neighbourhood $U(\alpha)$ of α in $(\mathscr{I}_\lambda^n,\tau)$ such that

$$\mathscr{I}_\lambda^n\setminus\mathscr{I}_\lambda^{n-1}\cap{\uparrow_{\preccurlyeq}}\alpha\nsubseteq U(\alpha)\cup{\uparrow_{\preccurlyeq}}\alpha_1\cup\cdots\cup{\uparrow_{\preccurlyeq}}\alpha_k,$$

for any finitely many $\alpha_1,\ldots,\alpha_k\in{\uparrow_{\preccurlyeq}}\alpha\setminus\{\alpha\}$. We observe that Lemma 2 implies that without loss of generality we may assume that $U(\alpha)\subseteq{\uparrow_{\preccurlyeq}}\alpha$.

Fix such an element α of $\mathscr{I}_\lambda^n$ and its open neighbourhood $U(\alpha)$ with the above determined property. Then our assumption implies that there exists $\alpha_1 \in {\uparrow_{\preccurlyeq}}\alpha \setminus U(\alpha)$ such that the set

$$\mathscr{I}_\lambda^n \setminus \mathscr{I}_\lambda^{n-1} \cap {\uparrow_{\preccurlyeq}}\alpha_1 \setminus U(\alpha)$$

is infinite and fix an arbitrary $\gamma_1 \in \mathscr{I}_\lambda^n \setminus \mathscr{I}_\lambda^{n-1} \cap {\uparrow_{\preccurlyeq}}\alpha_1 \setminus U(\alpha)$. Next, by induction using our assumption we construct sequences $\{\alpha_i\}_{i\in\mathbb{N}}$ and $\{\gamma_i\}_{i\in\mathbb{N}}$ of the distinct points of the set ${\uparrow_{\preccurlyeq}}\alpha$ such that the following conditions hold:

(*a*) $\alpha_{i+1} \in {\uparrow_{\preccurlyeq}}\alpha \setminus \big(U(\alpha) \cup {\uparrow_{\preccurlyeq}}\alpha_1 \cup \cdots \cup {\uparrow_{\preccurlyeq}}\alpha_i\big)$ and the set

$$\mathscr{I}_\lambda^n \setminus \mathscr{I}_\lambda^{n-1} \cap {\uparrow_{\preccurlyeq}}\alpha \cap {\uparrow_{\preccurlyeq}}\alpha_{1+1} \setminus \big(U(\alpha) \cup {\uparrow_{\preccurlyeq}}\alpha_1 \cup \cdots \cup {\uparrow_{\preccurlyeq}}\alpha_i\big)$$

is infinite;

(*b*) $\gamma_{i+1} \in \mathscr{I}_\lambda^n \setminus \mathscr{I}_\lambda^{n-1} \cap {\uparrow_{\preccurlyeq}}\alpha \cap {\uparrow_{\preccurlyeq}}\alpha_1 \setminus \big(U(\alpha) \cup {\uparrow_{\preccurlyeq}}\alpha_1 \cup \cdots \cup {\uparrow_{\preccurlyeq}}\alpha_i\big)$,

for all positive integers i.

By Lemma 2 and Theorem 14 of [4] the space ${\uparrow_{\preccurlyeq}}\alpha$ is feebly compact. Then Lemma 1 implies that the infinite family of non-empty open subsets $\mathscr{U} = \{\{\gamma_i\} : i \in \mathbb{N}\}$ is locally finite, which contradicts the feeble compactness of ${\uparrow_{\preccurlyeq}}\alpha$. The obtained contradiction implies the statement of the lemma. □

Theorem 3. *Let n be an arbitrary positive integer and λ be an arbitrary infinite cardinal. Then every shift-continuous semiregular feebly compact T_1-topology τ on $\mathscr{I}_\lambda^n$ is compact.*

Proof. We shall prove the statement of the theorem by induction. First we observe that for every element α of a semiregular feebly compact T_1-semitopological semigroup $(\mathscr{I}_\lambda^n, \tau)$ with $\operatorname{rank}\alpha = n-1, n$ the set ${\uparrow_{\preccurlyeq}}\alpha$ is compact. Indeed, by Lemma 2 for every $\beta \in \mathscr{I}_\lambda^n$ the set ${\uparrow_{\preccurlyeq}}\beta$ is open-and-closed in $(\mathscr{I}_\lambda^n, \tau)$, and hence we have that $\mathscr{I}_\lambda^n \setminus \mathscr{I}_\lambda^{n-1}$ is an open discrete subspace of $(\mathscr{I}_\lambda^n, \tau)$ and using Theorem 14 of [4] we obtain that ${\uparrow_{\preccurlyeq}}\alpha$ is feebly compact, which implies that the space ${\uparrow_{\preccurlyeq}}\alpha$ is compact.

Next we shall prove a more stronger step of induction: *if for every element α of a semiregular feebly compact T_1-semitopological semigroup $(\mathscr{I}_\lambda^n, \tau)$ with $\operatorname{rank}\alpha > l \leqslant n$ the set ${\uparrow_{\preccurlyeq}}\alpha$ is compact, then ${\uparrow_{\preccurlyeq}}\beta$ is compact for every $\beta \in \mathscr{I}_\lambda^n$ with $\operatorname{rank}\alpha = l$.*

Suppose to the contrary that there exists a semiregular feebly compact T_1-semitopological semigroup $(\mathscr{I}_\lambda^n, \tau)$ such that for some positive integer $l \leqslant n$ the set ${\uparrow_{\preccurlyeq}}\alpha$ is compact for every $\alpha \in \mathscr{I}_\lambda^n$ with $\operatorname{rank}\alpha > l$, but there exists $\beta \in \mathscr{I}_\lambda^n$ with $\operatorname{rank}\beta = l$ such that the set ${\uparrow_{\preccurlyeq}}\beta$ is not compact.

First we observe that our assumption that the set ${\uparrow_{\preccurlyeq}}\alpha$ is compact and Corollary 3.1.14 of [8] imply that the following family

$$\mathscr{B}_{\mathsf{c}}(\alpha) = \big\{U_\alpha(\alpha_1, \ldots, \alpha_k) = {\uparrow_{\preccurlyeq}}\alpha \setminus ({\uparrow_{\preccurlyeq}}\alpha_1 \cup \cdots \cup {\uparrow_{\preccurlyeq}}\alpha_k) \colon \alpha_i \in {\uparrow_{\preccurlyeq}}\alpha \setminus \{\alpha\}, i = 1, \ldots, k\big\}$$

is a base of topology at the point α of $(\mathscr{I}_\lambda^n, \tau)$ for every $\alpha \in \mathscr{I}_\lambda^n$ with $\operatorname{rank}\alpha > l$.

Then the Alexander Subbase Theorem (see [1, Theorem 1] or [8, p. 221, 3.12.2(a)]) and Lemma 2 imply that there exists a base $\mathscr{B}$ of the topology τ on $\mathscr{I}_\lambda^n$ with the following properties:

(*i*) $\mathscr{B} = \bigcup\{\mathscr{B}(\gamma) \colon \gamma \in \mathscr{I}_\lambda^n\}$ and for every $\gamma \in \mathscr{I}_\lambda^n$ the family $\mathscr{B}(\gamma)$ is a base at the point γ;

(*ii*) $U(\gamma) \subseteq {\uparrow_{\preccurlyeq}}\gamma$ for any $U(\gamma) \in \mathscr{B}(\gamma)$;

(*iii*) $\mathscr{B}(\gamma) = \mathscr{B}_{\mathsf{c}}(\gamma)$ for every $\gamma \in \mathscr{I}_\lambda^n$ with $\operatorname{rank}\gamma > l$;

(iv) there exists a cover $\mathscr{U}$ of the set ${\uparrow_{\preccurlyeq}}\beta$ by members of the base $\mathscr{B}$ which has not a finite subcover.

We claim that the subspace ${\uparrow_{\preccurlyeq}}\beta$ of $(\mathscr{I}_\lambda^n,\tau)$ contains an infinite closed discrete subspace X. Indeed, let $\mathscr{U}_0$ be a subfamily of $\mathscr{U}$ such that

$$\{\beta\}\cup\left({\uparrow_{\preccurlyeq}}\beta\cap\mathscr{I}_\lambda^n\setminus\mathscr{I}_\lambda^{n-1}\right)\subseteq\bigcup\mathscr{U}_0.$$

Since the set ${\uparrow_{\preccurlyeq}}\beta$ is not compact and ${\uparrow_{\preccurlyeq}}\gamma$ is compact for any $\gamma\in{\uparrow_{\preccurlyeq}}\beta\setminus\{\beta\}$, without loss of generality we may assume that there exists $k>\operatorname{rank}\beta$ such that the following conditions hold:

(a) there exist infinitely many elements $\zeta\in{\uparrow_{\preccurlyeq}}\beta$ with $\operatorname{rank}\zeta=k$ such that $\zeta\notin\bigcup\mathscr{U}_0$;
(b) $\varsigma\in\bigcup\mathscr{U}_0$ for all $\varsigma\in{\uparrow_{\preccurlyeq}}\beta$ with $\operatorname{rank}\varsigma<k$.

It is obvious that the set

$$X={\uparrow_{\preccurlyeq}}\beta\setminus\left(\bigcup\mathscr{U}_0\setminus\bigcup\left\{{\uparrow_{\preccurlyeq}}\varsigma\colon\operatorname{rank}\varsigma>k\right\}\right)$$

is requested.

Fix an arbitrary regular open neighbourhood $U(\beta)$ of the point β in $(\mathscr{I}_\lambda^n,\tau)$ such that $U(\beta)\cap X=\varnothing$. By Lemma 3 there exist finitely many $\beta_1,\ldots,\beta_s\in{\uparrow_{\preccurlyeq}}\beta$ such that

$$\mathscr{I}_\lambda^n\setminus\mathscr{I}_\lambda^{n-1}\cap{\uparrow_{\preccurlyeq}}\beta\subseteq U(\beta)\cup{\uparrow_{\preccurlyeq}}\beta_1\cup\cdots\cup{\uparrow_{\preccurlyeq}}\beta_s.$$

It is obvious that the set $X\setminus({\uparrow_{\preccurlyeq}}\beta_1\cup\cdots\cup{\uparrow_{\preccurlyeq}}\beta_s)$ is infinite. For every $\delta\in X$ the set ${\uparrow_{\preccurlyeq}}\delta$ is compact and open, and moreover by Lemma 1 the set ${\uparrow_{\preccurlyeq}}\delta\setminus\left({\uparrow_{\preccurlyeq}}\beta_1\cup\cdots\cup{\uparrow_{\preccurlyeq}}\beta_s\right)$ contains infinitely many points of the neighbourhood $U(\beta)$. This implies that $\operatorname{int}_{\mathscr{I}_\lambda^n}\left(\operatorname{cl}_{\mathscr{I}_\lambda^n}(U(\beta))\right)\cap X\neq\varnothing$, which contradicts the assumption that $U(0)\cap X=\varnothing$. The obtained contradiction implies that the subspace ${\uparrow_{\preccurlyeq}}\beta$ of $(\mathscr{I}_\lambda^n,\tau)$ is compact, which completes the proof of the theorem. □

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References

1. J. W. Alexander, *Ordered sets, complexes and the problem of compactification*, Proc. Natl. Acad. Sci. USA **25** (1939), 296–298.
2. A. V. Arhangel'skii, *On spaces with point-countable base*, Topological spaces and their mappings, Riga, 1985, P. 3–7 (in Russian).
3. A. V. Arkhangel'skii, *Topological function spaces*, Kluwer Publ., Dordrecht, 1992.
4. R. W. Bagley, E. H. Connell, and J. D. McKnight, Jr., *On properties characterizing pseudo-compact spaces*, Proc. Amer. Math. Soc. **9** (1958), no. 3, 500–506.
5. S. Bardyla and O. Gutik, *On a semitopological polycyclic monoid*, Algebra Discrete Math. **21** (2016), no. 2, 163–183.
6. J. H. Carruth, J. A. Hildebrant, and R. J. Koch, *The theory of topological semigroups*, Vols. **I** and **II**, Marcell Dekker, Inc., New York and Basel, 1983 and 1986.
7. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vols. **I** and **II**, Amer. Math. Soc. Surveys **7**, Providence, R.I., 1961 and 1967.
8. R. Engelking, *General topology*, 2nd ed., Heldermann, Berlin, 1989.
9. G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, and D. S. Scott, *Continuous lattices and domains*, Cambridge Univ. Press, Cambridge, 2003.

10. O. Gutik, *On closures in semitopological inverse semigroups with continuous inversion*, Algebra Discrete Math. **18** (2014), no. 1, 59-–85.
11. O. Gutik, J. Lawson, and D. Repovš, *Semigroup closures of finite rank symmetric inverse semigroups*, Semigroup Forum **78** (2009), no. 2, 326–336.
12. O. V. Gutik and K. P. Pavlyk, *On topological semigroups of matrix units*, Semigroup Forum **71** (2005), no. 3, 389–400.
13. O. V. Gutik and K. P. Pavlyk, *Topological semigroups of matrix units*, Algebra Discrete Math. (2005), no. 3, 1–17.
14. O. Gutik, K. Pavlyk, and A. Reiter, Topological semigroups of matrix units and countably compact Brandt λ^0-extensions, Mat. Stud. **32** (2009), no. 2, 115–131.
15. O. V. Gutik and O. V. Ravsky, *Pseudocompactness, products and topological Brandt λ^0-extensions of semitopological monoids*, Math. Methods and Phys.-Mech. Fields **58** (2015), no. 2, 20–37; reprinted version: J. Math. Sc. **223** (2017), no. 1, 18–38.
16. O. V. Gutik and A. R. Reiter, *Symmetric inverse topological semigroups of finite rank $\leqslant n$*, Math. Methods and Phys.-Mech. Fields **52** (2009), no. 3, 7–14; reprinted version: J. Math. Sc. **171** (2010), no. 4, 425–432.
17. O. Gutik and A. Reiter, *On semitopological symmetric inverse semigroups of a bounded finite rank*, Visnyk Lviv Univ. Ser. Mech. Math. **72** (2010), 94–106 (in Ukrainian).
18. O. Gutik and O. Sobol, *On feebly compact topologies on the semilattice* $\exp_n \lambda$, Mat. Stud. **46** (2016), no. 1, 29–43.
19. O. Gutik and O. Sobol, *On feebly compact shift-continuous topologies on the semilattice* $\exp_n \lambda$, Visnyk Lviv Univ. Ser. Mech. Math. **82** (2016), 128–136.
20. D. W. Hajek and A. R. Todd, *Compact spaces and infra H-closed spaces*, Proc. Amer. Math. Soc. **48** (1975), no. 2, 479–482.
21. H. Herrlich, *Wann sind alle stetigen Abbildungen in Y konstant?*, Math. Z. **90** (1965), 152–154.
22. M. Katětov, Über H-abgeschlossene und bikompakte Räume, Čas. Mat. Fys. **69** (1940), no. 2, 36–49.
23. M. V. Lawson, *Inverse semigroups. The theory of partial symmetries*, World Scientific, Singapore, 1998.
24. M. Matveev, *A survey of star covering properties*, Topology Atlas preprint, April 15, 1998.
25. R. A. McCoy, *A filter characterization of regular Baire spaces*, Proc. Amer. Math. Soc. **40** (1973), 268–270.
26. M. Petrich, *Inverse semigroups*, John Wiley & Sons, New York, 1984.
27. W. Ruppert, *Compact semitopological semigroups: an intrinsic theory*, Lect. Notes Math., **1079**, Springer, Berlin, 1984.
28. J. E. Vaughan, *Countably compact and sequentially compact spaces*, Handbook of set-theoretic topology, K. Kunen and J. E. Vaughan (eds.), North-Holland, Amsterdam, 1984, pp. 569–602.
29. V. V. Wagner, *Generalized groups*, Dokl. Akad. Nauk SSSR **84** (1952), 1119–1122 (in Russian).

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ПРО СЛАБКО КОМПАКТНІ НАПІВТОПОЛОГІЧНІ СИМЕТРИЧНІ ІНВЕРСНІ НАПІВГРУПИ ОБМЕЖЕНОГО СКІНЧЕННОГО РАНГУ

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Ми вивчаємо слабко компактні T_1-топології на симетричній інверсній напівгрупі $\mathscr{I}_\lambda^n$ скінченних перетворень рангу $\leqslant n$, які перетворюють її в напівтопологічну напівгрупу. Для довільного натурального числа $n \geqslant 2$ і для кожного нескінченного кардинала λ побудована гаусдорфова зліченно пракомпакна некомпактна топологія на напівгрупі $\mathscr{I}_\lambda^n$, яка перетворює її в напівтопологічну напівгрупу. Доведено, що для довільного натурального числа n і для довільного нескінченного кардинала λ для T_1-топології τ на напівгрупі $\mathscr{I}_\lambda^n$ такі умови є еквівалентними: (i) τ — зліченно пракомпактна; (ii) τ — слабко компактна; (iii) τ — d-слабко компактна; (iv) простір $(\mathscr{I}_\lambda^n, \tau)$ є H-замкненим; (v) простір $(\mathscr{I}_\lambda^n, \tau)$ є $\mathbb{N}_\mathfrak{d}$-компактним для дискретного зліченного простору $\mathbb{N}_\mathfrak{d}$; (vi) простір $(\mathscr{I}_\lambda^n, \tau)$ є $\mathbb{R}$-компактним; (vii) простір $(\mathscr{I}_\lambda^n, \tau)$ є інфра H-замкненим. Також доведено, що для довільного натурального числа n і для довільного нескінченного кардинала λ кожна напіврегулярна слабко компактна T_1-топологія на $\mathscr{I}_\lambda^n$, яка перетворює її в напівтопологічну напівгрупу, є компактною.

Ключові слова: напівгрупа, інверсна напівгрупа, напівтопологічна напівгрупа, компактний, зліченно компактний, зліченно пракомпактний, слабко компактний, H-замкнений, інфра H-замкнений, X-компактний, напіврегулярний простір.