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CONE AND JOIN IN THE ASYMPTOTIC CATEGORIES

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We will prove that the join $\mathbb{R}^n * \mathbb{R}_+$ is isomorphic to the half-space $\mathbb{R}^{n+1}_+$ and extend this result onto the class of γ-weakly convex and δ-weakly concave geodesic spaces. We will also show that the cone over divergent sequence is not isomorphic to the join of this sequence and $\mathbb{R}_+$ in the asymptotic category $\mathcal{A}$.

Key words: join, cone, asymptotic category.

1. Introduction.

Asymptotic topology is a part of mathematics dealing with large scale properties of metric spaces and, more generally, coarse spaces. Backgrounds of the asymptotic topology are described in [1]. In particular, this paper contains basic functorial constructions in the coarse categories.

Some of these constructions are considered in the present note. We establish relations between the cones and joins in the asymptotic categories.

2. Terminology and notation.

A metric space (X, d) is *proper* if every closed ball in X is compact.

A map $f\colon X \to Y$ is *proper* if the preimage of every compact subset is compact. A map $f\colon X \to Y$ is *coarsely proper*, if the preimage of every bounded set id bounded.

A map $f\colon (X, d) \to (Y, \rho)$ is *coarsely uniform*, if there is a non-decreasing function $\varphi\colon [0, \infty) \to [0, \infty)$ such that $\lim_{t\to\infty} \varphi(t) = \infty$ and $\rho(f(x), f(y)) \leqslant \varphi(d(x, y))$ for all $x, y \in X$.

A map f is *coarse*, if f is coarse uniform and coarse proper.

A map $f\colon (X, d) \to (Y, \rho)$ is *asymptotically Lipschitz*, if there are λ *i* s ($\lambda > 0,\ s \geqslant 0$) such that

$$\rho(f(x), f(y)) \leqslant \lambda d(x, y) + s, \quad x, y \in X.$$

The objects of the asymptotic category $\mathcal{A}$ are proper metric spaces, the morphisms of this category are proper asymptotically Lipschitz maps.

The objects of the asymptotic category $\overline{\mathcal{A}}$ are proper metric spaces (actually, one can consider all metric spaces), the morphisms of this category are coarsely proper, asymptotically Lipschitz maps.

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An *isomorphism* in the category $\overline{\mathcal{A}}$ is a homeomorphism $f\colon X\to Y$ such that f and f^{-1} are asymptotically Lipschitz. A morphism $f\colon X\to Y$ в $\overline{\mathcal{A}}$ is called a *coarse isomorphism*, if there is a morphism $g\colon Y\to X$ such that $f\circ g$ and $g\circ f$ are equivalent to the identity maps 1_X and 1_Y respectively. Metric spaces X and Y are *coarse isomorphic* (quasi-isometric), if there exists a coarse isomorphism $f\colon X\to Y$.

A map $f\colon X\to Y$ of metric spaces (X,d) and (Y,ρ) is called a *quasi-isometry* if there exist $C,D\geqslant 0$, $\lambda>0$ such that

$$\frac{1}{\lambda}d(x,y)-C\leqslant\rho(f(x),f(y))\leqslant\lambda d(x,y)+C,\quad x,y\in X$$

and the D-neighborhood of the set $f(X)$ equals Y.

Let $C>0$. A set in a metric space is called *C-connected* if, for every $x,y\in M$, there exist $x_0=x,x_1,\dots,x_{n-1},x_n=y\in M$ such that $d(x_i,x_{i-1})\leqslant C$, $i=1,\dots,n$.

3. Main result.

Let X be a metric space. The cone CX of X is defined as follows: $CX=X\tilde{\times}\mathbb{R}^2_+/i_+(X)$, where $i_+\colon X\to X\tilde{\times}\mathbb{R}^2_+$ is the embedding defined by the formula $i_+(x)=(x,\|x\|,0)$ (see [1]).

Lemma 1. *The cone $C\mathbb{R}$ is not isomorphic to the half-space $\mathbb{R}^2_+$ in the asymptotic category $\mathcal{A}$.*

Proof. Suppose that the cone $C\mathbb{R}$ is isomorphic (quasi-isometric) to the half-plane $\mathbb{R}^2_+$. Thus, there exists $f\colon C\mathbb{R}\to\mathbb{R}^2_+$ such that, for some $C\geqslant 0$ and $\lambda>0$,

$$\frac{1}{\lambda}d(x,y)-C\leqslant\rho(f(x),f(y))\leqslant\lambda d(x,y)+C,\quad x,y\in C\mathbb{R}.$$

Pick and infinite sequence $x_1,x_2,x_3,\dots$ in $C\mathbb{R}$ and $d(x_i,x_j)=2\lambda C+1$ for all $i,j\in\mathbb{N}$. This is easy to construct in $C\mathbb{R}$, let, e.g.,

$$x_k=\left(k(2\lambda C+1),(2\lambda C+1)\frac{1-8k^2}{8k},(2\lambda C+1)\frac{\sqrt{16k^2-1}}{8k}\right).$$

The images $x_1,x_2,x_3,\dots$ belong to the neighborhood $O_{f(x_1)}(2\lambda^2C+\lambda+C)$ and

$$d(f(x_i),f(x_j))\geqslant\frac{1}{\lambda}d(x_i,x_j)-C=\frac{1}{\lambda}(2\lambda C+1)-C=C+\frac{1}{\lambda},$$

which provides a contradiction. □

Lemma 1 contradicts to the statement from [1] that for geodesic spaces X the cone can be defined by the formula $CX=X\times\mathbb{R}_+$.

3.1. Kantorovich-Rubinstein metric on the join $X*\mathbb{R}_+$.

For any two pointed metric spaces X and Y one can define the bouquet $X\vee Y$. We endow the bouquet with the natural quotient metric. The join $X*\mathbb{R}_+$ is the subspace of $P_2(X\vee\mathbb{R}_+)$ of probability measures with supports of cardinality ≤ 2. Let us define the Kantorovich-Rubinstein distance on the join $X*\mathbb{R}_+$ between two probability measures μ and ν,

$$\mu=\alpha\delta_x+(1-\alpha)\delta_y$$
$$\nu=\beta\delta_{x'}+(1-\beta)\delta_{y'}$$
$$\|x\|=y,\ \|x'\|=y',\ \{x,x'\}\subset X,\ \{y,y'\}\subset\mathbb{R}_+,$$

$$d_{KP}(\mu,\nu) = \inf\{\varepsilon d(y',x) + (\alpha-\varepsilon)d(x,x') + (1-\beta-\varepsilon)d(y,y') + (\beta-\alpha+\varepsilon)d(x',y) \mid \varepsilon \geqslant 0, \varepsilon \geqslant \alpha-\beta\} =$$

$$\inf\{\varepsilon(d(y',x) - d(x,x') - d(y,y') + d(x',y)) + \alpha d(x,x') + (1-\beta)d(y,y') + (\beta-\alpha)d(x',y) \mid \varepsilon \geqslant 0, \varepsilon \geqslant \alpha-\beta\} =$$

$$= \begin{cases} (\beta-\alpha)d(x',y) + \alpha d(x',x) + (1-\beta)d(y,y'), & \beta > \alpha, \\ (\alpha-\beta)d(x,y') + \beta d(x,x') + (1-\alpha)d(y,y'), & \beta \leqslant \alpha, \end{cases} =$$

$$|\alpha-\beta|(y+y') + \min\{\alpha,\beta\}d(x,x') + (1-\max\{\alpha,\beta\})|y-y'|.$$

Lemma 2. *The join $\mathbb{R}^n * \mathbb{R}_+$ is isomorphic to the half-space $\mathbb{R}^{n+1}_+$ in the asymptotic topology $\mathcal{A}$.*

Proof. Consider the map $\varphi\colon \mathbb{R}^n * \mathbb{R}_+ \to \mathbb{R}^{n+1}_+$ defined by the formula

$$\varphi(\alpha\delta_x + (1-\alpha)\delta_y) = (\alpha x, (1-\alpha)y)$$
$$x \in \mathbb{R}^n, y \in \mathbb{R}_+.$$

The inverse map $\varphi^{-1}\colon \mathbb{R}^{n+1}_+ \to \mathbb{R}^n * \mathbb{R}_+$ is defined by the formula

$$\varphi^{-1}(l,t) = \frac{t}{\|l\|+t}\delta_{\frac{\|l\|+t}{t}l} + \frac{\|l\|}{\|l\|+t}\delta_{\frac{\|l\|+t}{\|l\|}t}.$$

Denoting $x = \frac{\|l\|+t}{t}l$, $\alpha = \frac{t}{\|l\|+t}$, $y = \frac{\|l\|+t}{\|l\|}t$, we obtain

$$\varphi^{-1}(\alpha x, (1-\alpha)y) = (\alpha\delta_x + (1-\alpha)\delta_y).$$

Show that the map φ^{-1} is Lipschitz.

$$d_{KP}(\varphi^{-1}(\alpha x,(1-\alpha)y); \varphi^{-1}\beta x',(1-\beta)y') = d_{KP}(\alpha\delta_x + (1-\alpha)\delta_y, \beta\delta_{x'} + (1-\beta)\delta_{y'}).$$

Without loss of generality one may assume that $y' > y$.

First, prove that φ^{-1} is Lipschitz for $\beta > \alpha$.

$$d_{KP}(\alpha\delta_x + (1-\alpha)\delta_y, \beta\delta_{x'} + (1-\beta)\delta_{y'}) =$$
$$= (\beta-\alpha)d(x',y) + \alpha d(x,x') + (1-\beta)d(y,y') \leqslant$$
$$\leqslant (\beta-\alpha)d(x',y) + d(\alpha x,\beta x') + (\beta-\alpha)\|x'\| + (1-\beta)d(y,y') =$$
$$= d(\alpha x,\beta x') + 2(\beta-\alpha)y' + (\beta-\alpha)y + (1-\beta)(y'-y)$$

Taking into account that

$$(1-\beta)(y'-y) \leqslant |(1-\alpha)y - (1-\beta)y'| + (\beta-\alpha)y$$

for $\beta > \alpha$ and $y' > y$, we obtain

$$\leqslant d(\alpha x,\beta x') + 2(\beta-\alpha)y' + 2(\beta-\alpha)y + |(1-\alpha)y - (1-\beta)y'| \leqslant$$
$$\leqslant d(\alpha x,\beta x') + 4(\beta-\alpha)y' + |(1-\alpha)y - (1-\beta)y'| \leqslant$$

Since $(\beta-\alpha)y' \leqslant \beta y' - \alpha y \leqslant d(\alpha x,\beta x')$, we obtain

$$\leqslant 5d(\alpha x,\beta x') + d((1-\alpha)y,(1-\beta)y') \leqslant$$
$$\leqslant 5d_{\mathbb{R}^{n+1}_+}((\alpha x,(1-\alpha)y);(\beta x',(1-\beta)y')).$$

Let us check that φ^{-1} is Lipschitz for $\beta \leqslant \alpha$.

$$d_{KP}(\alpha\delta_x + (1-\alpha)\delta_y, \beta\delta_{x'} + (1-\beta)\delta_{y'}) =$$

$$= (\alpha-\beta)d(x,y') + \beta d(x,x') + (1-\alpha)d(y,y') \leqslant$$

$$\leqslant (\alpha-\beta)d(x,y') + (\alpha-\beta)\|x\| + d(\alpha x,\beta x') + (1-\alpha)d(y,y') =$$

$$= d(\alpha x,\beta x') + (\alpha-\beta+1-\alpha)y' + (\alpha-\beta+\alpha-\beta-1+\alpha)y =$$

$$= d(\alpha x,\beta x') + (1-\beta)y' - (1-\alpha)y + 2(\alpha-\beta)y$$

Since $2(\alpha-\beta)y \leqslant 2d((1-\beta)y',(1-\alpha)y)$, we obtain

$$\leqslant 3d(\alpha x,\beta x') + 3d((1-\alpha)y,(1-\beta)y') =$$

$$= 3d_{\mathbb{R}^{n+1}_+}((\alpha x,(1-\alpha)y);(\beta x',(1-\beta)y')).$$

Show that φ is Lipschitz with constant 1. Suppose that $y' \geqslant y$. Consider two cases: 1. $\beta > \alpha$ and 2. $\beta \leqslant \alpha$.

$$1.\ d_{\mathbb{R}^{n+1}_+}(\varphi(\alpha\delta_x,(1-\alpha)\delta_y);\varphi(\beta\delta_{x'},(1-\beta)\delta_{y'})) =$$

$$= d_{\mathbb{R}^n\times\mathbb{R}_+}((\alpha x,(1-\alpha)y);(\beta x',(1-\beta)y')) =$$

$$= d_{\mathbb{R}^n}(\alpha x,\beta x') + d_{\mathbb{R}_+}((1-\alpha)y,(1-\beta)y') \leqslant$$

$$\leqslant \alpha d(x,x') + (\beta-\alpha)\|x'\| + (1-\beta)(y'-y) + (\beta-\alpha)y =$$

The latter inequality is a consequence of the following two inequalities:

$$d(\alpha x,\beta x') \leqslant \alpha d(x,x') + (\beta-\alpha)\|x'\|$$

$$d((1-\alpha)y,(1-\beta)y') \leqslant (1-\beta)(y'-y) + (\beta-\alpha)y$$

$$= \alpha d(x,x') + (\beta-\alpha)d(x',y) + (1-\beta)d(y,y') =$$

$$= d_{KP}(\alpha\delta_x + (1-\alpha)\delta_y, \beta\delta_{x'} + (1-\beta)\delta_{y'}).$$

$$2.\ \ d_{\mathbb{R}^n}(\alpha x,\beta x') + d_{\mathbb{R}_+}((1-\alpha)y,(1-\beta)y') \leqslant$$

Since $d(\alpha x,\beta x') \leqslant \beta d(x,x') + (\alpha-\beta)\|x\|$, we obtain

$$\beta d(x,x') + (\alpha-\beta)\|x\| + (1-\beta)y' - (1-\alpha)y =$$

$$= \beta d(x,x') + (\alpha-\beta)\|x\| + \alpha y' - \beta y' - \alpha y' + y' - (1-\alpha)y =$$

$$= \beta d(x,x') + (\alpha-\beta)(\|x\| + y') + (1-\alpha)(y'-y) =$$

$$= d_{KP}(\alpha\delta_x + (1-\alpha)\delta_y, \beta\delta_{x'} + (1-\beta)\delta_{y'}).$$

□

Lemmas 1 and 2 imply the following

Corollary 1. *The join $R * \mathbb{R}_+$ is not isomorphic to the cone $C\mathbb{R}$ in the asymptotic category $\overline{\mathcal{A}}$.*

Lemma 3. *Let $X = \{n^2 \mid n \in \mathbb{N}\} \subset \mathbb{R}$. The join $X * \mathbb{R}_+$ is not isomorphic to the cone CX in the asymptotic category $\overline{\mathcal{A}}$.*

Proof. Let $f\colon CX \to X * \mathbb{R}_+$ be a coarse map. The image of every segment $[x, y]$ from CX is contained in the ball $O_{s(1)}(f(x))$.

Consider in CX a segment $[x_0, x], d(x_0, x) = n$. Construct a sequence

$$x_0, x_1, x_2, \ldots, x_n = x,\ d(x_i, x_{i+1}) = 1.$$

Since the image of every segment $[x_i, x_{i+1}]$ is contained in the ball $O_{s(1)}(f(x_i))$, the image $[x_0, x]$ is a bounded set, since $X * \mathbb{R}_+$ is not $S(1)$-connected for all $S(1) > 0$.

Thus, there is no coarse uniform map of the cone CX into the join $X * \mathbb{R}_+$ with unbounded image. □

Lemma 3 also holds in the asymptotic category $\mathcal{A}$.

4. Weakly γ-convex and weakly δ-concave geodesic spaces.

A metric space (X, d) is called geodesic if, for any $x, y \in X$, there is an isometric embedding $c\colon [0, d(x, y)] \to X$ such that $c(0) = x$ and $c(d(x, y)) = y$. Any isometric embedding $c_{xy}\colon [0, d(x, y)] \to X$ such that $c_{xy}(0) = x$, $c_{xy}(d(x, y)) = y$, will be called an isometric segment connecting $x \in X$ and $y \in Y$.

A geodesic space (X, d) is called γ-weakly convex ($\gamma \geqslant 1$), if every pair of geodesic segments, c_{xy} and c_{xz}, satisfies the inequality

$$d(c_{xy}(t \cdot d(x, y)), c_{xz}(t \cdot d(x, z))) \leqslant \gamma \cdot t \cdot d(y, z).$$

A geodesic space (X, d) is called weakly δ-concave ($0 < \delta \leqslant 1$), if every pair of geodesic segments, c_{xy} and c_{xz}, satisfies the inequality

$$d(c_{xy}(t \cdot d(x, y)), c_{xz}(t \cdot d(x, z))) \geqslant \delta \cdot t \cdot d(y, z).$$

Lemma 4. *Let X be a weakly γ-convex and weakly δ-concave geodesic space. The join $X * \mathbb{R}_+$ is isomorphic to the space $X \times \mathbb{R}_+$ in the asymptotic category $\mathcal{A}$.*

Proof. The proof is similar to that of Lemma 2. Let $x_0 \in X$ be a fixed point. Define the norm $\|x\|$ of $x \in X$ as $d_X(x, x_0)$. For the sake of brevity, $c_{x_0 x}(\alpha \cdot d(x_0, x))$ will be denoted x_α in the sequel.

Consider the map $\varphi\colon X * \mathbb{R}_+ \to X \times \mathbb{R}_+$, defined by the formula

$$\varphi(\alpha\delta_x + (1-\alpha)\delta_y) = (x_\alpha, (1-\alpha)y)$$

$$x \in X, y \in \mathbb{R}_+.$$

The inverse map $\varphi^{-1}\colon X \times \mathbb{R}_+ \to X * \mathbb{R}_+$ is defined by the formula

$$\varphi^{-1}(x_\alpha, (1-\alpha)y) = (\alpha\delta_x + (1-\alpha)\delta_y).$$

Show that φ^{-1} is a Lipschitz map.

$$d_{KP}(\varphi^{-1}(x_\alpha, (1-\alpha)y); \varphi^{-1}(x'_\beta, (1-\beta)y') = d_{KP}(\alpha\delta_x + (1-\alpha)\delta_y, \beta\delta_{x'} + (1-\beta)\delta_{y'}).$$

Without loss of generality we may assume that $y' > y$.

We first prove that φ^{-1} is Lipschitz for $\beta > \alpha$.

$$\begin{aligned} &d_{KP}(\alpha\delta_x + (1-\alpha)\delta_y, \beta\delta_{x'} + (1-\beta)\delta_{y'}) \\ &= (\beta-\alpha)d(x', y) + \alpha d(x, x') + (1-\beta)d(y, y') \\ &\leqslant (\beta-\alpha)d(x', y) + \frac{1}{\delta}d(x_\alpha, x'_\beta) + \frac{1}{\delta}(\beta-\alpha)\|x'\| + (1-\beta)d(y, y') \end{aligned}$$

$$= \frac{1}{\delta}d(x_\alpha, x'_\beta) + (1+\frac{1}{\delta})(\beta-\alpha)y' + (\beta-\alpha)y + (1-\beta)(y'-y)$$

Taking into account that $(1-\beta)(y'-y) \leqslant |(1-\alpha)y-(1-\beta)y'| + (\beta-\alpha)y$ for $\beta > \alpha$ *i* $y' > y$, the inequality cam be continued:

$$\leqslant \frac{1}{\delta}d(x_\alpha, x'_\beta) + (1+\frac{1}{\delta})(\beta-\alpha)y' + 2(\beta-\alpha)y + |(1-\alpha)y-(1-\beta)y'|$$

$$\leqslant \frac{1}{\delta}d(x_\alpha, x'_\beta) + (3+\frac{1}{\delta})(\beta-\alpha)y' + |(1-\alpha)y-(1-\beta)y'|$$

Since $(\beta-\alpha)y' \leqslant \beta y' - \alpha y \leqslant d(x_\alpha, x'_\beta)$, we obtain

$$\leqslant (3+\frac{2}{\delta})d(x_\alpha, x'_\beta) + d((1-\alpha)y, (1-\beta)y')$$

$$\leqslant (3+\frac{2}{\delta})d_{X\times\mathbb{R}_+}((x_\alpha, (1-\alpha)y); (x'_\beta, (1-\beta)y')).$$

Let us check that φ^{-1} is Lipschitz for $\beta \leqslant \alpha$.

$$d_{KP}(\alpha\delta_x + (1-\alpha)\delta_y, \beta\delta_{x'} + (1-\beta)\delta_{y'})$$
$$= (\alpha-\beta)d(x,y') + \beta d(x,x') + (1-\alpha)d(y,y')$$
$$\leqslant (\alpha-\beta)d(x,y') + \frac{1}{\delta}(\alpha-\beta)\|x\| + \frac{1}{\delta}d(x_\alpha, x'_\beta) + (1-\alpha)d(y,y')$$
$$= \frac{1}{\delta}d(x_\alpha, x'_\beta) + (\alpha-\beta+1-\alpha)y' + (\alpha-\beta+\frac{1}{\delta}(\alpha-\beta)-1+\alpha)y$$
$$= \frac{1}{\delta}d(x_\alpha, x'_\beta) + (1-\beta)y' - (1-\alpha)y + (1+\frac{1}{\delta})(\alpha-\beta)y$$

Taking into account that $(\alpha-\beta)y \leqslant d((1-\beta)y', (1-\alpha)y)$, we obtain

$$\leqslant \frac{1}{\delta}d(x_\alpha, x'_\beta) + (2+\frac{1}{\delta})d((1-\alpha)y, (1-\beta)y')$$

$$\leqslant (2+\frac{1}{\delta})d_{X\times\mathbb{R}_+}((x_\alpha, (1-\alpha)y); (x'_\beta, (1-\beta)y')).$$

Show that φ is a Lipschitz map with constant γ. Take $y' \geqslant y$. Consider two cases: 1. $\beta > \alpha$ and 2. $\beta \leqslant \alpha$.

$$1.\ d_{X\times\mathbb{R}_+}(\varphi(\alpha\delta_x, (1-\alpha)\delta_y); \varphi(\beta\delta_{x'}, (1-\beta)\delta_{y'}))$$
$$= d_{X\times\mathbb{R}_+}((x_\alpha, (1-\alpha)y); (x'_\beta, (1-\beta)y'))$$
$$= d_X(x_\alpha, x'_\beta) + d_{\mathbb{R}_+}((1-\alpha)y, (1-\beta)y')$$
$$\leqslant \gamma\alpha d(x,x') + (\beta-\alpha)\|x'\| + (1-\beta)(y'-y) + (\beta-\alpha)y$$

The latter inequality is a consequence of the following two inequalities:

$$d(x_\alpha, x'_\beta) \leqslant \gamma\alpha d(x,x') + (\beta-\alpha)\|x'\|$$

$$d((1-\alpha)y, (1-\beta)y') \leqslant (1-\beta)(y'-y) + (\beta-\alpha)y$$
$$= \gamma\alpha d(x,x') + (\beta-\alpha)d(x',y) + (1-\beta)d(y,y')$$
$$\leqslant \gamma d_{KP}(\alpha\delta_x + (1-\alpha)\delta_y, \beta\delta_{x'} + (1-\beta)\delta_{y'}).$$

$$2.\ d_X(x_\alpha, x'_\beta) + d_{\mathbb{R}_+}((1-\alpha)y, (1-\beta)y')$$

Since $d_X(x_\alpha, x'_\beta) \leqslant \gamma\beta d(x,x') + (\alpha-\beta)\|x\|$, we obtain

$$\begin{aligned} &\leqslant \gamma\beta d(x,x') + (\alpha-\beta)\|x\| + (1-\beta)y' - (1-\alpha)y \\ &= \gamma\beta d(x,x') + (\alpha-\beta)\|x\| + \alpha y' - \beta y' - \alpha y' + y' - (1-\alpha)y \\ &= \gamma\beta d(x,x') + (\alpha-\beta)(\|x\| + y') + (1-\alpha)(y'-y) \\ &\leqslant \gamma d_{KP}(\alpha\delta_x + (1-\alpha)\delta_y, \beta\delta_{x'} + (1-\beta)\delta_{y'}). \end{aligned}$$

□

5. Remarks and open questions.

It was asked in [1] whether the cone CX and the join $X * \mathbb{R}_+$ are isomorphic. Corollary 1 provides a negative answer to this question; in particular, these spaces are not isomorphic for $X = \mathbb{R}$. Lemma 3 contains an example of a non-geodesic space X for which these spaces are not isomorphic.

This leads to the following questions.

1. Is there a non-bounded metric space X for which the CX and the join $X * \mathbb{R}_+$ are isomorphic in the asymptotic category $\mathcal{A}$?
2. Are the join $X * \mathbb{R}_+$ and $X \times \mathbb{R}_+$ isomorphic for all geodesic spaces X? For $X = \mathbb{R}^n_+$, the answer is given by Lemma 2.
3. Let $\mathbb{H}$ denote the hyperbolic space. Are the spaces $\mathbb{H}\times\mathbb{R}_+$ and $\mathbb{H}*\mathbb{R}_+$ isomorphic?

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КОНУС І ДЖОЙ В АСИМПТОТИЧНИХ КАТЕГОРІЯХ

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Доведено, що джойн $\mathbb{R}^n * \mathbb{R}_+$ ізоморфний півпросторові $\mathbb{R}^{n+1}_+$. Цей результат поширено на клас γ-слабко опуклих і δ-слабко вгнутих геодезійних просторів. Доведено також, що конус над розбіжною послідовністю не ізоморфний джойнові цієї послідовності і $\mathbb{R}_+$ в асимптотичній категорії $\mathcal{A}$.

Ключові слова: джойн, конус, асимптотична категорія.