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ON EXISTENCE AND UNIQUENESS OF VARIATIONAL SOLUTIONS TO DIRICHLET BOUNDARY VALUE PROBLEM FOR NONLINEAR ELLIPTIC EQUATION WITH NONSTANDARD GROWTH CONDITIONS

Pavlo TKACHENKO

Oles Honchar Dnipropetrovsk National University,
Haharina av., 72, Dnipro, Ukraine
e-mail: cool.phenom@mail.ru

The Dirichlet boundary value problem for a nonlinear elliptic equation with nonstandard growth conditions in the main part of operator is considered. There is a peculiarity of this problem, which means that without a preliminary definition of an intermediate space, where the solution is searched, a Lavrentiev effect may be observed. Existence and uniqueness of variational solutions for each intermediate weighted Sobolev-Orlicz space are proven.

Key words: nonlinear elliptic equation, variable exponent, Lavrentiev effect, Sobolev-Orlicz space.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with Lipschitz boundary, let $p\colon \Omega \to \mathbb{R}$ be a Lebesgue measurable function such that $1 < \alpha \leq p(x) \leq \beta < +\infty$ for a.e. $x \in \Omega$. Let also $\mu\colon \Omega \to \mathbb{R}$ be a measurable function such that $\mu \in L^1(\Omega)$, $\mu(x) > 0$ for a.e. $x \in \Omega$ and $\mu(x)^{-\frac{1}{p(x)}} \in L^{q(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$ for a.e. $x \in \Omega$. Here $L^{q(x)}(\Omega)$ is a well-known variable Lebesgue space.

Let $L^{p(x)}(\Omega, \mu\,\mathrm{d}x)$ be a functional space which is defined as follows:

$$L^{p(x)}(\Omega, \mu\,\mathrm{d}x) = \left\{ v\colon \Omega \to \mathbb{R}\colon \int\limits_\Omega |v(x)|^{p(x)}\, \mu(x)\,\mathrm{d}x < +\infty \right\}.$$

Unlike $L^{p(x)}(\Omega)$, these spaces are far less known, but, nonetheless, under the given constraints on p they have almost the same properties as $L^{p(x)}(\Omega)$: reflexivity, separability

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2010 Mathematics Subject Classification: 35J60 and completeness with respect to the Luxemburg norm [1]

$$\|u\|_{L^{p(x)}(\Omega,\mu\,\mathrm{d}x)} = \inf\left\{\lambda > 0\colon \int\limits_{\Omega} |u(x)/\lambda|^{p(x)}\,\mu(x)\,\mathrm{d}x \le 1\right\}.$$

These spaces are usually called weighted variable Lebesgue spaces or weighted Lebesgue-Orlicz spaces. We will also take into consideration another type of spaces, namely, weighted Sobolev-Orlicz spaces, which are separable reflexive Banach spaces generally defined as follows:

$$W_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x) = \left\{u \in W_0^{1,1}(\Omega) : \int\limits_{\Omega} |\nabla u(x)|^{p(x)}\,\mu(x)\,\mathrm{d}x < \infty\right\},$$

$$\|u\|_{W_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x)} = \||\nabla u|\|_{L^{p(x)}(\Omega,\mu\,\mathrm{d}x)}\,.$$

The aim of this paper is to establish some existence and uniqueness theorems for the following boundary value problem (BVP)

$$\begin{cases} -\operatorname{div}\Big(\mu(x)\,|\nabla u|^{p(x)-2}\,\nabla u\Big) = -\operatorname{div}\Big(\mu(x)\vec{F}(x)\Big) \text{ on } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\vec{F} = (f_1, ..., f_n) \in \left[L^{q(x)}(\Omega,\mu\,\mathrm{d}x)\right]^n$ is given, $u : \overline{\Omega} \to \mathbb{R}$ is uknown. It is worth mentioning that we do not state anything rigorous beforehand about the exact functional space to which the function u belongs. To be more precise, we can only assert that $u \in W_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x)$ as in the widest possible case.

The reason for such a vague explanation is based on the structure of spaces $W_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x)$ and $H_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x)$, where $H_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x)$ is a closure of $C_0^{\infty}(\Omega)$ with respect to the given norm of $W_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x)$. It is obvious that $H_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x) \subset W_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x)$, but there exist functions such that $H_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x) \neq W_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x)$ [2], which makes the boundary value problem (1) far more challenging.

Let us consider a closed subspace V of the space $W_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x)$ such that

$$H_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x) \subset V.$$

It is obvious that $C_0^{\infty}(\Omega) \subset V$, but if $V \neq H_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x)$, then $C_0^{\infty}(\Omega)$ is not dense in V. It is also clear that a functional

$$v \mapsto \int\limits_{\Omega} \mu(x)\sum_{k=1}^{n} f_k(x)v_{x_k}(x)\,\mathrm{d}x \equiv \int\limits_{\Omega} \mu(x)\vec{F}(x)\nabla v(x)\,\mathrm{d}x \tag{2}$$

is an element of V^* (to make sure, see Theorem 8, Section 3), a dual space of V. This allows us to give the following definition of solutions to problem (1).

Definition 1. *A function $u \in V$ is said to be a V-solution (variational solution with respect to the space V) to the problem* (1) *if the integral equality*

$$\int\limits_{\Omega} \mu(x)\,|\nabla u|^{p(x)-2}\,\nabla u\nabla v\,\mathrm{d}x = \int\limits_{\Omega} \mu(x)\vec{F}(x)\nabla v\,\mathrm{d}x \tag{3}$$

holds true for all $v \in V$.

Similarly, another well-known definition of solutions to (1) should also be recalled.

Definition 2. *A function $u \in W_0^{1,p(x)}(\Omega, \mu\,\mathrm{d}x)$ is said to be a weak solution to the boundary value problem* (1) *if an integral equality from* (3) *holds true only for those v which belong to $C_0^\infty(\Omega)$.*

Remark 1. Each V-solution to (1) is also a weak solution to (1).

The main result of this paper is the following statement.

Theorem 1. *Boundary value problem* (1) *has a unique V-solution for each intermediate space $H_0^{1,p(x)}(\Omega, \mu\,\mathrm{d}x) \subseteq V \subseteq W_0^{1,p(x)}(\Omega, \mu\,\mathrm{d}x)$.*

The proof of this theorem will be based on considering an operator equation

$$A(u) = f, \tag{4}$$

where $u \in V, f \in V^*, A : V \to V^*$, V is a Banach space, V^* is a dual space. We will show that BVP (1) is equivalent to the equation (4), which allows to apply a theorem of existence and uniqueness of solutions to this equation [3]. This idea will be implemented in Section 3.

To be more rigorous, we should provide a historical review and some information about physical sense of the given problem, because it is not just abstract one and has some important background. To start with, the boundary value problem (1) became widely known after the paper [4] by V.V. Zhikov in 1986, which was followed by a numerous series of researches in, for instance, articles [5, 17]. Namely, it was shown that a functional

$$I(u) = \int_\Omega f(x, \nabla u)\,\mathrm{d}x,$$

where the function $f(x,\xi) : \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfies the growth conditions

$$-c_0 + c_1 |\xi|^p \le f(x,\xi) \le c_0 + c_2 |\xi|^q, \quad q > p,$$

can attain different minimums for different test function spaces. In other words, it means that we can observe Lavrentiev phenomenon for some functional spaces. Afterwards, the question of solvability of the corresponding Euler-Lagrange equation, which also was considered separately as a degenerate elliptic equation, was broached in papers of scientists such as Xian-Ling Fan, Qi-Hu Zhang [6], V.V. Zhikov, S.Ye. Pastukhova (for instance, [9]), Yu.A. Alkhutov, O.V. Krashennikova (for instance, [10]), P. Marcellini [11], M. Giaquinta [12], M. Růžička [13] and others.

Generally, the first studies on solvability of problem (1) in terms of weak solutions (see Definition 2) were devoted to the case $\mu(x) = 1$ (see [6]). Furthermore, a series of researches into the problem (1) was conducted by the group of Russian scientists (V.V. Zhikov, S.Ye. Pastukhova, Yu.A. Alkhutov, O.V. Krashennikova). These researches include both variations of problem (1) with $\mu(x) \neq 1$ and parabolic generalizations of this problem, not to mention that topics of these scientists' papers also include some case studies of Sobolev-Orlicz spaces.

A substantial contribution to the theory of equations with variable exponents was also brought by some Ukrainian mathematicians, mostly by M.M. Bokalo and

O.M. Buhriy, who conducted various researches into parabolic extensions of BVP (1) (see their latest papers [7, 8]).

The key issue of the problem (1) is that generally it may have an infinite number of solutions. This issue is based on the fact that $C_0^\infty(\Omega)$ may be either dense or not dense in $W_0^{1,p(x)}(\Omega, \mathrm{d}x)$ depending on regularity properties of $p(x)$. There were also some studies on the equality

$$H_0^{1,p(x)}(\Omega, \mu\,\mathrm{d}x) = W_0^{1,p(x)}(\Omega, \mu\,\mathrm{d}x),$$

which subsequently turned out to be guaranteed by the density of $C_0^\infty(\Omega)$ in the space $W_0^{1,p(x)}(\Omega, \mu\,\mathrm{d}x)$. In this case, the density of $C_0^\infty(\Omega)$ in $W_0^{1,p(x)}(\Omega, \mu\,\mathrm{d}x)$ may be violated not only because of the lack of regularity for $p(x)$, but also due to violation of the Muckenhoupt condition by $\mu(x)$, which in turn is the corresponding condition for density of $C_0^\infty(\Omega)$ in $W_0^{1,p}(\Omega, \mu\,\mathrm{d}x)$ [14, p.1].

The main result of the paper is similar to those mentioned above, most of all to Theorem 2.1 from [5], but it has a certain difference: unlike the results on BVPs for degenerate elliptic equations in [5], the following result encompasses those cases of weight $\mu(x)$ which do not satisfy conditions from 2.2 [5].

As for the physical applicability, the BVP (1) is a certain variation of the classical thermistor problem [15]. It can be reduced from the system of PDEs to a single equation in the same way as it was shown in [15]. By and large, this BVP can be used for modeling electrorheological and thermoelectric characteristics of various processes [13], which makes us ascertain of actuality of the given problem.

2. Some Preliminary Results

Theorem 2. *The space $W_0^{1,p(x)}(\Omega, \mu\,\mathrm{d}x)$ is continuously imbedded into the space $W_0^{1,1}(\Omega)$; in other words,*

$$\forall u \in W_0^{1,p(x)}(\Omega, \mu\,\mathrm{d}x): \quad \|u\|_{W_0^{1,1}(\Omega)} \le K^* \|u\|_{W_0^{1,p(x)}(\Omega, \mu\,\mathrm{d}x)},$$

where $K^* = \mathrm{const} > 0$.

Proof. Firstly, by definition, $W_0^{1,p(x)}(\Omega, \mu\,\mathrm{d}x) \subset W_0^{1,1}(\Omega)$, which implies that for arbitrarily chosen $u \in W_0^{1,p(x)}(\Omega, \mu\,\mathrm{d}x)$ we have $\|u\|_{W_0^{1,1}(\Omega)} = \int\limits_\Omega |\nabla u(x)|\,\mathrm{d}x < \infty$. Secondly, by the Hölder inequality for variable Lebesgue spaces [16, p.14] we obtain

$$\int\limits_\Omega |\nabla u|\,\mathrm{d}x = \int\limits_\Omega |\nabla u|\,\mu^{1/p(x)} \cdot \mu^{-1/p(x)}\,\mathrm{d}x \le K \left\|\mu^{1/p(x)}\,|\nabla u|\right\|_{L^{p(x)}(\Omega)} \left\|\mu^{-1/p(x)}\right\|_{L^{q(x)}(\Omega)}.$$

Since

$$\left\|\mu^{\frac{1}{p(x)}}\,|\nabla u|\right\|_{L^{p(x)}(\Omega)} = \inf\left\{\lambda > 0 : \int\limits_\Omega \left|\frac{\mu^{\frac{1}{p(x)}}(x)\,|\nabla u|}{\lambda}\right|^{p(x)} \mathrm{d}x \le 1\right\} =$$

$$= \inf\left\{\lambda > 0 : \int\limits_\Omega \left|\frac{\nabla u}{\lambda}\right|^{p(x)} \mu(x)\,\mathrm{d}x \le 1\right\} = \||\nabla u|\|_{L^{p(x)}(\Omega, \mu\,\mathrm{d}x)},$$

then

$$\|u\|_{W_0^{1,1}(\Omega)} \le \underbrace{K \left\|\mu^{-1/p(x)}\right\|_{L^{q(x)}(\Omega)}}_{K^*} \||\nabla u|\|_{L^{p(x)}(\Omega,\mu\,\mathrm{d}x)} = K^* \|u\|_{W_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x)},$$

which provides an imbedding $W_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x) \hookrightarrow W_0^{1,1}(\Omega)$. $\square$

Theorem 3. *The space $W_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x)$ is a separable reflexive Banach space with respect to the given norm.*

Proof. The normability of $W_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x)$ is almost obvious. The next step is to verify the completeness, reflexivity and separability for this space. To start with, we draw our attention to the completeness property. With that in mind, we consider a fundamental sequence $\{u_k\}_{k=1}^{+\infty}$ and substantiate its convergence.

Firstly, by the Hölder inequality for variable Lebesgue spaces, since

$$\|u\|_{[L^1(\Omega)]^n} = \int\limits_{\Omega} |u(x)|\,\mathrm{d}x,$$

then

$$\int\limits_{\Omega} |\nabla u_k - \nabla u_m|\,\mathrm{d}x \le K \|u_k - u_m\|_{W_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x)} \left\|\mu^{-1/p(x)}\right\|_{L^{q(x)}(\Omega)}, \tag{5}$$

which implies that $\{\nabla u_k\}_{n=1}^{+\infty}$ is fundamental in $\left[L^1(\Omega)\right]^n$. The space $\left[L^1(\Omega)\right]^n$ is complete, that is why there exists $\psi \in \left[L^1(\Omega)\right]^n$ such that $\nabla u_k \to \psi$ strongly in $\left[L^1(\Omega)\right]^n$. In addition, as $\{u_k\}_{k=1}^{+\infty}$ is fundamental in $W_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x)$, therefore, the sequence $\{\nabla u_k\}_{k=1}^{+\infty}$ is fundamental with respect to $\left[L^{p(x)}(\Omega,\mu\,\mathrm{d}x)\right]^n$, thereby, due to completeness we establish an existence of a function $\psi' \in \left[L^{p(x)}(\Omega,\mu\,\mathrm{d}x)\right]^n$ with a property $\nabla u_k \to \psi'$. Thus, as $\left[L^{p(x)}(\Omega,\mu\,\mathrm{d}x)\right]^n \hookrightarrow \left[L^1(\Omega)\right]^n$ by (5), then it follows that $\psi = \psi'$.

Secondly, basing on imbedding $W_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x) \hookrightarrow W_0^{1,1}(\Omega)$ (by Theorem 2) we infer that ψ is a weak gradient for some function $u \in L^{p(x)}(\Omega,\mu\,\mathrm{d}x)$. In conclusion, as every function $\{u_k\}_{k=1}^{+\infty}$ has a zero trace, it indicates that u also has zero trace, from where we state that $\{u_k\}_{k=1}^{+\infty}$ converges to u in the space $W_0^{1,1}(\Omega)$. Since $\{\nabla u_k\}_{k=1}^{+\infty}$ is convergent to ∇u with respect to $\left[L^{p(x)}(\Omega,\mu\,\mathrm{d}x)\right]^n$, we conclude that $u \in W_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x)$ to finish the proof of completeness for $W_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x)$.

Now let us prove reflexivity and separability for $W_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x)$. In order to establish these statements, we define a function

$$f : W_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x) \to M \subsetneq \left[L^{p(x)}(\Omega,\mu\,\mathrm{d}x)\right]^n,$$

where $f(u) = \nabla u$, M is an image of $W_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x)$. The next stage is to show that f is an injective operator. We assume to the contrary that for distinct $u_1 \ne u_2 \in W_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x)$ an equality $\nabla u_1 = \nabla u_2$ holds true. If it holds, then $u_1 = u_2 + C$, but also $u_1 \in W_0^{1,1}(\Omega)$, which is contrary to $u_2 \in W_0^{1,1}(\Omega)$, hence, we arrive at contradiction. The operator f is also surjective by the definition. Moreover, since

$$\|u\|_{\left[L^{p(x)}(\Omega,\mu\,\mathrm{d}x)\right]^n} = \||u|\|_{L^{p(x)}(\Omega,\mu\,\mathrm{d}x)},$$

$$\|u_1 - u_2\|_{W_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x)} = \|\nabla u_1 - \nabla u_2\|_{\left[L^{p(x)}(\Omega,\mu\,\mathrm{d}x)\right]^n},$$

that is why f is an isometry. As $W_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x)$ is a complete space, then the image M is closed in $\left[L^{p(x)}(\Omega,\mu\,\mathrm{d}x)\right]^n$. The space $\left[L^{p(x)}(\Omega,\mu\,\mathrm{d}x)\right]^n$ is separable and reflexive [1], and because of the fact that f is an isometry, $W_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x)$ is a separable reflexive space by the properties of isometry. □

Theorem 4. *Let $u \in L^{p(x)}(\Omega,\mu\,\mathrm{d}x)$. Then the following inequality holds true*

$$\min\left\{\|u\|^{\alpha}, \|u\|^{\beta}\right\} \le \rho_{p,\mu}(u) \le \max\left\{\|u\|^{\alpha}, \|u\|^{\beta}\right\},$$

where

$$\rho_{p,\mu}(u) = \int\limits_{\Omega} |u(x)|^{p(x)}\,\mu(x)\,\mathrm{d}x,$$

$$\|u\| = \|u\|_{L^{p(x)}(\Omega,\mu\,\mathrm{d}x)}.$$

Proof. If $u = 0$, then the inequality is obvious. To start with, we mention that if $\|u\| = a \ne 0$, then $\rho_{p,\mu}\left(\frac{u}{a}\right) = 1$ ([16, p.4]). To proceed, let $\|u\| \ge 1$. Then

$$\frac{1}{a^{\beta}}\rho_{p,\mu}(u) \le \rho_{p,\mu}\left(\frac{u}{a}\right) \le \frac{1}{a^{\alpha}}\rho_{p,\mu}(u),$$

which implies that

$$\|u\|^{\alpha} \le \rho_{p,\mu}(u) \le \|u\|^{\beta}.$$

The inequality

$$\|u\|^{\beta} \le \rho_{p,\mu}(u) \le \|u\|^{\alpha},$$

if $0 < \|u\| < 1$, can be confirmed in the same way. Now, if we combine these inequalities, then the given result is obvious. □

Definition 3 ([3, p. 182])**.** *Let V be a Banach space. An operator $A : V \to V^*$ is said to be coercive if*

$$\lim_{\|u\|\to\infty} \frac{\langle A(u), u\rangle_{V^*,V}}{\|u\|_V} = +\infty.$$

Definition 4 ([3, p. 168])**.** *Let V be a Banach space. An operator $A : V \to V^*$ is said to be hemicontinuous if a function*

$$f(\lambda) = \langle A(u + \lambda v), w\rangle_{V^*,V}$$

is continuous on $\mathbb{R}$ for all $u, v, w \in V$.

3. Existence and Uniqueness of Variational Solutions

To prove the main theorem of this paper, we will use the following statement.

Theorem 5 ([3, p. 182]). *Let V be a separable reflexive Banach space, let V^* be a dual space of V and let $A\colon V \to V^*$ be a bounded hemicontinuous coercive monotone operator. Then for every $f \in V^*$ an equation $A(u) = f$ has a solution. If the operator is strictly monotone, then this solution is unique.*

Before proving the main theorem, we provide some additional theorems in order to make the proof clearer.

Theorem 6. *Every intermediate space V is a separable reflexive Banach space equipped with the norm $\|\cdot\|_{W_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x)}$.*

Proof. By definition, V is a closed linear manifold in the space $W_0^{1,p(x)}(\Omega,\mu\,\mathrm{d}x)$, which is a separable reflexive Banach space, therefore, V is also a separable reflexive Banach space. □

Let us consider a form

$$a(u,v) = \int_\Omega \mu(x)\,|\nabla u|^{p(x)-2}\,\nabla u \nabla v\,\mathrm{d}x, \quad u,v \in V. \tag{6}$$

Theorem 7. *The form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ (see (6)) is well defined and the following inequality*

$$|a(u,v)| \le K\,\|v\|_V \max\left\{\|u\|_V^{\alpha-1}, \|u\|_V^{\beta-1}\right\} \quad \forall u,v \in V,$$

holds, where $K = \mathrm{const} > 0$.

Proof. By the Hölder inequality, we have

$$\int_\Omega \mu(x)\,|\nabla u|^{p(x)-1}\,|\nabla v|\,\mathrm{d}x \le K \left\|\mu^{\frac{1}{p(x)}}\,|\nabla v|\right\|_{L^{p(x)}(\Omega)} \left\|\mu^{\frac{1}{q(x)}}\,|\nabla u|^{p(x)-1}\right\|_{L^{q(x)}(\Omega)}.$$

By the arguments from Theorem 2,

$$\left\|\mu^{\frac{1}{p(x)}}\,|\nabla v|\right\|_{L^{p(x)}(\Omega)} = \|v\|_V\,. \tag{7}$$

Let us show that

$$\left\|\mu^{\frac{1}{q(x)}}\,|\nabla u|^{p(x)-1}\right\|_{L^{q(x)}(\Omega)} \le K \max\left\{\|u\|_V^{\alpha-1}, \|u\|_V^{\beta-1}\right\}. \tag{8}$$

We will take into account the following chain of transformations:

$$\left\|\mu^{\frac{1}{q(x)}}\,|\nabla u|^{p(x)-1}\right\|_{L^{q(x)}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{\mu^{\frac{p(x)-1}{p(x)}}(x)\,|\nabla u|^{p(x)-1}}{\lambda}\right|^{\frac{p(x)}{p(x)-1}} \mathrm{d}x \le 1\right\} =$$

$$= \inf\left\{\lambda > 0 : \int_\Omega \frac{|\nabla u|^{p(x)}}{\lambda^{q(x)}}\mu(x)\,\mathrm{d}x \le 1\right\}.$$

If $u = 0$, then the inequality (8) holds true. Assuming that $\|u\|_V = C < +\infty$, let us demonstrate that infimum in the last block of equalities is finite. In order to substantiate it, we consider two cases: $C \geq 1, 0 < C < 1$. In the first case let $\lambda = C^{\beta-1}$. Hence,

$$\int\limits_\Omega \frac{|\nabla u|^{p(x)}}{\lambda^{q(x)}}\mu(x)\,\mathrm{d}x = \int\limits_\Omega \frac{|\nabla u|^{p(x)}}{C^{(\beta-1)\frac{p(x)}{p(x)-1}}}\mu(x)\,\mathrm{d}x \leq$$

$$\leq \int\limits_\Omega \frac{|\nabla u|^{p(x)}}{C^{(p(x)-1)\frac{p(x)}{p(x)-1}}}\mu(x)\,\mathrm{d}x = \int\limits_\Omega \frac{|\nabla u|^{p(x)}}{C^{p(x)}}\mu(x)\,\mathrm{d}x = 1 \implies$$

$$\implies \inf\left\{\lambda > 0 : \int\limits_\Omega \frac{|\nabla u|^{p(x)}}{\lambda^{q(x)}}\mu(x)\,\mathrm{d}x \leq 1\right\} \leq C^{\beta-1} = \|u\|_V^{\beta-1}.$$

In the second case $0 < C < 1$, that is why we take $\lambda = C^{\alpha-1}$ and use the same idea as for the first case with $C \geq 1$ to show that

$$\inf\left\{\lambda > 0 : \int\limits_\Omega \frac{|\nabla u|^{p(x)}}{\lambda^{q(x)}}\mu(x)\,\mathrm{d}x \leq 1\right\} \leq C^{\alpha-1} = \|u\|_V^{\alpha-1}.$$

Thus, the inequality (8) holds true, since we may combine two cases by choosing maximum value between them. Ultimately, after combining (8) and (7) we draw a conclusion that the statement of this theorem holds. □

It is clear that by fixing an argument $u \in V$ the form from Theorem 7 can be restricted to the domain V, whereby we define a functional $v \mapsto a(u, v)$, $v \in V$, which is also linear and continuous on V. Therefore, we have operator $A : V \to V^*$ defined by the rule

$$\langle A(u), v\rangle_{V^*,V} = a(u, v) \quad \forall u, v \in V. \tag{9}$$

Theorem 8. *The operator $A : V \to V^*$ is bounded, coercive, hemicontinuous and strictly monotone.*

Proof. From Theorem 7 we have

$$\|A(u)\|_{V^*} \leq K \max\left\{\|u\|_V^{\alpha-1}, \|u\|_V^{\beta-1}\right\},$$

hence, the operator A is bounded.

The next step of the proof is to confirm that the operator A is coercive. Owing to the equality

$$\langle A(u), u\rangle_{V^*,V} = \int\limits_\Omega \mu(x)\,|\nabla u(x)|^{p(x)}\,\mathrm{d}x,$$

and Definition 3, we ensure that

$$\lim_{\|u\|\to\infty} \frac{\langle A(u), u\rangle_{V^*,V}}{\|u\|_V} = \lim_{\|u\|\to\infty} \frac{\int\limits_\Omega \mu(x)\,|\nabla u|^{p(x)}\,\mathrm{d}x}{\|u\|_V} = \lim_{\|u\|\to\infty} \frac{\rho_{p,\mu}(|\nabla u|)}{\|u\|_V} \geq$$

$$\geq \lim_{\|u\|\to\infty} \frac{\min\left\{\|u\|_V^\alpha, \|u\|_V^\beta\right\}}{\|u\|_V} = \lim_{\|u\|\to\infty} \min\left\{\|u\|_V^{\alpha-1}, \|u\|_V^{\beta-1}\right\} = +\infty,$$

because $\alpha, \beta > 1$. In the last line we have applied an inequality from Theorem 4.

Now, we prove that the operator A is hemicontinuous. In order to ensure hemicontinuity, it is sufficient to make sure of the equality

$$\lim_{\lambda\to 0}\langle A(u+\lambda v), w\rangle_{V^*,V} = \langle A(u), w\rangle_{V^*,V} \quad \forall u, v, w \in V. \tag{10}$$

Without the loss of generality, we may consider $\lambda \in [-1, 1]$. By definition,

$$\langle A(u+\lambda v), w\rangle_{V^*,V} = \int\limits_{\Omega} \mu(x) \left|\nabla u + \lambda \nabla v\right|^{p(x)-2} \nabla\left(u+\lambda v\right) \nabla w \, \mathrm{d}x.$$

The integrated expression can be estimated as follows:

$$\left|\mu(x) \left|\nabla u + \lambda \nabla v\right|^{p(x)-2} \nabla\left(u+\lambda v\right) \nabla w\right| \leq \mu(x) \left|\nabla u + \lambda \nabla v\right|^{p(x)-1} \left|\nabla w\right| \leq$$

$$\leq \mu(x) \left(\left|\nabla u\right| + \left|\nabla v\right|\right)^{p(x)-1} \left|\nabla w\right| \in L^1(\Omega),$$

where the last statement may be confirmed by the same arguments as those used in Theorem 7 while proving the first inequality. Hence, since

$$\left|\nabla u + \lambda \nabla v\right|^{p(x)-2} \nabla\left(u+\lambda v\right) \nabla w \mu(x) \to \left|\nabla u\right|^{p(x)-2} \nabla u \nabla w \mu(x) \text{ a.e. in } \Omega \quad \forall u, v, w \in V,$$

then (by the Lebesgue dominated convergence theorem) the operator A is hemicontinuous.

Finally, it remains to guarantee that the operator A is strictly monotone.

Let $r \in \mathbb{R}$ be an arbitrary number such that $r > 1$. It is well-known that the function $s \mapsto |s|^{r-2} s : \mathbb{R} \to \mathbb{R}$ is strictly monotone, i.e.,

$$\left(|s_1|^{r-2} s_1 - |s_2|^{r-2} s_2\right)\left(s_1 - s_2\right) > 0 \quad \forall s_1, s_2 \in \mathbb{R}, s_1 \neq s_2.$$

With this in mind, we can deduce that

$$|\xi|^r + |\eta|^r > |\xi|\,|\eta| \left(|\xi|^{r-2} + |\eta|^{r-2}\right) \quad \forall \xi, \eta \in \mathbb{R}^n : |\xi|\,, |\eta| \neq 0,\ |\xi| \neq |\eta|\,.$$

By the Cauchy-Schwarz inequality, $\xi\eta \leq |\xi|\,|\eta|$, which can be applied to the previous inequality:

$$|\xi|^r + |\eta|^r > \xi\eta \left(|\xi|^{r-2} + |\eta|^{r-2}\right) \Longrightarrow$$

$$\Longrightarrow \left(|\xi|^{r-2} \xi - |\eta|^{r-2} \eta, \xi - \eta\right) > 0 \quad \forall \xi, \eta \in \mathbb{R}^n : |\xi|\,, |\eta| \neq 0,\ |\xi| \neq |\eta|\,.$$

Now let $\xi, \eta \in \mathbb{R}^n : |\xi| = |\eta| = \rho > 0$ and $\xi \neq \pm\eta$. Then it is clear that

$$\left(|\xi|^{r-2} \xi - |\eta|^{r-2} \eta, \xi - \eta\right) = |\xi|^r + |\eta|^r - \xi\eta \left(|\xi|^{r-2} + |\eta|^{r-2}\right) =$$

$$= 2\rho^r - 2\xi\eta\rho^{r-2} = 2\rho^r(1 - \xi'\eta'), \tag{11}$$

where $\xi' = \rho^{-1}\xi$, $\eta' = \rho^{-1}\eta$. From here we infer that $|\xi'| = |\eta'| = 1$ and $\xi' \neq \pm\eta'$. Hence, it is obvious that

$$|\xi' - \lambda\eta'| > 0 \quad \forall \lambda \in \mathbb{R},$$

because otherwise in the case $\xi' = \lambda_0\eta'$ for some $\lambda_0 \in \mathbb{R}$, we have $\lambda_0 = \pm 1$, which is impossible. As a result, this implies that

$$1 - 2\lambda\xi'\eta' + \lambda^2 > 0 \quad \forall \lambda \in \mathbb{R},$$

guaranteeing that $D = 4(\xi'\eta')^2 - 4 < 0$ and $\xi'\eta' < 1$, which provides that the expression (11) is always positive. The same holds true for $\eta = -\xi$ as well as when one of these vectors is zero.

Taking into account all these arguments, we draw a conclusion that

$$\Big(|\xi|^{r-2}\, \xi - |\eta|^{r-2}\, \eta, \xi - \eta \Big) > 0 \quad \forall \xi, \eta \in \mathbb{R}^n : \xi \neq \eta. \tag{12}$$

To show that the operator A is also strictly monotone, we first recall the condition of monotonicity:

$$\langle A(u) - A(v), u - v\rangle_{V^*,V} > 0 \quad \forall u, v \in V, u \neq v.$$

If $u, v \in V : u \neq v$, then there exists a positive measure set $E \subset \Omega$ such that $\nabla u(x) \neq \nabla v(x)$ for $x \in E \subset \Omega$ and $\nabla u(x) = \nabla v(x)$ for $x \in \Omega \setminus E$. Since $p(x) > 1$ and $\mu(x) > 0$ for almost all $x \in \Omega$, then from this and (12) it follows

$$\langle A(u) - A(v), u - v\rangle_{V^*,V} =$$
$$= \int\limits_E \mu(x) \big(|\nabla u(x)|^{p(x)-2}\, \nabla u(x) - |\nabla v(x)|^{p(x)-2}\, \nabla v(x), \nabla u(x) - \nabla v(x) \big)\, \mathrm{d}x > 0,$$

from where we ascertain that operator A is strictly monotone. □

Now we define functional $f : V \to \mathbb{R}$ by the following rule:

$$v \mapsto f(v) = \int\limits_\Omega \mu(x)\vec{F}(x)\nabla v(x)\, \mathrm{d}x, \quad v \in V. \tag{13}$$

Theorem 9. *This functional $f\colon V \to \mathbb{R}$ is well defined, linear and continuous, that is, $f \in V^*$, and also $f(v) = \langle f, v\rangle_{V^*,V}, \quad v \in V$.*

Proof. Indeed, after using again the Hölder inequality, we get

$$\left| \int\limits_\Omega \mu(x)\vec{F}(x)\nabla v(x)\, \mathrm{d}x \right| \leq \int\limits_\Omega \mu(x) \left| \vec{F}(x) \right| |\nabla v(x)|\, \mathrm{d}x \leq K \left\| \left| \vec{F}(x) \right| \right\|_{L^{q(x)}(\Omega,\mu\, \mathrm{d}x)} \|v\|_V . \tag{14}$$

From the last inequality we may infer that $f\colon V \to \mathbb{R}$ is a bounded functional. Since f is linear, hence, it is also continuous, that is, $f \in V^*$. □

Proof of Theorem 1. After the previous definitions and statements we may make conclusion that the definition of V-solution to the problem (1) is correct, and this problem is equivalent to abstract equation (4), where

$$H_0^{1,p(x)}(\Omega, \mu\, \mathrm{d}x) \subseteq V \subseteq W_0^{1,p(x)}(\Omega, \mu\, \mathrm{d}x),$$

V^* is dual to V, and $f \in V^*$, $A : V \to V^*$ are defined above. To this operator equation we can apply Theorem 5. Therefore, by Theorem 5, the equation (4) has a unique solution, which in turn is equivalent to the existence of a unique V-solution to the BVP (1). □

Consequently, we have proven that the boundary value problem (1) has a unique V-solution with respect to each intermediate space

$$H_0^{1,p(x)}(\Omega, \mu\, \mathrm{d}x) \subseteq V \subseteq W_0^{1,p(x)}(\Omega, \mu\, \mathrm{d}x).$$

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ПРО ІСНУВАННЯ ТА ЄДИНІСТЬ ВАРІАЦІЙНИХ РОЗ'ЯЗКІВ ЗАДАЧІ ДІРІХЛЕ ДЛЯ НЕЛІНІЙНОГО ЕЛІПТИЧНОГО РІВНЯННЯ З НЕСТАНДАРТНИМИ УМОВАМИ ЗРОСТАННЯ

Павло ТКАЧЕНКО

Дніпропетровський національний університет ім. О. Гончара,
проспект Гагаріна, 72, Дніпро, Україна
e-mail: cool.phenom@mail.ru

Розглянуто задачу Діріхле для нелінійного еліптичного рівняння з нестандартними умовами зростання в головній частині оператора. Існує особливість цієї задачі, вона полягає в тому, що без попереднього зазначення простору Соболєва-Орліча, в якому шукаємо розв'язок, простежується ефект Лаврентьєва. Доведено існування та єдиність варіаційних розв'язків для проміжних просторів Соболєва-Орліча.

Ключові слова: нелінійне еліптичне рівняння, змінний показник, ефект Лаврентьєва, простір Соболєва-Орліча.