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CLEAN DUO RINGS

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A ring with unit element is called a duo ring if every one-sided ideal is two-sided. A ring is called clean if every its element is the sum of a unit and an idempotent. This article describes some of the properties of clean duo-rings.

Key words: Bezout ring, clean ring, avoidable ring, duo ring

Helmer [6] was the first to start considering the adequate domains with the idea to obtain abstract characterization of the rings of integer functions. The adequate rings with zero-divisors in Jacobson radical were studied by Kaplansky [7]. Gillman and Henriksen have shown that a von Neumann regular ring is adequate [4]. The first example of Bezout nonadequate domain, which is a domain of elementary divisors, was constructed by Henriksen [5].

Zabavsky and Kuznitska [8] have introduced for consideration a new class of so-called avoidable rings that contains the adequate ring.

Gatalevych [3] was the first, who studied noncommutative adequate rings and their generalizations. He has proved that the generalized right adequate duo Bezout domain is an elementary divisor domain.

All rings are associative rings with nonzero identity.

Definition 1. *A ring R is said to be a clean ring if for any element $x \in R$ there exists a converse element $u \in R$ and an idempotent $e \in R$ such that $x = u + e$.*

Definition 2. *A ring R is called a Bezout right (left) ring if every finitely generated right (left) ideal in R is principal.*

Definition 3. *An element $a \neq 0$ of the Bezout domain R is called a right adequate element if for any element $b \in R$ there exist the elements $r, s \in R$ so that:*

1) $a = rs$;
2) $bR + rR = R$;
3) $\forall\, s' \in R,\ sR \subset s'R \neq R \Rightarrow bR + s'R \neq R$.

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Definition 4. *A ring, in which every nonzero element is right adequate, is called a right adequate ring.*

A ring R is called everywhere right adequate ring ring if every its element is adequate

Definition 5. *A ring R is called a right (left) duo-ring if the following equivalent statements hold:*

1) Every right (left) ideal in R is an ideal;

2) For every $x \in R, Rx \subset xR$ $(Rx \supset xR)$.

Such rings were investigated by E. Feller [2] and G. Thierrin [11]. Trivial examples of duo rings are, of course, commutative rings and division rings. Nontrivial duo rings are not difficult to come by (e.g., any noncommutative special primary ring is duo, since the only right or left ideals are powers of the unique maximal ideal). In fact some interesting examples of duo rings have already occurred in the literature: M. Auslander and O. Goldman have shown in [1] that there exist noncommutative maximal orders which are both duo rings and Noetherian domains. Further investigations of such rings have been carried out by G. Maury in [9].

Definition 6. *An element a of a duo ring R is said to be right avoidable if for any elements $b, c \in R$ such that $aR + bR + cR = R$ there exist elements $r, s \in R$ such that $a = rs$, $rR+bR = R$, $sR+cR = R$, and $rR+sR = R$. A ring R is called right avoidable if every its nonzero element is right avoidable.*

A ring R is called everywhere right avoidable if every element of R is right avoidable.

Theorem 1. *Any right adequate element of Bezout duo-ring is right avoidable.*

Proof. Let R be a Bezout duo-ring and a is an adequate element of R. Let $aR+bR+cR = R$ and $a = rs$ where $rR + bR = R$ and $\acute{s}R + bR \neq R$ for every element $\acute{s}$ such that $sR \subset \acute{s}R \neq R$. Obviously $rR+sR = R$. Let $sR+cR = dR \neq R$. Since d is a noninvertible divisor of an element s, we see that $dR + bR = hR \neq R$. Since $cR \subset dR \subset hR, bR \subset hR$ and $aR \subset sR \subset hR$. We have $aR + bR + cR \subset hR \neq R$. This is impossible, since $aR + bR + cR = R$. □

As an obvious consequence we obtain the following result.

Corollary 1. *Any right adequate Bezout duo-ring is right avoidable ring.*

Theorem 2. *Let a be an right avoidable element of a duo-ring R. Then zero is right avoidable element of the factor-ring R/aR.*

Proof. Let $\overline{R} = R/aR$, and $\overline{b}\overline{R}+\overline{c}\overline{R} = \overline{R}$, where $\overline{b} = b+aR$, $\overline{c} = c+aR$. By the assumption we have $a = rs$, where $rR + bR = R$, $sR + cR = R$ and $rR + sR = R$. Now we have $\overline{0} = \overline{r} \cdot \overline{s}$, where $\overline{r}\overline{R} + \overline{b}\overline{R} = \overline{R}$, $\overline{s}\overline{R} + \overline{c}\overline{R} = \overline{R}$ and $\overline{r}\overline{R} + \overline{s}\overline{R} = \overline{R}$. □

Theorem 3. *Duo-ring R is clean if and only if the zero element is right avoidable.*

Proof. Let $bR+cR = R$ and $0 = rs$, where $rR+bR = R$, $sR+cR = R$ and $rR+sR = R$. Since $rR+sR = R$, we see that $ru+sv = 1$ for some elements $u, v \in R$. Since $0 = rs$, we have $r^2u = r$, $s^2v = s$. Denote $ru = e$, then $e^2 = e$ and $1 - e = sv$. Since $rR + bR = R$, we obtain $r\alpha + b\beta = 1$ for some elements $\alpha, \beta \in R$. Here $svb\beta = sv$, i.e. $1 - e \in bR$.

Similarly, $e \in cR$. By [10] R is an exchange ring. In the case of duo rings classes of exchange and clean rings coincide, so R is a clean ring. N ecessity is proved.

We will prove that any clean ring is a ring in which zero is right avoidable. Let $bR + cR = R$. There exists an idempotent $e \in R$ such that $e \in bR$ and $1 - e \in cR$. Since $0 = e(1-e)$, we obtain $(1-e)R + bR = R$, $eR + cR = R$, and $eR + (1-e)R = R$. Putting $1 - e = r, e = s$ we obtain an appropriate representation of the zero element. □

Theorem 4. *Let R be a duo Bezout domain. If $\overline{0}$ is right avoidable element of R/aR then a is right avoidable element of R.*

Proof. Denote $\overline{R} = R/aR$. Since zero is right avoidable element, for any elements $\overline{b}, \overline{c} \in \overline{R}$ such that $\overline{bR} + \overline{cR} = \overline{R}$ there exists such $\overline{r}, \overline{s} \in \overline{R}$ that $\overline{0} = \overline{r} \cdot \overline{s}$ and $\overline{rR} + \overline{bR} = \overline{R}$, $\overline{sR} + \overline{cR} = \overline{R}$ and $\overline{rR} + \overline{sR} = \overline{R}$. Denote $\overline{r} = r + aR$, $\overline{s} = s + aR$, $\overline{b} = b + aR$, $\overline{c} = c + aR$.

Let b and c be arbitrary elements of R such that $aR + bR + cR = R$. Since $\overline{rR} + \overline{bR} = \overline{R}$, there are elements $u, v, t \in R$ such that $ru + bv = 1 + at$. Let $aR + rR = \delta R$. Then $a = \delta a_0$, $r = \delta r_0$ for some elements $a_0, r_0 \in R$ moreover $a_0R + r_0R = R$. It follows that $\delta R + bR = R$.

Since $\overline{0} = \overline{r} \cdot \overline{s}$, we obtain $rs = a\alpha$ for some element $\alpha \in R$. Then $\delta r_0 s = \delta a_0 \alpha$. Since R is a domain we have $r_0 s = a_0 \alpha$. Since $r_0R + a_0R = R$, there exist elements $t, k \in R$ such that $r_0 t + a_0 k = 1$. This means that $a_0 \beta = s$ for some element $\beta \in R$. Therefore $a = \delta a_0$, where $\delta R + bR = R$ and $sR \subset a_0R$. Since $\overline{sR} + \overline{cR} = \overline{R}$, it is obvious that $a_0R + cR = R$. Since $\overline{rR} + \overline{sR} = \overline{R}$ then, obviously, $\delta R + a_0R = R$. Thus we have shown that a is right avoidable element. □

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ЧИСТІ ДУО КІЛЬЦЯ

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Кільце з одиницею називається дуо кільцем, якщо кожен його однобічний ідеал є двобічним. Кільце називається чистим, якщо кожен його елемент є сумою оборотного й ідемпотента. Описано деякі з властивостей чистих дуо кілець.

Ключові слова: кільце Безу, чисте кільце, роздільне кільце, дуо кільце