ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2016. Випуск 82. С. 163–173
Visnyk of the Lviv Univ. Series Mech. Math. 2016. Issue 82. P. 163–173

УДК 512.628.2

ON THE RADICAL OF A DIFFERENTIAL SEMIRING IDEAL

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Some new examples and properties of differential semiring ideals are given. Radical differential ideals of commutative differential semirings are studied. It is shown that a radical differential subtractive ideal is an intersection of prime differential subtractive ideals. Differential semirings in which the radical of every differential subtractive ideal is again differential are characterized.

Key words: Differential semiring, differential semiring ideal, radical differential ideal.

1. Introduction and preliminaries. In 1935 Vandiver [9] introduced a notion of semiring as a generalization of associative rings and distributive lattices. Semiring derivations, differential semirings and their differential ideals were considered by Golan in [4], where he gave few simple examples and properties. Thierrin [8] proved that the semiring of languages over some alphabet forms a differential additively idempotent semiring under the operations of union as the addition and catenation as the product. He gave a number of other interesting examples of differential semirings of languages and studied some of their properties, proving that differential semirings are of great interest due to their possible applications. Recently in [2] the authors investigated some further properties of semiring derivations and differential semiring ideals. This motivates a study of differential semirings as semirings, not necessarily idempotent, with an abstract derivation, not connected with formal languages.

The objective of this paper is to provide a study of differential semirings, mostly concerning basic properties of differential semiring ideals. A number of new examples and properties of differential semiring ideals are given. In the paper, radical differential ideals of commutative differential semirings are investigated. It is shown that a radical differential subtractive ideal is an intersection of prime differential subtractive ideals (Theorem 2). The paper also touches the question as to when the radical of every differential semiring ideal is differential. Theorem 3 lists conditions equivalent to the last-mentioned one. Differential semirings in which the previously stated property holds for every differential ideal are studied.

2010 Mathematics Subject Classification: Primary 16Y60, Secondary 13N99

For the sake of completeness some definitions and properties used in the paper will be given here. For more information on semirings see [4] or [5].

Let R be a nonempty set and let $+$ and $\cdot$ be binary operations on R named addition and multiplication respectively. An algebraic system $(R, +, \cdot)$ is called a *semiring* if $(R, +)$ is a commutative semigroup and $(R, \cdot)$ is a semigroup such that multiplication distributes over addition from either side. A semiring which is not a ring is called a *proper semiring.* A semiring $(R, +, \cdot)$ is called *commutative* if multiplication is commutative.

An element $0 \in R$ is called *zero* if $a + 0 = 0 + a = a$ for all $a \in R$. An element $1 \in R$ is called *identity* if $a \cdot 1 = 1 \cdot a = a$ for all $a \in R$. Zero $0 \in R$ is called (*multiplicatively*) *absorbing* if $a \cdot 0 = 0 \cdot a = 0$ for all $a \in R$.

A *semifield* is a semiring in which non-zero elements form a group under multiplication.

An element $a \in R$ is called *additively idempotent* if $a + a = a$. An element $a \in R$ is called *multiplicatively idempotent* if $a \cdot a = a$. Denote by $I^+(R)$ the set of all additively idempotent elements of R, and by $I^\times(R)$ the set of all multiplicatively idempotent elements of R. The set $I^+(R)$ is an ideal of R, and $I^\times(R)$ is a submonoid of $(R, \cdot)$, if $1 \in R$.

A semiring R is called *additively (multiplicatively) idempotent* if every element of R is additively (multiplicatively) idempotent. Additively idempotent semirings are of great interest due to their applications. They are widely known as idempotent semirings.

A non-empty subset of R, closed under addition and multiplication, is called a *subsemiring* of R. A nonempty subset $I \neq \emptyset$ of R is called a *(semiring) ideal* of R, if it is closed under addition and both $ra \in I$ and $ar \in I$ hold for any $r \in R$ and $a \in I$. Note that according to this definition a semiring ideal is not necessarily proper.

An ideal I of R is called a *subtractive ideal* (or *k-ideal*) if $a + b \in I$ and $a \in I$ imply that $b \in I$. The *k-closure* $cl(I)$ of an ideal I is defined as the set $cl(I) = \{a \in R | a + b \in I \text{ for some } b \in I\}$. It is an ideal of R satisfying $I \subseteq cl(I)$ and $cl(cl(I)) = cl(I)$. An ideal I of R is subtractive if and only if $I = cl(I)$.

An ideal I of the semiring R is called *strong* if $a + b \in I$ implies $a \in I$ and $b \in I$ for every $a, b \in R$. Every strong ideal is subtractive.

A *prime ideal* of R is a proper ideal P of R in which $a \in P$ or $b \in P$ whenever $ab \in P$. So P is prime if and only if for ideals A and B in R the inclusion $AB \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$, where $AB = \{ab | a \in A \text{ and } b \in B\} \subseteq A \bigcap B$.

A proper ideal I of R is called *maximal* if $I \subsetneqq J$ for any ideal J of R implies $J = R$.

In a commutative semiring R the *radical* of an ideal I is denoted by $\sqrt{I}$ and defined to be the set $\sqrt{I} = \{r \in R | r^n \in I \text{ for some } n \in \mathbb{N}_0\}$. According to [3] and [1] $I \subseteq \sqrt{I}$. If I is a subtractive ideal of R, then so is $\sqrt{I}$. Moreover, $\sqrt{I}$ is an intersection of all the prime ideals of R containing I, whenever $1 \in R$.

An ideal I of R is said to be *radical* (or *perfect*) if $I = \sqrt{I}$.

Throughout the paper R denotes a commutative semiring in the above sense with identity 1 and absorbing zero $0 \neq 1$, unless stated otherwise. $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$ the set of non-negative integers.

2. Differential semiring ideals and homomorphisms. Let R be a semiring, not necessarily commutative. A map $\delta \colon R \to R$ is called a *derivation* [4] *on* R if $\delta(a + b) = \delta(a) + \delta(b)$ and $\delta(ab) = \delta(a)b + a\delta(b)$ for any $a, b \in R$. A semiring R equipped with a derivation δ is called *differential* with respect to the derivation δ, or a *δ-semiring*, and denoted by (R, δ) [2].

An ideal I of the δ-semiring R is called *differential* [4] if $\delta(a) \in I$ whenever $a \in I$.

It is easily seen that $\{0\}$ *is a differential subtractive ideal of any differential semiring* R. As noted in [2], in a differential semiring R with absorbing zero the set $V(R)$ of all additively invertible elements forms a differential ideal.

Example 1. The set $I^{+}(R)$ of all additively idempotent elements of a differential semiring (R, δ) is a differential ideal of R.

The set $I^{\times}(R)$ of all multiplicatively idempotent elements of the commutative differential semiring (R, δ) is generally not an ideal. Moreover, $I^{\times}(R)$ is not differentially closed, but it can be easily proved that *if R is a commutative differential semiring and $I^{\times}(R)$ is an ideal of R, then it is a differential ideal.*

Example 2. In a polynomial ring $R = \mathbb{N}_0[x]$ together with one derivation $\delta = \frac{d}{dx}$, defined by $\delta(n) = 0$ for all $n \in \mathbb{N}_0$ and $\delta(x) = 1$, the ideal $I = (x^n, n)$, $n \in \mathbb{N}$, is differential.

In what follows R denotes a differential semiring under the derivation δ.

Proposition 1. *Every multiplicatively idempotent two-sided ideal I of a differential semiring R (i. e. such that $I^2 = I$) is differential.*

Proof. Let (R, δ) be a differential semiring and $I^2 = I$. If $a \in I$, then it is a finite sum $a = \sum_{i=1}^{k} r_i s_i$, where $r_i, s_i \in I$. Then $\delta(a) = \sum_{i=1}^{k} \delta(r_i) s_i + \sum_{i=1}^{k} r_i \delta(s_i) \in I$. Hence I is a differential ideal of R. □

Proposition 2. *If I is a differential ideal of R, then its k-closure $cl(I)$ is a differential subtractive ideal of R.*

Proof. It is well known that $cl(I)$ is a subtractive ideal. If $a \in cl(I)$, then there exists $b \in I$ such that $a + b \in I$. It follows that $\delta(a) + \delta(b) \in I$ and $\delta(b) \in I$. Therefore $\delta(a) \in cl(I)$, and $cl(I)$ is differential ideal. □

Proposition 3.
(1) *An intersection of any family of subtractive differential ideals of R is a subtractive differential ideal of R;*
(2) *A sum of any family of differential ideals of R is a differential ideal of R;*
(3) *A product of any finite family of differential ideals of R is a differential ideal of R.*

A semiring R is called *ideally differential* if all of its ideals are differential.

Every additively idempotent differential semiring is ideally differential. Proposition 1 implies that every multiplicatively idempotent commutative differential semiring is ideally differential.

In what follows let R be a commutative differential semiring with respect to the derivation δ.

Lemma 1. *If I is a radical differential subtractive ideal of R and $ab \in I$, then $\delta(a) b \in I$ and $a\delta(b) \in I$.*

Proof. It is clear that $\delta(ab) = \delta(a)b + a\delta(b) \in I$. Moreover, $\delta(a)b \in R$. By multiplicative commutativity we have that $\delta(a)b \cdot \delta(ab) = (\delta(a)b)^2 + ab \cdot \delta(a)\delta(b) \in I$. Since I is an ideal of R, $ab \in I$ implies $ab \cdot \delta(a)\delta(b) \in I$, and by subtractiveness $(\delta(a)b)^2 \in I$. Hence $\delta(a)b \in I$. As a result, the subtractiveness of I implies $a\delta(b) \in I$. □

Proposition 4. *If I is a radical differential subtractive ideal of R and A is an arbitrary nonempty subset of R, then*

$$(I : A) = \{r \in R \,|\, ra \in I \text{ for all } a \in A\}$$

is a radical differential subtractive ideal of R.

Proof. Under given conditions $(I : A)$ is an ideal of R [5]. If $r \in (I : A)$, then $ra \in I$ for all $a \in A$. Therefore $\delta(ra) = \delta(r)a + r\delta(a) \in I$. It follows $\delta(r)a \cdot \delta(ra) = (\delta(r)a)^2 + ra \cdot \delta(r)\delta(a) \in I$. It is clear that $ra \cdot \delta(r)\delta(a) \in I$. Since the ideal I is subtractive, we have $(\delta(r)a)^2 \in I$, and $\delta(r)a \in I$ for all $a \in A$. Therefore $\delta(r) \in (I : A)$. Hence $(I : A)$ is a differential ideal of R.

Let $r^n \in (I : A)$ for some $r \in R$, $n \in \mathbb{N}$. Then $r^n a \in I$ for all $a \in A$. It follows that $(ra)^n = (r^n a)\, a^{n-1} \in I$. Since I is radical, we have $ra \in I$ for all $a \in A$. Hence $r \in (I : A)$, and $(I : A)$ is radical.

Let $r, r + s \in (I : A)$. Then $ra \in I$ and $(r + s)a \in I$ for all $a \in A$. By subtractiveness of I, $ra + sa \in I$ and $ra \in I$ follow $sa \in I$. Hence $s \in (I : A)$, so $(I : A)$ is a subtractive ideal of R. □

A subset A of R is called *differentially closed*, if $a \in A$ implies $\delta(a) \in A$. Differential ideals are differentially closed.

Proposition 5. *Let (R, δ) be a differential semiring, not necessarily commutative. If I is a differential subtractive left ideal of R and $A \subseteq R$ is a nonempty differentially closed subset of R, then $(I : A) = \{r \in R \,|\, ra \in I\}$ is a differential subtractive left ideal of R.*

Proof. Under given conditions $(I : A)$ is a subtractive left ideal of R [5]. Let $r \in (I : A)$. Then $ra \in I$ for all $a \in A$. Since I and A are differentially closed $\delta(a) \in A$ and $\delta(ra) = \delta(r)a + r\delta(a) \in I$. It follows that $\delta(r)a \in I$, since I is subtractive and $r\delta(a) \in I$. Hence $\delta(r) \in (I : A)$, and $(I : A)$ is differential. □

Let $A \subseteq R$ be a non-empty subset of a semiring R. The *annihilator ideal of* A is defined as the set $(0 : A) = \{r \in R \,|\, ra = 0 \text{ for all } a \in A\}$.

Corollary 1. *Let (R, δ) be a differential semiring, not necessarily commutative. If $A \subseteq R$ is a nonempty differentially closed subset of R, then $(0 : A)$ is a differential subtractive ideal of R.*

For an element $a \in R$ denote $a^{(0)} = a$, $a' = \delta(a)$, $a'' = \delta(\delta(a))$, $\ldots a^{(n)} = \delta(a^{(n-1)})$, $n \in \mathbb{N}_0$, and $a^{(\infty)} = \{a^{(n)} | n \in \mathbb{N}_0\}$. The set $a^{(\infty)}$ of all derivatives of $a \in R$ is differentially closed in R, so we have the following result.

Corollary 2. *Let (R, δ) be a differential semiring, not necessarily commutative. If I is a differential subtractive left ideal of R and $a \in R$, then $\left(I : a^{(\infty)}\right)$ is a differential subtractive left ideal of R.*

Note the following properties, which are straightforward to prove.

Proposition 6. *Let (R, δ) be a differential semiring, not necessarily commutative.*

(1) *If $I \subseteq J$, then $\left(I : a^{(\infty)}\right) \subseteq \left(J : a^{(\infty)}\right)$ for any $a \in R$.*
(2) *$\left(I : a^{(\infty)}\right) \subseteq (I : a)$ for any $a \in R$.*
(3) *$\left(\left(I : a^{(\infty)}\right) : b^{(\infty)}\right) = \left(I : (ab)^{(\infty)}\right)$ for any $a, b \in R$.*

Proposition 7. *An intersection of an arbitrary family of radical (resp. subtractive) differential ideals of R is a radical (resp. subtractive) differential ideal of R.*

Proof. In any differential semiring R an intersection of an arbitrary family of (subtractive) differential ideals of R is a (subtractive) differential ideal of R by Proposition 3. In any commutative semiring R an intersection of any family of radical semiring ideals is a radical ideal of R. □

Let A be a subset of R. Denote the smallest differential ideal containing the set A by $[A]$, the smallest radical differential ideal containing A by $\{A\}$, the smallest differential subtractive ideal containing the set A by $|A|$, and the smallest radical differential subtractive ideal containing A by $\langle A \rangle$.

Lemma 2. *For any element $r \in R$ and any subset A of R, $r\langle A \rangle \subseteq \langle rA \rangle$.*

Proof. By Proposition 4, $(\langle rA \rangle : r)$ is a radical differential subtractive ideal of R. Since $rA \subseteq \langle rA \rangle$, then $A \subseteq (\langle rA \rangle : r)$. It follows $\langle A \rangle \subseteq (\langle rA \rangle : r)$. Hence $r\langle A \rangle \subseteq \langle rA \rangle$. □

Lemma 3. *For any subsets A and B of R, $\langle A \rangle \langle B \rangle \subseteq \langle AB \rangle$.*

Proof. By Lemma 2, $A \subseteq (\langle AB \rangle : \langle B \rangle) = \{x \in R | x\langle B \rangle \subseteq \langle AB \rangle\}$. By Proposition 4, $(\langle AB \rangle : \langle B \rangle)$ is a radical differential subtractive ideal of R. It all implies that $\langle A \rangle \subseteq (\langle AB \rangle : \langle B \rangle)$. Hence $\langle A \rangle \langle B \rangle \subseteq \langle AB \rangle$. □

Theorem 1. *Let S be a multiplicatively closed subset of R ($0 \notin S$). If I is a radical differential subtractive ideal of R maximal among radical differential subtractive ideals disjoint from S, then I is prime.*

Proof. Let $S \subseteq R$ be a multiplicatively closed subset of R and let I be a radical differential ideal of R maximal among those not meeting S. Suppose that there exist $a, b \in R$ such that $a \cdot b \in I$, $a \notin I$ and $b \notin I$. Then $I \subsetneq \langle I, a \rangle$ and $I \subsetneq \langle I, b \rangle$, moreover $\langle I, a \rangle \bigcap S \neq \varnothing$ and $\langle I, a \rangle \bigcap S \neq \varnothing$. Thus there exist $u, v \in S$ such that $u \in \langle I, a \rangle$ and $v \in \langle I, b \rangle$. Thus $uv \in \langle I, a \rangle \langle I, b \rangle \subseteq I$ by Lemma 3. Therefore $I \bigcap S \neq \varnothing$, which is a contradiction. □

Corollary 3. *Let $S \subseteq R$ be a multiplicatively closed subset of R and let I be any radical differential subtractive ideal disjoint from S. Then there exists a prime differential subtractive ideal P containing I which is disjoint from S.*

A semiring ideal I of R is called *quasi-prime* if it is maximal among the differential ideals disjoint from some multiplicatively closed subset S of R.

Every prime differential ideal is quasi-prime.

Theorem 2. *If I is a radical differential subtractive ideal of R, then it is an intersection of all the prime differential subtractive ideals containing I.*

Proof. Let I be a radical differential subtractive ideal of R. It is clear that any radical differential subtractive ideal is contained in the intersection of all the prime differential subtractive ideals containing it.

To prove the inclusion $\bigcap_{I\subseteq P} P \subseteq I$ take some $a \notin I$ and denote $S = \{a^n | n \in \mathbb{N}_0\}$. Since I is radical, $S \bigcap I = \varnothing$. There exists some radical differential subtractive ideal P of R which is maximal among radical differential subtractive ideals disjoint from S. By Theorem 1, P is a prime differential subtractive ideal of R containing I and $S \bigcap P = \varnothing$. It follows that $a^n \notin P$ for any $n \in \mathbb{N}_0$, and therefore $a \notin P$. Hence $a \notin \bigcap_{I\subseteq P} P$. □

Corollary 4. *Let A be a non-empty subset of R. Then $\langle A\rangle$ is the intersection of all the prime differential subtractive ideals P containing A.*

A map $f\colon R_1 \to R_2$ is called a semiring *homomorphism* if $f(a+b) = f(a) + f(b)$ and $f(ab) = f(a)\cdot f(b)$ for all $a, b \in R$. The *kernel* of f is defined as the set $\operatorname{Ker} f = \{r \in R | f(r) = 0_{R_2}\}$, and the image of f is the set $\operatorname{Im} f = \{r \in R_2 : \exists s \in R_1\ f(s) = r\}$.

A homomorphism of differential semirings $f\colon R_1 \to R_2$ is called a *differential homomorphism* if $f(\delta(r)) = \delta(f(r))$ for all $r \in R_1$.

Proposition 8. *Let R_1 and R_2 be differential semirings, and let $f\colon R_1 \to R_2$ be a differential semiring homomorphism. Then*

(1) $\operatorname{Ker} f$ *is a differential subtractive ideal of* R_1*;*
(2) $\operatorname{Im} f$ *is a differential subsemiring of* R_2*;*
(3) *If I is a differential ideal of R_1, then I^e is a differential ideal of R_2;*
(4) *If I is a differential subtractive ideal of R_2, then I^c is a differential subtractive ideal of R_1.*

Proof. (1) Clearly, if $r \in \operatorname{Ker} f$, then $f(r) = 0_{R_2}$ and $f(\delta(r)) = \delta(f(r)) = \delta(0_{R_2}) = 0_{R_2}$. Hence $\delta(r) \in \operatorname{Ker} f$, and $\operatorname{Ker}(f)$ is differential.

(2) If $r \in \operatorname{Im} f$ then there exists $s \in R_1$ such that $f(s) = r$. It follows that $\delta(r) = \delta(f(s)) = f(\delta(s)) \in \operatorname{Im} f$.

(3) If $r \in I^e$ then $r = \sum_{i=1}^{k} r_i f(s_i)$, $s_i \in I$. Then we have $\delta(r) = \delta\left(\sum_{i=1}^{n} r_i f(s_i)\right) = \sum_{i=1}^{n} \left(\delta(r_i)\cdot f(s_i) + r_i \cdot f(\delta(s_i))\right) \in I^e$, because $s_i \in I$.

(4) If $r \in I^c$ then $f(r) \in I$. Since $\delta(f(r)) = f(\delta(r)) \in I$, then $\delta(r) \in I^c$. □

Corollary 5. *Let R_1 and R_2 be differential semirings. If $f\colon R_1 \to R_2$ is a differential semiring homomorphism and P is a prime differential subtractive ideal of R_2, then $f^{-1}(P)$ is a prime differential subtractive ideal of R_1.*

Proof. Follows by [3, Proposition 3.2].

The following proposition is straightforward to prove.

Proposition 9. *Let R_1 and R_2 be differential semirings, and let $f\colon R_1 \to R_2$ be a differential semiring homomorphism. Then f induces a differential isomorphism $\bar{f}\colon R_1/\operatorname{Ker} f \to \operatorname{Im} f$ for which $\bar{f}(r + \operatorname{Ker} f) = f(r)$ for all $r \in R_1$.*

3. Differential semirings in which the radical of each differential ideal is differential. For a subset A of R we define its *differential* $A_\#$ to be the set

$$A_\# = \left\{ a \in R \,\middle|\, a^{(n)} \in A \,\text{for all}\, n \in \mathbb{N}_0 \right\}.$$

Proposition 10. *Let A, B, A_i, $i \in I$, be subsets of R. Then $(\ )_\#$ has the following properties:*

(1) $A_\# \subseteq A$;
(2) $(A_\#)_\# = A_\#$;
(3) $A_\# = A$ *if and only if A is differentially closed in R;*
(4) *If $A \subseteq B$ then $A_\# \subseteq B_\#$;*
(5) $\left(\bigcap_{i\in I} A_i\right)_\# = \bigcap_{i\in I} (A_i)_\#$;
(6) $\bigcup_{i\in I} (A_i)_\# \subseteq \left(\bigcup_{i\in I} A_i\right)_\#$;
(7) $A_\# + B_\# \subseteq (A + B)_\#$;
(8) $A_\# \cdot B_\# \subseteq (AB)_\#$.

Proposition 11. *The operator $(\ )_\#$ has the following properties.*

(1) *If I is an ideal of R, then $I_\#$ is a differential ideal of R.*
(2) *If I is a strong ideal of R, then $I_\#$ is a differential strong ideal of R.*
(3) *If I is a subtractive ideal of R, then $I_\#$ is a differential subtractive ideal of R.*
(4) *If I is a subsemiring of R, then $I_\#$ is a differential subsemiring of R.*
(5) *If I is a differential ideal of R, then $I_\# = I$.*

Proof. (1) Let $a, b \in I_\#$. Then $a^{(n)} \in I$ and $b^{(n)} \in I$ for any $n \in \mathbb{N}_0$, thus $(a + b)^{(n)} = a^{(n)} + b^{(n)} \in I$. Hence $a + b \in I_\#$. If $a \in I_\#$ and $r \in R$ then $a^{(k)} \in I$ for any $k \in \mathbb{N}_0$. By the Leibnitz rule, $(ra)^{(n)} = \sum_{k=0}^{n} C_n^k r^{(n-k)} a^{(k)} \in I$. It means that $ra \in I_\#$. Hence $I_\#$ is an ideal of R. The ideal $I_\#$ is differential since $I_\#$ is differentially closed for any subset I of R.

(2) Suppose that $a + b \in I_\#$. Then $(a + b)^{(n)} = a^{(n)} + b^{(n)} \in I$ for any $n \in \mathbb{N}_0$. The ideal I being strong implies that $a^{(n)} \in I$ and $b^{(n)} \in I$. Thus $a \in I_\#$ and $b \in I_\#$, so $I_\#$ is strong.

(3) Follows from (2) since every strong ideal is subtractive. (4) Follows from (1). (5) follows from Proposition 10. □

Proposition 12. *Let I be an arbitrary subtractive semiring ideal of R and let A be a differentially closed subset of R. Then the following equality holds:*

$$(I : A)_\# = (I_\# : A).$$

Proof. Suppose $r \in (I : A)_\#$. Then $r^{(n)} \in (I : A)$ for all $n \in \mathbb{N}_0$, so $r^{(n)}a \in I$ for all $a \in A$. Since A is differentially closed, then $ra' \in I$. Therefore $(ra)' = r'a + ra' \in I$. By induction we obtain that $(ra)^{(n)} \in I$ for all $n \in \mathbb{N}_0$. Hence $r \in (I_\# : A)$.

Conversely, let $r \in (I_\# : A)$. Then $(ra)^{(n)} \in I$ for all $a \in A$, $n \in \mathbb{N}_0$, i. e. $ra \in I$, $(ra)' = r'a+ra' \in I$, $(ra)'' = r''a+2r'a'+ra'' \in I, \dots, (ra)^{(n)} = \sum_{k=0}^{n} C_n^k r^{(n-k)} a^{(k)} \in I$. Since A is differentially closed, by subtractiveness of I, $(ra)' \in I$ and $ra' \in I$ imply $r'a \in I$. We may infer by induction that $r^{(n)}a \in I$ for all $a \in A$, $n \in \mathbb{N}_0$. It follows that $r^{(n)} \in (I : A)$, i. e. $r \in (I : A)_\#$. □

Corollary 6. *If I is a subtractive ideal of R and A is a differentially closed subset of R, then $(I_{\#} : A)$ is a differential subtractive ideal of R.*

Corollary 7. *Let I be an arbitrary subtractive ideal of R and $a \in R$. Then $\left(I : a^{(\infty)}\right)_{\#} = \left(I_{\#} : a^{(\infty)}\right)$.*

Proposition 13. *Let R_1 and R_2 be differential semirings, and let $f \colon R_1 \to R_2$ be a differential semiring homomorphism. Then $(\quad)_{\#}$ has the following properties:*

(1) *If A is a subset of R_1, then $f(A_{\#}) \subseteq (f(A))_{\#}$;*
(2) *If A is a subset of R_1 and $f \colon R_1 \to R_2$ is a differential semiring monomorphism, then $f(A_{\#}) = (f(A))_{\#}$;*
(3) *If B is a subset of R_2 and $f \colon R_1 \to R_2$ is a differential semiring epimorphism, then $f^{-1}(B_{\#}) = \left(f^{-1}(B)\right)_{\#}$.*

Proposition 14. *In any differential semiring R for any prime ideal P of R the differential ideal $P_{\#}$ is quasi-prime.*

Proof. Suppose P is a prime ideal of R and $S = R \setminus P$. Then S is multiplicatively closed, and, by Propositions 10 and 11, $P_{\#}$ is a differential ideal of R disjoint from S. If I is any differential ideal disjoint from S, then $I \subseteq P$. Thus $I = I_{\#} \subseteq P_{\#}$. Hence $P_{\#}$ is quasi-prime. □

It is known even in the case of differential rings that the radical of a differential ideal is not necessarily differential. This is also true for semirings. For example, for an ideal (x^n, n) of the semiring $\mathbb{N}_0[x]$ its radical is not differential.

Theorem 3. *The following conditions are equivalent:*

(1) *If I is a differential subtractive ideal of R, then so is $\sqrt{I}$;*
(2) *If $S \subseteq R$ is a multiplicatively closed subset of R ($0 \notin S$) and I is a differential subtractive ideal of R disjoint from S, then every differential subtractive ideal of R which is maximal among differential subtractive ideals containing I and not meeting S is prime.*
(3) *If I is a prime subtractive ideal of R, then $I_{\#}$ is a differential prime subtractive ideal of R.*
(4) *Any prime subtractive ideal, minimal over some differential subtractive ideal, is differential.*
(5) *If A is any subset of R then $\langle A \rangle = \sqrt{|A|}$.*
(6) *Any quasi-prime subtractive ideal I in R is prime.*
(7) *Any quasi-prime subtractive ideal I in R is radical.*

Proof. (1) ⇒ (2) Let the radical of each differential subtractive ideal of R be differential. Suppose $S \subseteq R$ is a multiplicatively closed subset of R ($0 \notin S$), I is a differential subtractive ideal of R such that $I \cap S = \varnothing$, and K is an arbitrary differential ideal of R such that $I \subseteq K$, $K \cap S = \varnothing$, and for any differential subtractive ideal L such that $K \subseteq L$ we have $K = L$. Under given conditions $\sqrt{K}$ is a differential subtractive ideal of R. Moreover, $I \subseteq K \subseteq \sqrt{K}$. Since K is maximal, we have $K = \sqrt{K}$. Thus K is a radical differential subtractive ideal of R, maximal with respect to the exclusion of S. Hence, by Theorem 1, K is prime.

(2) $\Rightarrow$ (3) Suppose $S \subseteq R$ is a multiplicatively closed subset of R ($0 \notin S$), I is a differential subtractive ideal of R such that $I \cap S = \varnothing$, and every differential subtractive ideal K of R, maximal among those containing I and not meeting S is prime. Let P be any prime subtractive ideal. Under given conditions $S = R \backslash P$ is a multiplicatively closed subset of R and $\{0\}$ is a differential subtractive ideal disjoint from S. Moreover, $P_{\#} \subseteq P$ follows $S \cap P_{\#} = \varnothing$. Thus $P_{\#}$ is a differential subtractive ideal of R disjoint from S. If I is an arbitrary differential subtractive ideal of R such that $P_{\#} \subseteq I$ and $I \cap S = \varnothing$, then $I \subseteq P$. It follows that $I = I_{\#} \subseteq P_{\#}$. Thus $P_{\#}$ is prime.

(3) $\Rightarrow$ (4) Let I be an arbitrary differential subtractive ideal of R, and let P be a prime subtractive ideal of R minimal among those containing I. Then we have $I = I_{\#} \subseteq P_{\#} \subseteq P$. Since P is prime, moreover it is minimal among prime subtractive ideals containing I, $P_{\#}$ is prime by assumption and $P_{\#} = P$. Thus P is differential.

(4) $\Rightarrow$ (1) Follows from Theorem 2 and Proposition 3. Let I be an differential subtractive ideal of R. The radical of each differential subtractive ideal is the intersection of all prime differential subtractive ideals containing it, moreover this intersection is a prime ideal and it is minimal over I. It follows by assumption that $\sqrt{I}$ is differential.

(1) $\Leftrightarrow$ (5) If a radical of each differential subtractive ideal is a differential subtractive ideal, then the same holds for the differential subtractive ideal $|A|$. Then $\sqrt{|A|}$ is differential and obviously coincides with $\langle A\rangle$. Conversely, let I be a differential ideal. Then $\langle I\rangle = \sqrt{|I|} = \sqrt{I}$ is a differential ideal of R.

(3) $\Leftrightarrow$ (6) Obviously, since (0) is a differential subtractive ideal contained in any other differential subtractive ideal not meeting S.

(6) $\Leftrightarrow$ (7) Obviously follows from definition and Theorem 2. □

A differential semiring satisfying one of the equivalent conditions stated in the Theorem 3 is called a *dmsp-semiring.* Note that differential rings in which the radical of each differential ideal is differential were studied in 1973 by H. Gorman, who coined the term of a *d-MP*-ring; rings satisfying the same property were studied by Keigher [6] in 1977, who named them special rings. Nowadays in differential algebraic geometry the term of a *Keigher ring* is generally used instead. It is therefore easy to see that in a *dmsp*-semiring maximal among differential subtractive ideals are prime. Every differentially trivial semiring is a *dmsp*-semiring. $\{0\}$ is a *dmsp*-semiring. Any differential semifield is a *dmsp*-semiring. Any Keigher ring is a *dmsp*-semiring.

Corollary 8. *In a dmsp-semiring the radical of an arbitrary differential subtractive ideal is the intersection of all the prime differential subtractive ideals containing I.*

Proof. Since I is differential by Theorem 3 (5) we have $\langle I\rangle = \sqrt{|I|} = \sqrt{I}$. From Theorem 2, $\langle I\rangle = \bigcap_{I\subseteq P} P$, and the result follows. □

Note that this corollary can be proved directly using the argument similar to the proof of Theorem 2.

Let I be an ideal of a semiring R and let $a, b \in R$. Define the equivalence $a \sim b$ if and only if there exist $x, y \in I$ such that $a + x = b + y$. Then $\sim$ is an equivalence relation on R. Let $[a]_I^R$ or $[a]$ be the equivalence class of $a \in R$. Then $R/I = \{[a]_I^R | a \in R\}$ is a semiring under the binary operations defined as follows: $[a] + [b] = [a + b]$ and $[a][b] = [ab]$ for all $a, b \in R$. This semiring is called the *Bourne factor semiring* of R by I.

Let (R,δ) be a differential semiring and let I be a differential semiring ideal of R. Then it can be easily proved that the Bourne factor semiring R/I is a differential semiring under the derivation $d\colon R/I \longrightarrow R/I$ given by $d\left([a]_I^R\right) = [d(a)]_I^R$ for any $a \in R$.

Proposition 15. *If R is a dmsp-semiring and I is a differential subtractive ideal of R, then R/I is a dmsp-semiring.*

Proof. The statement follows easily from the structure of prime ideals of the Bourne factor semiring R/I and the definition of *dmsp*-semiring. □

Proposition 16. *If I is a radical differential subtractive ideal of the* dmsp-*semiring R, then $I_\#$ is a radical differential subtractive ideal of R.*

Proof. Let I be a radical differential subtractive ideal of R. By Theorem 2, I coincides with the intersection of all prime differential subtractive ideals of R which contain it. Thus $I = \bigcap_{I\subseteq P} P$. By Propositions 10 and 11 the operator $(\quad)_\#$ preserves intersections, inclusion and subtractive ideals. Therefore $(\quad)_\#$ also preserves radical ideals. □

Proposition 17. *If R_1 is a dmsp-semiring and $f\colon R_1 \to R_2$ is a differential semiring epimorphism, then R_2 is a dmsp-semiring.*

Proof. Denote $A = \{\mathcal{P} \in \operatorname{Spec} R_1 \mid \operatorname{Ker} f \subseteq \mathcal{P}\} \subseteq R_1$. It is clear that the differential epimorphism $f\colon R_1 \to R_2$ induces a differential isomorphism $\bar{f} = f\,|_A : A \to \operatorname{Spec} R_2$ between prime ideals $\mathcal{P}$ of R_1, containing the kernel of the homomorphism $\operatorname{Ker} f$ and prime ideals of R_2.

Let $\mathcal{Q} \in \operatorname{Spec} R_2$. Since R_1 is a *dmsp*-semiring and $\bar{f}^{-1}(\mathcal{Q}) = f^{-1}(\mathcal{Q}) \in A$ is a prime subtractive ideal of R_1 by Corollary 5, then so is $f^{-1}(\mathcal{Q})_\#$. It follows from the properties of $(\quad)_\#$ (Proposition 13) that $\mathcal{Q}_\# = \bar{f}\left(\bar{f}^{-1}(\mathcal{Q}_\#)\right) = \bar{f}\left(\left(\bar{f}^{-1}(\mathcal{Q})\right)_\#\right)$. Therefore $\mathcal{Q}_\#$ is a prime differential subtractive ideal of R_2. Hence R_2 is a *dmsp*-semiring. □

Proposition 18. *Let $R_1, \ldots, R_n$ be differential semirings and let $R = R_1 \times \cdots \times R_n$. Then R is a dmsp-semiring if and only if R_i is a dmsp-semiring for each i.*

Proof. Let R be a *dmsp*-semiring. Then for every i the canonical projection $\pi_i\colon R \to R_i$ is a differential epimorphism. By Proposition 17, every R_i is a *dmsp*-semiring.

Conversely, suppose all R_i are *dmsp*-semirings and $\mathcal{P}$ is a prime subtractive ideal of R. Consider the canonical projections $\pi_i\colon P \to P_i$ for all $i = 1, 2, \ldots, n$. It follows that $\pi_k(\mathcal{P}) = \mathcal{P}_k$ is a prime subtractive ideal of R_k for some k, $k \in \{1, 2, \ldots, n\}$, and $\pi_j(\mathcal{P}) = R_l$ for $l \neq k$. Then $\pi_k^{-1}(\mathcal{P}_k) = \mathcal{P}$. Therefore $\mathcal{P}_\# = (\pi_k^{-1}(\mathcal{P}_k))_\# = \pi_k^{-1}((\mathcal{P}_k)_\#)$. Since $\pi_k^{-1}((\mathcal{P}_k)_\#)$ is a prime subtractive ideal of R by Corollary 5, so is $\mathcal{P}_\#$. Hence R is a *dmsp*-semiring. □

Acknowledgement. I would like to thank Professor M. Ya. Komarnytskyi for helpful discussions.

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Стаття: надійшла до редколегії 04.06.2015
доопрацьована 28.11.2016
прийнята до друку 14.12.2016

ПРО РАДИКАЛ ДИФЕРЕНЦІАЛЬНОГО ІДЕАЛУ НАПІВКІЛЬЦЯ

Іванна МЕЛЬНИК

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Наведено нові приклади та властивості диференціальних ідеалів у напівкільцях. Досліджуємо радикал диференціального ідеалу комутативного диференціального напівкільця. Доведено, що радикальний диференціальний напівстрогий ідеал є перетином первинних диференціальних напівстрогих ідеалів. Подано характеризацію диференціальних напівкілець, в яких радикал кожного диференціального напівстрогого ідеалу є диференціальним.

Ключові слова: диференціальне напівкільце, диференціальний ідеал напівкільця, радикальний диференціальний ідеал.