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ON SEMITOPOLOGICAL INTERASSOCIATES OF THE BICYCLIC MONOID

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Semitopological interassociates $\mathscr{C}_{m,n}$ of the bicyclic semigroup $\mathscr{C}(p,q)$ are studied. In particular, we show that for arbitrary non-negative integers m, n and every Hausdorff topology τ on $\mathscr{C}_{m,n}$ such that $(\mathscr{C}_{m,n},\tau)$ is a semitopological semigroup, is discrete. Also, we prove that if an interassociate of the bicyclic monoid $\mathscr{C}_{m,n}$ is a dense subsemigroup of a Hausdorff semitopological semigroup $(S,\cdot)$ and $I = S\setminus\mathscr{C}_{m,n} \neq \varnothing$ then I is a two-sided ideal of the semigroup S and show that for arbitrary non-negative integers m, n, any Hausdorff locally compact semitopological semigroup $\mathscr{C}^0_{m,n} = \mathscr{C}_{m,n}\sqcup\{0\}$ is either discrete or compact.

Key words: semigroup, interassociate of a semigroup, semitopological semigroup, topological semigroup, bicyclic extension, locally compact space, discrete space, remainder.

We shall follow the terminology of [9, 10, 14, 27]. In this paper all spaces will be assumed to be Hausdorff. By $\mathbb{N}_0$ and $\mathbb{N}$ we denote the sets of non-negative integers and positive integers, respectively. If A is a subset of a topological space X then by $\operatorname{cl}_X(A)$ and $\operatorname{int}_X(A)$ we denote the closure and interior of A in X, respectively.

A *semigroup* is a non-empty set with a binary associative operation.

The *bicyclic semigroup* (or the *bicyclic monoid*) $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subject only to the condition $pq = 1$. The bicyclic monoid $\mathscr{C}(p,q)$ is a combinatorial bisimple F-inverse semigroup (see [23]) and it plays an important role in the algebraic theory of semigroups and in the theory of topological semigroups. For example the well-known O. Andersen's result [1] states that a (0–)simple semigroup is completely (0–)simple if and only if it does not contain the bicyclic semigroup. The bicyclic semigroup cannot be embedded into the stable semigroups [22].

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An interassociate of a semigroup $(S,\cdot)$ is a semigroup $(S,*)$ such that for all $a,b,c\in S$, $a\cdot(b*c)=(a\cdot b)*c$ and $a*(b\cdot c)=(a*b)\cdot c$. This definition of interassociativity was studied extensively in 1996 by Boyd et al [8]. Certain classes of semigroups are known to give rise to interassociates with various properties. For example, it is very easy to show that if S is a monoid, every interassociate must satisfy the condition $a*b=acb$ for some fixed element $c\in S$ (see [8]). This type of interassociate was called a variant by Hickey [20]. In addition, every interassociate of a completely simple semigroup is completely simple [8]. Finally, it is relatively easy to show that every interassociate of a group is isomorphic to the group itself.

In the paper [16] the bicyclic semigroup $\mathscr{C}(p,q)$ and its interassociates are investigated. In particular, if p and q are generators of the bicyclic semigroup $\mathscr{C}(p,q)$ and m and n are fixed nonnegative integers, the operation $a*_{m,n}b=aq^mp^nb$ is known to be an interassociate. It was shown that for distinct pairs (m,n) and (s,t), the interassociates $(\mathscr{C}(p,q),*_{m,n})$ and $(\mathscr{C}(p,q),*_{s,t})$ are not isomorphic. Also in [16] the authors generalized a result regarding homomorphisms on $\mathscr{C}(p,q)$ to homomorphisms on its interassociates.

Later for fixed non-negative integers m and n the interassociate $(\mathscr{C}(p,q),*_{m,n})$ of the bicyclic monoid $\mathscr{C}(p,q)$ will be denoted by $\mathscr{C}_{m,n}$.

A (*semi*)*topological semigroup* is a topological space with a (separately) continuous semigroup operation.

The bicyclic semigroup admits only the discrete semigroup topology and if a topological semigroup S contains it as a dense subsemigroup then $\mathscr{C}(p,q)$ is an open subset of S [13]. Bertman and West in [7] extend this result for the case of Hausdorff semitopological semigroups. Stable and Γ-compact topological semigroups do not contain the bicyclic semigroup [2, 21]. The problem of an embedding of the bicyclic monoid into compact-like topological semigroups studied in [5, 6, 19]. Also in the paper [15] it was proved that the discrete topology is the unique topology on the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ such that the semigroup operation on $\mathscr{C}_{\mathbb{Z}}$ is separately continuous. Amazing dichotomy for the bicyclic monoid with adjoined zero $\mathscr{C}^0=\mathscr{C}(p,q)\sqcup\{0\}$ was proved in [18]: every Hausdorff locally compact semitopological bicyclic semigroup with adjoined zero $\mathscr{C}^0$ is either compact or discrete.

In this paper we study semitopological interassociates $(\mathscr{C}(p,q),*_{m,n})$ of the bicyclic monoid $\mathscr{C}(p,q)$ for arbitrary non-negative integers m and n. Some results from [7, 13, 18] obtained for the bicyclic semigroup are extended to its interassociate $(\mathscr{C}(p,q),*_{m,n})$. In particular, we show that for arbitrary non-negative integers m, n and every Hausdorff topology τ on $\mathscr{C}_{m,n}$ such that $(\mathscr{C}_{m,n},\tau)$ is a semitopological semigroup, is discrete. Also, we prove that if an interassociate of the bicyclic monoid $\mathscr{C}_{m,n}$ is a dense subsemigroup of a Hausdorff semitopological semigroup $(S,\cdot)$ and $I=S\setminus\mathscr{C}_{m,n}\neq\varnothing$ then I is a two-sided ideal of the semigroup S and show that for arbitrary non-negative integers m, n, any Hausdorff locally compact semitopological semigroup $\mathscr{C}^0_{m,n}$ ($\mathscr{C}^0_{m,n}=\mathscr{C}_{m,n}\sqcup\{0\}$) is either discrete or compact.

For arbitrary $m,n\in N$ we denote

$$\mathscr{C}^*_{m,n}=\left\{q^{n+k}p^{m+l}\in\mathscr{C}_{m,n}\colon k.l\in\mathbb{N}_0\right\}.$$

The semigroup operation $*_{m,n}$ of $\mathscr{C}_{m,n}$ implies that $\mathscr{C}^*_{m,n}$ is a subsemigroup of $\mathscr{C}_{m,n}$.

We need the following trivial lemma.

Lemma 1. *For arbitrary non-negative integers m and n the subsemigroup $\mathscr{C}^*_{m,n}$ of $\mathscr{C}_{m,n}$ is isomorphic to the bicyclic semigroup $\mathscr{C}(p,q)$ under the map $\iota\colon \mathscr{C}(p,q)\to\mathscr{C}^*_{m,n}\colon q^ip^j\mapsto q^{n+i}p^{m+j}$, $i,j\in\mathbb{N}_0$.*

Proof. It is sufficient to show that the map $\iota\colon \mathscr{C}(p,q)\to\mathscr{C}^*_{m,n}$ is a homomorphism, because ι is bijective. Then for arbitrary $i,j,k,l\in\mathbb{N}_0$ we have that

$$\iota(q^ip^j\cdot q^kp^l)=\begin{cases}\iota(q^{i-j+k}p^l), & \text{if } j<k;\\ \iota(q^ip^{j-k+l}), & \text{if } j\geqslant k\end{cases}=\begin{cases}q^{n+i-j+k}p^{m+l}, & \text{if } j<k;\\ q^{n+i}p^{m+j-k+l}, & \text{if } j\geqslant k\end{cases}$$

and

$$\begin{aligned}\iota(q^ip^j)*_{m,n}\iota(q^kp^l)&=q^{n+i}p^{m+j}*_{m,n}q^{n+k}p^{m+l}=\\&=q^{n+i}p^{m+j}\cdot q^mp^n\cdot q^{n+k}p^{m+l}=\\&=q^{n+i}p^j\cdot q^kp^{m+l}=\\&=\begin{cases}q^{n+i-j+k}p^{m+l}, & \text{if } j<k;\\ q^{n+i}p^{m+j-k+l}, & \text{if } j\geqslant k,\end{cases}\end{aligned}$$

which completes the proof of the lemma. □

Lemma I.1 from [13] and the definition of the semigroup operation in $\mathscr{C}_{m,n}$ imply the following:

Lemma 2. *For arbitrary non-negative integers m and n and for each elements $a,b\in\mathscr{C}_{m,n}$ both sets*

$$\{x\in\mathscr{C}_{m,n}\colon a*_{m,n}x=b\}\qquad\textit{and}\qquad\{x\in\mathscr{C}_{m,n}\colon x*_{m,n}a=b\}$$

are finite; that is, both left and right translation by a are finite-to-one maps.

The following theorem generalizes the Eberhart–Selden result on semigroup topologization of the bicyclic semigroup (see [13, Corollary I.1]) and the corresponding statement for the case semitopological semigroups in [7].

Theorem 1. *For arbitrary non-negative integers m, n, every Hausdorff semitopological semigroup $(\mathscr{C}_{m,n},\tau)$ is discrete.*

Proof. By Proposition 1 of [7] every Hausdorff semitopological semigroup $\mathscr{C}(p,q)$ is discrete. Hence Lemma 1 implies that for any element $x\in\mathscr{C}^*_{m,n}$ there exists an open neighbourhood $U(x)$ of the point x in $(\mathscr{C}_{m,n},\tau)$ such that $U(x)\cap\mathscr{C}^*_{m,n}=\{x\}$. Fix an arbitrary open neighbourhood $U(q^np^m)$ of the point q^np^m in $(\mathscr{C}_{m,n},\tau)$ such that $U(q^np^m)\cap\mathscr{C}^*_{m,n}=\{q^np^m\}$. Then the separate continuity of the semigroup operation in $(\mathscr{C}_{m,n},\tau)$ implies that there exists an open neighbourhood $V(q^np^m)\subseteq U(q^np^m)$ of the point q^np^m in the space $(\mathscr{C}_{m,n},\tau)$ such that

$$V(q^np^m)*_{m,n}q^np^m\subseteq U(q^np^m)\qquad\text{and}\qquad q^np^m*_{m,n}V(q^np^m)\subseteq U(q^np^m).$$

Suppose to the contrary that the neighbourhood $V(q^np^m)$ is an infinite set. Then at least one of the following conditions holds:

(i) there exists a non-negative integer $i_0<n$ such that the set $A=\left\{q^{i_0}p^l\colon l\in N\right\}\cap V(q^np^m)$ is infinite;

(ii) there exists a non-negative integer $j_0<m$ such that the set $B=\left\{q^lp^{j_0}\colon l\in N\right\}\cap V(q^np^m)$ is infinite.

In case (i) for arbitrary $q^{i_0}p^l \in A$ we have that

$$q^np^m *_{m,n} q^{i_0}p^l = q^np^mq^mp^nq^{i_0}p^l = q^np^nq^{i_0}p^l = \\ = q^np^{n-i_0+l} \notin U(q^np^m) \quad \text{for sufficiently large } l;$$

and similarly in case (ii) we obtain that

$$q^lp^{j_0} *_{m,n} q^np^m = q^lp^{j_0}q^mp^nq^np^m = q^lp^{j_0}q^mp^m = \\ = q^{m-j_0+l}p^m \notin U(q^np^m) \quad \text{for sufficiently large } l;$$

for each $q^lp^{j_0} \in B$, which contradicts the separate continuity of the semigroup operation in $(\mathscr{C}_{m,n}, \tau)$. The obtained contradiction implies that q^np^m is an isolated point in the space $(\mathscr{C}_{m,n}, \tau)$.

Now, since the semigroup $\mathscr{C}_{m,n}$ is simple (see [16, Section 2]) for arbitrary $a, b \in \mathscr{C}_{m,n}$ there exist $x, y \in \mathscr{C}_{m,n}$ such that $xay = b$. The above argument implies that for arbitrary element $u \in \mathscr{C}_{m,n}$ there exist $x_u, y_u \in \mathscr{C}_{m,n}$ such that $x_uuy_u = q^np^m$. Now, by Lemma 2 we get that the equation $x_uxy_u = q^np^m$ has finitely many solutions. This and the separate continuity of the semigroup operation in $(\mathscr{C}_{m,n}, \tau)$ imply that the point u has an open finite neighbourhood in $(\mathscr{C}_{m,n}, \tau)$, and hence, by the Hausdorffness of $(\mathscr{C}_{m,n}, \tau)$, u is an isolated point in $(\mathscr{C}_{m,n}, \tau)$. Then the choice of u implies that all elements of the semigroup $\mathscr{C}_{m,n}$ are isolated points in $(\mathscr{C}_{m,n}, \tau)$. □

The following theorem generalizes Theorem I.3 from [13].

Theorem 2. *If m and n are arbitrary non-negative integers, the interassociate $\mathscr{C}_{m,n}$ of the bicyclic monoid $\mathscr{C}(p, q)$ is a dense subsemigroup of a Hausdorff semitopological semigroup $(S, \cdot)$, and $I = S \setminus \mathscr{C}_{m,n} \neq \varnothing$ then I is a two-sided ideal of the semigroup S.*

Proof. Fix an arbitrary element $y \in I$. If $x \cdot y = z \notin I$ for some $x \in \mathscr{C}_{m,n}$ then there exists an open neighbourhood $U(y)$ of the point y in the space S such that $\{x\} \cdot U(y) = \{z\} \subset \mathscr{C}_{m,n}$. The neighbourhood $U(y)$ contains infinitely many elements of the semigroup $\mathscr{C}_{m,n}$ which contradicts Lemma 2. The obtained contradiction implies that $x \cdot y \in I$ for all $x \in \mathscr{C}_{m,n}$ and $y \in I$. The proof of the statement that $y \cdot x \in I$ for all $x \in \mathscr{C}_{m,n}$ and $y \in I$ is similar.

Suppose to the contrary that $x \cdot y = w \notin I$ for some $x, y \in I$. Then $w \in \mathscr{C}_{m,n}$ and the separate continuity of the semigroup operation in S implies that there exist open neighbourhoods $U(x)$ and $U(y)$ of the points x and y in S, respectively, such that $\{x\} \cdot U(y) = \{w\}$ and $U(x) \cdot \{y\} = \{w\}$. Since both neighbourhoods $U(x)$ and $U(y)$ contain infinitely many elements of the semigroup $\mathscr{C}_{m,n}$, both equalities $\{x\} \cdot U(y) = \{w\}$ and $U(x) \cdot \{y\} = \{w\}$ contradict the mentioned above part of the proof, because $\{x\} \cdot (U(y) \cap \mathscr{C}_{m,n}) \subseteq I$. The obtained contradiction implies that $x \cdot y \in I$. □

We recall that a topological space X is said to be:

- *compact* if every open cover of X contains a finite subcover;
- *countably compact* if each closed discrete subspace of X is finite;
- *feebly compact* if each locally finite open cover of X is finite;
- *pseudocompact* if X is Tychonoff and each continuous real-valued function on X is bounded;

- *locally compact* if every point x of X has an open neighbourhood $U(x)$ with the compact closure $\mathrm{cl}_X(U(x))$;
- *Čech-complete* if X is Tychonoff and there exists a compactification cX of X such that the remainder of X is an F_σ-set in cX.

According to Theorem 3.10.22 of [14], a Tychonoff topological space X is feebly compact if and only if X is pseudocompact. Also, a Hausdorff topological space X is feebly compact if and only if every locally finite family of non-empty open subsets of X is finite. Every compact space and every sequentially compact space are countably compact, every countably compact space is feebly compact (see [4]).

A topological semigroup S is called Γ-*compact* if for every $x \in S$ the closure of the set $\{x, x^2, x^3, \ldots\}$ is compact in S (see [21]). Since by Lemma 1 the semigroup $\mathscr{C}_{m,n}$ contains the bicyclic semigroup as a subsemigroup the results obtained in [2], [5], [6], [19], [21] imply the following corollary

Corollary 1. *Let m and n be arbitrary non-negative integers. If a Hausdorff topological semigroup S satisfies one of the following conditions:*

(i) *S is compact;*
(ii) *S is Γ-compact;*
(iii) *the square $S \times S$ is countably compact; or*
(iv) *the square $S \times S$ is a Tychonoff pseudocompact space,*

then S does not contain the semigroup $\mathscr{C}_{m,n}$.

Proposition 1. *Let m and n be arbitrary non-negative integers. Let S be a Hausdorff topological semigroup which contains a dense subsemigroup $\mathscr{C}_{m,n}$. Then for every $c \in \mathscr{C}_{m,n}$ the set*

$$D_c = \{(x, y) \in \mathscr{C}_{m,n} \times \mathscr{C}_{m,n} \colon x *_{m,n} y = c\}$$

is an open-and-closed subset of $S \times S$.

Proof. By Theorem 1, $\mathscr{C}_{m,n}$ is a discrete subspace of S and hence Theorem 3.3.9 of [14] implies that $\mathscr{C}_{m,n}$ is an open subspace of S. Then the continuity of the semigroup operation of S implies that D_c is an open subset of $S \times S$ for every $c \in \mathscr{C}_{m,n}$.

Suppose that there exists $c \in \mathscr{C}_{m,n}$ such that D_c is a non-closed subset of $S \times S$. Then there exists an accumulation point $(a, b) \in S \times S$ of the set D_c. The continuity of the semigroup operation in S implies that $a \cdot b = c$. But $\mathscr{C}_{m,n} \times \mathscr{C}_{m,n}$ is a discrete subspace of $S \times S$ and hence by Theorem 2 the points a and b belong to the two-sided ideal $I = S \setminus \mathscr{C}_{m,n}$ and hence the product $a \cdot b \in S \setminus \mathscr{C}_{m,n}$ cannot be equal to the element c. □

Theorem 3. *Let m and n be arbitrary non-negative integers. If a Hausdorff topological semigroup S contains $\mathscr{C}_{m,n}$ as a dense subsemigroup then the square $S \times S$ is not feebly compact.*

Proof. By Proposition 1 for every $c \in \mathscr{C}_{m,n}$ the square $S \times S$ contains an open-and-closed discrete subspace D_c. In the case when $c = q^n p^m$, the subspace D_c contains an infinite subset $\left\{\left(q^n p^{m+i}, q^{n+i} p^m\right) : i \in \mathbb{N}_0\right\}$ and hence D_c is infinite. This implies that the square $S \times S$ is not feebly compact. □

For arbitrary non-positive integers m and n by $\mathscr{C}^0_{m,n}$ we denote the interassociate $\mathscr{C}_{m,n}$ with an adjoined zero 0 of the bicyclic monoid $\mathscr{C}(p,q)$, i.e., $\mathscr{C}^0_{m,n} = \mathscr{C}_{m,n} \sqcup \{0\}$.

Example 1. On the semigroup $\mathscr{C}^0_{m,n}$ we define a topology τ_{Ac} in the following way:

(i) every element of the semigroup $\mathscr{C}_{m,n}$ is an isolated point in the space $\left(\mathscr{C}^0_{m,n}, \tau_{\mathsf{Ac}}\right)$;
(ii) the family $\mathscr{B}(0) = \left\{U \subseteq \mathscr{C}^0_{m,n} \colon U \ni 0 \text{ and } \mathscr{C}_{m,n} \setminus U \text{ is finite}\right\}$ determines a base of the topology τ_{Ac} at zero $0 \in \mathscr{C}^0_{m,n}$,

i.e., τ_{Ac} is the topology of the Alexandroff one-point compactification of the discrete space $\mathscr{C}_{m,n}$ with the remainder $\{0\}$. The semigroup operation in $\left(\mathscr{C}^0_{m,n}, \tau_{\mathsf{Ac}}\right)$ is separately continuous, because all elements of the interassociate $\mathscr{C}_{m,n}$ of the bicyclic semigroup $\mathscr{C}(p,q)$ are isolated points in the space $(\mathscr{C}^0_{m,n}, \tau_{\mathsf{Ac}})$ and the left and right translations in the semigroup $\mathscr{C}_{m,n}$ are finite-to-one maps (see Lemma 2).

Remark 1. By Theorem 1 the discrete topology τ_{d} is a unique Hausdorff topology on the interassociate $\mathscr{C}_{m,n}$ of the bicyclic monoid $\mathscr{C}(p,q)$, $m,n \in \mathbb{N}_0$, such that $\mathscr{C}_{m,n}$ is a semitopological semigroup. So τ_{Ac} is the unique compact topology on $\mathscr{C}^0_{m,n}$ such that $(\mathscr{C}^0_{m,n}, \tau_{\mathsf{Ac}})$ is a Hausdorff compact semitopological semigroup for any non-negative integers m and n.

The following theorem generalized Theorem 1 from [18].

Theorem 4. *Let m and n be arbitrary non-negative integers. If $\left(\mathscr{C}^0_{m,n}, \tau\right)$ is a Hausdorff locally compact semitopological semigroup, then τ is either discrete or $\tau = \tau_{\mathsf{Ac}}$.*

Proof. Let τ be a Hausdorff locally compact topology on $\mathscr{C}^0_{m,n}$ such that $\left(\mathscr{C}^0_{m,n}, \tau\right)$ is a semitopological semigroup and the zero 0 of $\mathscr{C}^0_{m,n}$ is not an isolated point of the space $\left(\mathscr{C}^0_{m,n}, \tau\right)$. By Lemma 1 the subsemigroup $\mathscr{C}^*_{m,n}$ of $\mathscr{C}_{m,n}$ is isomorphic to the bicyclic semigroup $\mathscr{C}(p,q)$ and hence the subsemigroup $\left(\mathscr{C}^*_{m,n}\right)^0 = \mathscr{C}^*_{m,n} \sqcup \{0\}$ of $\mathscr{C}^0_{m,n}$ is isomorphic to the bicyclic semigroup with adjoined zero $\mathscr{C}^0 = \mathscr{C}(p,q) \sqcup \{0\}$. Theorem 1 implies that $\mathscr{C}_{m,n}$ is a dense discrete subspace of $\left(\mathscr{C}^0_{m,n}, \tau\right)$, so it is open by Corollary 3.3.10 of [14]. This Corollary also implies that the subspace $\left(\mathscr{C}^*_{m,n}\right)^0$ of $\left(\mathscr{C}^0_{m,n}, \tau\right)$ is locally compact.

We claim that for every open neighbourhood $V(0)$ of zero 0 in $\left(\mathscr{C}^0_{m,n}, \tau\right)$ the set $V(0) \cap \left(\mathscr{C}^*_{m,n}\right)^0$ is infinite. Suppose to the contrary that there exists an open neighbourhood $V(0)$ of zero 0 in $\left(\mathscr{C}^0_{m,n}, \tau\right)$ such that the set $V(0) \cap \left(\mathscr{C}^*_{m,n}\right)^0$ is finite. Since the space $\left(\mathscr{C}^0_{m,n}, \tau\right)$ is Hausdorff, without loss of generality we may assume that $V(0) \cap \left(\mathscr{C}^*_{m,n}\right)^0 = \{0\}$. Then by the separate continuity of the semigroup operation of $\left(\mathscr{C}^0_{m,n}, \tau\right)$ there exists an open neighbourhood $W(0)$ of zero in $\left(\mathscr{C}^0_{m,n}, \tau\right)$ such that $W(0) \subseteq V(0)$ and

$$(q^n p^m *_{m,n} W(0)) \cup (W(0) *_{m,n} q^n p^m) \subseteq V(0).$$

Since 0 is a non-isolated point of $\left(\mathscr{C}^0_{m,n}, \tau\right)$, at least one of the following conditions holds:

(a) the set $W(0) \cap \left\{q^i p^j \colon i \in \mathbb{N}_0, j = 0,1,\ldots,m-1\right\}$ is infinite;
(b) the set $W(0) \cap \left\{q^i p^j \colon i = 0,1,\ldots,n-1, j \in \mathbb{N}_0\right\}$ is infinite.

If (a) holds then the neighbourhood $W(0)$ contains infinitely many elements of the form q^ip^j, where $j < m$, for which we have that

$$q^ip^j *_{m,n} q^np^m = q^ip^jq^mp^nq^np^m = q^ip^jq^mp^m = q^{i-j+m}p^m \in \mathscr{C}^*_{m,n}.$$

Similarly, if (b) holds then the neighbourhood $W(0)$ contains infinitely many elements of the form q^ip^j, where $i < n$, for which we have that

$$q^np^m *_{m,n} q^ip^j = q^np^mq^mp^nq^ip^j = q^np^nq^ip^j = q^np^{n-i+j} \in \mathscr{C}^*_{m,n}.$$

The above arguments imply that the set $V(0) \cap \left(\mathscr{C}^*_{m,n}\right)^0$ is infinite. Hence we have that the zero 0 is a non-isolated point in the subspace $\left(\mathscr{C}^*_{m,n}\right)^0$ of $\left(\mathscr{C}^0_{m,n}, \tau\right)$.

By Lemma 1 the subsemigroup $\mathscr{C}^*_{m,n}$ of $\mathscr{C}_{m,n}$ is isomorphic to the bicyclic semigroup and hence by Theorem 1 from [18] we obtain that the space $\left(\mathscr{C}^*_{m,n}\right)^0$ is compact. Then for every open neighbourhood $U(0)$ of the zero 0 in $\left(\mathscr{C}^0_{m,n}, \tau\right)$ we have that the set $\left(\mathscr{C}^*_{m,n}\right)^0 \setminus U(0)$ is finite.

Now, the semigroup operation of $\mathscr{C}^0_{m,n}$ implies that

$$p^m *_{m,n} q^ip^j = p^mq^mp^nq^ip^j = p^nq^ip^j = q^{i-n}p^j$$

and

$$q^ip^j *_{m,n} q^n = q^ip^jq^mp^nq^n = q^ip^jq^m = q^ip^{j-m},$$

for arbitrary element $q^ip^j \in \mathscr{C}^*_{m,n}$. This and the definition of $\mathscr{C}^*_{m,n}$ imply that

$$p^m *_{m,n} \mathscr{C}^*_{m,n} = \left\{q^{i-n}p^j : i \geqslant n, j \geqslant m\right\}$$

and

$$\mathscr{C}^*_{m,n} *_{m,n} q^n = \left\{q^ip^{j-m} : i \geqslant n, j \geqslant m\right\}.$$

Thus the set $\mathscr{C}^0_{m,n} \setminus \left(p^m *_{m,n} \left(\mathscr{C}^*_{m,n}\right)^0 \cup \left(\mathscr{C}^*_{m,n}\right)^0 *_{m,n} q^n\right)$ is finite, and hence the above arguments imply that every open neighbourhood $U(0)$ of the zero 0 in $\left(\mathscr{C}^0_{m,n}, \tau\right)$ has a finite complement in the space $\left(\mathscr{C}^0_{m,n}, \tau\right)$. Thus the space $\left(\mathscr{C}^0_{m,n}, \tau\right)$ is compact and by Remark 1 the semitopological semigroup $\mathscr{C}^0_{m,n}$ is topologically isomorphic to the semitopological semigroup $\left(\mathscr{C}^0_{m,n}, \tau_{\mathsf{Ac}}\right)$. □

Since by Corollary 1 the interassociate $\mathscr{C}_{m,n}$ of the bicyclic monoid $\mathscr{C}(p,q)$ does not embed into any Hausdorff compact topological semigroup, Theorem 4 implies the following corollary.

Corollary 2. *If m and n are arbitrary non-negative integers and $\mathscr{C}^0_{m,n}$ is a Hausdorff locally compact topological semigroup, then $\mathscr{C}^0_{m,n}$ is discrete.*

The following example shows that a counterpart of the statement of Corollary 2 does not hold when $\mathscr{C}^0_{m,n}$ is a Čech-complete metrizable topological semigroup for any non-negative integers m and n.

Example 2. Fix arbitrary non-negative integers m and n. On the semigroup $\mathscr{C}^0_{m,n}$ we define a topology τ_1 in the following way:

(i) every element of the interassociate $\mathscr{C}_{m,n}$ of the bicyclic monoid is an isolated point in the space $(\mathscr{C}^0_{m,n}, \tau_1)$;

(ii) the family $\mathscr{B}_1(0) = \{U_s \colon s \in \mathbb{N}_0\}$, where

$$U_s = \{0\} \cup \left\{q^{n+i}p^{m+j} \in \mathscr{C}^0_{m,n} \colon i, j > s\right\},$$

is a base of the topology τ_1 at the zero.

It is obvious that $(\mathscr{C}^0_{m,n}, \tau_1)$ is first countable. Then the definition of the semigroup operation of $\mathscr{C}^0_{m,n}$ and the arguments presented in [17, p. 68] show that $(\mathscr{C}^0_{m,n}, \tau_1)$ is a Hausdorff topological semigroup.

First we observe that each element of the family $\mathscr{B}_1(0)$ is an open-and-closed subset of $(\mathscr{C}^0_{m,n}, \tau_1)$, and hence the space $(\mathscr{C}^0_{m,n}, \tau_1)$ is regular. Since the space $\mathscr{C}^0_{m,n}$ is countable and first countable, it is second countable and hence by Theorem 4.2.9 from [14] it is metrizable. Also, by Theorem 4.3.26 from [14] the space $(\mathscr{C}^0_{m,n}, \tau_1)$ is Čech-complete, as a completely metrizable space.

Also the following example presents an interassociate of the bicyclic semigroup with adjoined zero $\mathscr{C}^0 = \mathscr{C}(p,q) \sqcup \{0\}$ for which a counterpart of the statements of Theorem 4 and of Corollary 2 do not hold.

Example 3. The interassociate of the bicyclic semigroup with adjoined zero $\mathscr{C}^0$ with the operation $a * b = a \cdot 0 \cdot b$ is a countable semigroup with zero-multiplication. It is well known that this semigroup endowed with any topology is a topological semigroup (see [9, Vol. 1, Chapter 1]).

Later we shall need the following notions. A continuous map $f \colon X \to Y$ from a topological space X into a topological space Y is called:

- *quotient* if the set $f^{-1}(U)$ is open in X if and only if U is open in Y (see [26] and [14, Section 2.4]);
- *hereditarily quotient* or *pseudoopen* if for every $B \subset Y$ the restriction $f|_B \colon f^{-1}(B) \to B$ of f is a quotient map (see [24, 25, 3] and [14, Section 2.4]);
- *closed* if $f(F)$ is closed in Y for every closed subset F in X;
- *perfect* if X is Hausdorff, f is a closed map and all fibers $f^{-1}(y)$ are compact subsets of X (see [28] and [14, Section 3.7]).

Every closed map and every hereditarily quotient map are quotient [14]. Moreover, a continuous map $f \colon X \to Y$ from a topological space X onto a topological space Y is hereditarily quotient if and only if for every $y \in Y$ and every open subset U in X which contains $f^{-1}(y)$ we have that $y \in \operatorname{int}_Y(f(U))$ (see [14, 2.4.F]).

We need the following trivial lemma, which follows from separate continuity of the semigroup operation in semitopological semigroups.

Lemma 3. *Let S be a Hausdorff semitopological semigroup and I be a compact ideal in S. Then the Rees-quotient semigroup S/I with the quotient topology is a Hausdorff semitopological semigroup.*

The following theorem generalizes Theorem 2 from [18].

Theorem 5. *Let $(\mathscr{C}^I_{m,n}, \tau)$ be a Hausdorff locally compact semitopological semigroup, $\mathscr{C}^I_{m,n} = \mathscr{C}_{m,n} \sqcup I$ and I is a compact ideal of $\mathscr{C}^I_{m,n}$. Then either $(\mathscr{C}^I_{m,n}, \tau)$ is a compact semitopological semigroup or the ideal I is open.*

Proof. Suppose that I is not open. By Lemma 3 the Rees-quotient semigroup $\mathscr{C}^I_{m,n}/I$ with the quotient topology τ_{q} is a semitopological semigroup. Let $\pi\colon \mathscr{C}^I_{m,n} \to \mathscr{C}^I_{m,n}/I$ be the natural homomorphism, which is a quotient map. It is obvious that the Rees-quotient semigroup $\mathscr{C}^I_{m,n}/I$ is isomorphic to the semigroup $\mathscr{C}^0_{m,n}$ and the image $\pi(I)$ is zero of $\mathscr{C}^I_{m,n}/I$. Now we shall show that the natural homomorphism $\pi\colon \mathscr{C}^I_{m,n} \to \mathscr{C}^I_{m,n}/I$ is a hereditarily quotient map. Since $\pi(\mathscr{C}_{m,n})$ is a discrete subspace of $(\mathscr{C}^I_{m,n}/I, \tau_{\mathsf{q}})$, it is sufficient to show that for every open neighbourhood $U(I)$ of the ideal I in the space $(\mathscr{C}^I_{m,n}, \tau)$ the image $\pi(U(I))$ is an open neighbourhood of the zero 0 in the space $(\mathscr{C}^I_{m,n}/I, \tau_{\mathsf{q}})$. Indeed, $\mathscr{C}^I_{m,n} \setminus U(I)$ is an open subset of $(\mathscr{C}^I_{m,n}, \tau)$, because the elements of the semigroup $\mathscr{C}_{m,n}$ are isolated points of the space $(\mathscr{C}^I_{m,n}, \tau)$. Also, since the restriction $\pi|_{\mathscr{C}_{m,n}}\colon \mathscr{C}_{m,n} \to \pi(\mathscr{C}_{m,n})$ of the natural homomorphism $\pi\colon \mathscr{C}^I_{m,n} \to \mathscr{C}^I_{m,n}/I$ is one-to-one, $\pi(\mathscr{C}^I_{m,n} \setminus U(I))$ is a closed subset of $(\mathscr{C}^I_{m,n}/I, \tau_{\mathsf{q}})$. So $\pi(U(I))$ is an open neighbourhood of the zero 0 of the semigroup $(\mathscr{C}^I_{m,n}/I, \tau_{\mathsf{q}})$, and hence the natural homomorphism $\pi\colon \mathscr{C}^I_{m,n} \to \mathscr{C}^I_{m,n}/I$ is a hereditarily quotient map. Since I is a compact ideal of the semitopological semigroup $(\mathscr{C}^I_{m,n}, \tau)$, $\pi^{-1}(y)$ is a compact subset of $(\mathscr{C}^I_{m,n}, \tau)$ for every $y \in \mathscr{C}^I_{m,n}/I$. By Din' N'e T'ong's Theorem (see [12] or [14, 3.7.E]), $(\mathscr{C}^I_{m,n}/I, \tau_{\mathsf{q}})$ is a Hausdorff locally compact space. Since I is not open, by Theorem 4 the semitopological semigroup $(\mathscr{C}^I_{m,n}/I, \tau_{\mathsf{q}})$ is topologically isomorphic to $(\mathscr{C}^0_{m,n}, \tau_{\mathsf{Ac}})$ and hence it is compact. We claim that the space $(\mathscr{C}^I_{m,n}, \tau)$ is compact. Indeed, let $\mathscr{U} = \{U_\alpha\colon \alpha \in \mathscr{I}\}$ be an arbitrary open cover of the topological space $(\mathscr{C}^I_{m,n}, \tau)$. Since I is compact, there exists a finite family $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\} \subset \mathscr{U}$ such that $I \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$. Put $U = U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$. Then $\mathscr{C}^I_{m,n} \setminus U$ is a closed-and-open subset of $(\mathscr{C}^I_{m,n}, \tau)$. Also, since the restriction $\pi|_{\mathscr{C}_{m,n}}\colon \mathscr{C}_{m,n} \to \pi(\mathscr{C}_{m,n})$ of the natural homomorphism π is one-to-one, $\pi(\mathscr{C}^I_{m,n} \setminus U(I))$ is an open-and-closed subset of $(\mathscr{C}^I_{m,n}/I, \tau_{\mathsf{q}})$, and hence the image $\pi(\mathscr{C}^I_{m,n} \setminus U(I))$ is finite, because the semigroup $(\mathscr{C}^I_{m,n}/I, \tau_{\mathsf{q}})$ is compact. Thus, the set $\mathscr{C}^I_{m,n} \setminus U$ is finite as well and hence the space $(\mathscr{C}^I_{m,n}, \tau)$ is also compact. □

Corollary 3. *If $(\mathscr{C}^I_{m,n}, \tau)$ is a Hausdorff locally compact topological semigroup, $\mathscr{C}^I_{m,n} = \mathscr{C}_{m,n} \sqcup I$ and I is a compact ideal of $\mathscr{C}^I_{m,n}$, then the ideal I is open.*

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ПРО НАПІВТОПОЛОГІЧНІ ІНТЕРАСОЦІАТИВНОСТІ БІЦИКЛІЧНОГО МОНОЇДА

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Вивчаємо напівтопологічні інтерасоціативності $\mathscr{C}_{m,n}$ біциклічного моноїда $\mathscr{C}(p,q)$. Доведено, що для довільних невід'ємних цілих чисел m, n кожна гаусдорфова топологія τ на $\mathscr{C}_{m,n}$ така, що $(\mathscr{C}_{m,n},\tau)$ — напівтопологічна напівгрупа, є дискретною. Доведено таке: якщо інтерасоціативність біциклічного моноїда $\mathscr{C}_{m,n}$ є щільною піднапівгрупою гаусдорфової напівтопологічної напівгрупи $(S,\cdot)$ та $I=S\setminus\mathscr{C}_{m,n}\neq\varnothing$, то I — двобічний ідеал в S, а також, що для довільних невід'ємних цілих чисел m, n кожна гаусдорфова локально компактна напівтопологічна напівгрупа $\mathscr{C}^0_{m,n}=\mathscr{C}_{m,n}\sqcup\{0\}$ є або дискретною, або компактною.

Ключові слова: напівгрупа, інтерасоціативність напівгрупи, напівтопологічна напівгрупа, топологічна напівгрупа, біциклічний моноїд, локально компактний простір, дискретний простір, наріст.