ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2016. Випуск 82. С. 34–38
Visnyk of the Lviv Univ. Series Mech. Math. 2016. Issue 82. P. 34–38

УДК 512.624.5

PERIODIC WORDS CONNECTED WITH THE k-FIBONACCI NUMBERS

Galyna BARABASH, Yaroslav KHOLYAVKA, Iryna TYTAR

Ivan Franko National University of Lviv,
Universytetska Str., 1, 79000, Lviv
e-mails: galynabarabash71@gmail.com,
ya_khol@franko.lviv.ua,
iratytar1217@gmail.com

We introduce periodic words that are connected with the k-Fibonacci numbers and investigated their properties.

Key words: k-Fibonacci numbers, k-Fibonacci words.

1. Introduction. The Fibonacci numbers F_n are defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, for all integer $n > 1$, and with initial values $F_0 = 0$ and $F_1 = 1$. These numbers and their generalizations have interesting properties. Different kinds of the Fibonacci sequence and their properties have been presented in the literature, see, e.g., [1, 2, 3]. In particular, the k-Fibonacci numbers are generalizations of the Fibonacci numbers [4].

The k-Fibonacci numbers $F_{k,n}$ defined for any integer number $k \geqslant 1$ by the recurrence relation $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$, for all integer $n > 1$, and with initial values $F_{k,0} = 0$ and $F_{k,1} = 1$, see [4, 5, 6]. These numbers have been studied in several papers, see [7, 8, 9].

Many properties of k-Fibonacci numbers require the full ring structure of the integers. However, generalizations to the ring $\mathbb{Z}_m$ have been considered, see, e.g., [10].

In analogy to the definition of the Fibonacci numbers, one defines the Fibonacci finite words as the contatenation of the two previous terms $f_n = f_{n-1}f_{n-2}$, $n > 1$, with initial values $f_0 = 1$ and $f_1 = 0$ and defines the infinite Fibonacci word f, $f = \lim f_n$ [11]. It is the archetype of a Sturmian word [12]. The properties of the Fibonacci infinite word have been studied extensively by many authors, see, e.g., [12, 13, 14, 15, 16, 17].

The k-Fibonacci words are defined as the contatenation of the previous terms $f_{k,n} = f_{k,n-1}^k f_{k,n-2}$, $n > 1$, with initial values $f_{k,0} = 0$ and $f_{k,1} = 0^{k-1}1$ and one defines the infinite k-Fibonacci word f_k^*, $f_k^* = \lim f_{k,n}$ [18]. It is the archetype of a Sturmian word [12, 18].

2010 Mathematics Subject Classification: 08A50, 11B39, 11B83

Using k-Fibonacci words, in the present article we introduce new kind of the infinite word, namely k-FLP word, and investigate some of its properties.

For any notations not explicitly defined in this article we refer to [2, 10, 12, 18, 19].

2. k-Fibonacci sequence modulo m. The letter p is reserved to designate a prime, m and k are arbitrary integers, $m \geqslant 2$, $k \geqslant 1$.

We reduce $F_{k,n}$ modulo m taking the least nonnegative residues. Let $F^*_{k,n}(m)$ denote the n-th member of the sequence of integers $F_{k,n} \equiv kF_{k,n-1} + F_{k,n-2} \pmod m$, $0 \leqslant F^*_{k,n}(m) < m$, for all integer $n > 1$, and with initial values $F_{k,0} = 0$ and $F_{k,1} = 1$ ($F^*_{k,0}(m) = 0$ and $F^*_{k,1}(m) = 1$).

For any fixed m and k the sequence $F^*_{k,n}(m)$ is periodic. The Pisano period, written $\pi_k(m)$, is the period for which the sequence $F^*_{k,n}(m)$ of k-Fibonacci numbers modulo m repeats [10].

The problem of determining the length of the period of the recurring sequence arose in connection with methods for generating random numbers. A few properties of the $\pi_k(m)$ are in the following theorem [10].

Theorem 1. *In $\mathbb{Z}_m$ the following hold:*

1) Any k-Fibonacci sequence modulo m is periodic and period less than m^2.
2) If m has prime factorization $m = \prod_{i=1}^{n} p_i^{e_i}$, then $\pi_k(m) = \mathrm{lcm}(\pi_k(p_1^{e_1}), \dots, \pi_k(p_n^{e_n}))$.
3) If $m_1|m_2$, then $\pi_k(m_1)|\pi_k(m_2)$.
4) If k is an odd number, then $\pi_k(k^2+4) = 4(k^2+4)$.
5) If k is an odd number, then $\pi_k(2) = 3$ and if k is an even number, then $\pi_k(2) = 2$.

3. k-Fibonacci words.

Definition 1. *The n-th finite k-Fibonacci words are words over $0, 1$ defined inductively as follows*

$$f_{k,0} = 0, \quad f_{k,1} = 0^{k-1}1, \quad f_{k,n} = f^k_{k,n-1} f_{k,n-2}, \quad n > 1. \tag{1}$$

The infinite word f^*_k is the limit $f^*_k = \lim f_{k,n}$ and is called the infinite k-Fibonacci word.

For example, the successive initial finite 3-Fibonacci words are:

$f_{3,0} = 0, f_{3,1} = 001, f_{3,2} = 0010010010, f_{3,3} = 001001001000100100100010010010001, \dots,$
$f^*_3 = 001001001000100100100010010010001\dots.$

We denote as usual by $|f_{k,n}|$ the length (the number of symbols) of $f_{k,n}$ (see [12]). The following proposition summarizes basic properties of k-Fibonacci words [18].

Theorem 2. *The infinite k-Fibonacci word and the finite k-Fibonacci words satisfy the following properties:*

1) The word 11 is not a subword of the infinite k-Fibonacci word.
2) For all $n > 1$ let ab be the last two symbols of $f_{k,n}$, then we have $ab = 10$ if n is even and $ab = 01$ if n is odd.
3) For all k, n $|f_{k,n}| = F_{k,n+1}$.
4) The number of 1s in $f_{k,n}$ equals $F_{k,n}$.

4. Periodic k-FLP words. Let us start with the classical definition of periodicity on words over arbitrary alphabet $\{a_0, a_1, a_2, \dots\}$ (see [19]).

Definition 2. *Let $w = a_0a_1a_2\ldots$ be an infinite word. We say that w is*

1) a periodic word if there exists a positive integer t such that $a_i = a_{i+t}$ for all $i \geqslant 0$. The smallest t satisfying the previous condition is called the period of w;

2) an eventually periodic word if there exist two positive integers r, s such that $a_i = a_{i+s}$, for all $i > r$;

3) an aperiodic word if it is not eventually periodic.

Theorem 3. *For any k the infinite k-Fibonacci word is aperiodic.*

Proof. This statement is proved in [18]. □

We consider the finite k-Fibonacci words $f_{k,n}$ (1) as numbers written in the binary system and denote them by $b_{k,n}$. Denote by $d_{k,n}$ the value of the number $b_{k,n}$ in usual decimal numeration system. We write $d_{k,n} = b_{k,n}$ meaning that $b_{k,n}$ and $d_{k,n}$ are writing of the same number in different numeration systems.

For example, for 3-Fibonacci words we obtain:

$$f_{3,0} = 0, f_{3,1} = 001, f_{3,2} = 0010010010, f_{3,3} = 0010010010001001001000100100010001, \ldots,$$

$$b_{3,0} = 0,\ b_{3,1} = 1,\ b_{3,2} = 10010010,\ b_{3,3} = 10010010001001001000100100010001, \ldots,$$

$$d_{3,0} = 0,\ d_{3,1} = 1,\ d_{3,2} = 146,\ d_{3,3} = 1225933969, \ldots.$$

Formally, $f_{k,n}$, $n > 0$, coincide with the $b_{k,n}$, taken with prefix 0^{k-1}: $f_{k,n} = 0^{k-1}b_{k,n}$.

Theorem 4. *For any finite k-Fibonacci word $f_{k,n}$ in decimal numeration system we have*

$$d_{k,n} = d_{k,n-1}\sum_{t=0}^{k-1} 2^{tF_{k,n}+F_{k,n-1}} + d_{k,n-2}, \quad n > 1,$$

with $d_{k,0} = 0$ and $d_{k,1} = 1$.

Proof. See [20] for a proof for FLP-words. The same argument applies to the k-FLP words. □

Theorem 5. *Let $d_{k,n}(p) = d_{k,n} \pmod p$, $0 \leqslant d_{k,n}(p) < p$. For any fixed k and p the sequence $d_{k,n}(p)$ is periodic.*

Proof. There are only a finite number of $d_{k,n}(p)$ and $2^{F_{k,n}} \pmod p$ possible, and the recurrence of the first few terms sequence $d_{k,n}(p)$ and $2^{F_{k,n}} \pmod p$ gives recurrence of all subsequent terms. The statement follows from Theorem 4. □

Let $T(k, m)$ denote the length of the period of the repeating sequence $d_{k,n}(m)$.

Theorem 6. *For any p and k $T(k + p(p-1), p) = T(k, p)$.*

Proof. This follows from the congruence $k + p(p-1) \equiv k \pmod p$, Euler's theorem and Theorem 4. □

Let $w_{k,0}(m) = 0$ and for arbitrary integer n, $n \geqslant 1$, let $b_{k,n}(m)$ be $d_{k,n}(m)$ in the binary numeration system, $w_{k,n}(m) = w_{k,n-1}(m)b_{k,n}(m)$. Denote by $w_k(m)$ the limit $w_k(m) = \lim_{n\to\infty} w_{k,n}(m)$.

Definition 3. *We say that*

1) $w_{k,n}(m)$ is a finite FLP-word type 1 by modulo m;

2) $w_k(m)$ is a infinite FLP-word type 1 by modulo m.

Theorem 7. *The infinite FLP-word type 1 $w_k(p)$ is periodic.*

Proof. The statement follows from Theorem 5. □

Using k-Fibonacci words we define a periodic FLP-word $v_k(m)$ (infinite FLP-word type 2 by modulo m).

As usual, we denote by ϵ the empty word [12].

First we define words $t_{k,n}(m)$. Let $t_{k,n}(m)$ be the last $F^*_{k,n+1}(m)$ symbols of the word $f_{k,n}$. If $F^*_{k,n+1}(m) = 0$ for some k, n, then $t_{k,n}(m) = \epsilon$. Since $F^*_{k,n}(m)$ is a periodic sequence, the sequence $|t_{k,n}(m)|$ is periodic with the same period.

Theorem 8. *The word length $|t_{k,n}(m)|$ coincides with $F^*_{k,n+1}(m)$.*

Proof. This is clear by construction of $t_{k,n}(m)$.

Let $v_{k,0}(m) = 0$ and for arbitrary integer n, $n \geqslant 1$, $v_{k,n}(m) = v_{k,n-1}(m)t_{k,n}(m)$. Denote by $v_k(m)$ the limit $v_k(m) = \lim_{n\to\infty} v_{k,n}(m)$.

Definition 4. *We say that*

1) $v_{k,n}(m)$ is a finite FLP-word of type 2 by modulo m;
2) $v_k(m)$ is an infinite FLP-word of type 2 by modulo m.

Theorem 9. *The infinite FLP-word of type 2 $v_k(m)$ is a periodic word.*

Proof. The proof is a direct corollary of Theorem 2 and Theorem 8. □

Acknowledgement. The authors thank Taras Banakh for fruitful discussions.

References

1. *Atanassov K.T., Atanassova V., Shannon A.G., Turner J.C.* New visual perspectives on Fibonacci numbers. — London: World Scientific, 2002.
2. *Koshy T.* Fibonacci and Lucas numbers with applications. — New York: Wiley-Interscience, 2001.
3. *Rami'rez J.L., Rubiano G.N., de Castro R.* A generalization of the Fibonacci word fractal and the Fibonacci snowflake // Theor. Comput. Sci. — 2014 — **528**. — P. 40–56.
4. *Falco'n S., Plaza A.* On the Fibonacci k-numbers // Chaos Solitons Fractals. — 2007. — **32**, №5. — P. 1615–1624.
5. *Bolat C., Kose H.* On the properties of k-Fibonacci numbers // Int. J. Contemp. Math. Sci. — 2010. — **5**, №21-24. — P. 1097–1105.
6. *Catarino P.* On Some Identities for k-Fibonacci sequence // Int. J. Contemp. Math. Sci. — 2014. — **9**, №1-4. — P. 37–42.
7. *Falcón S., Plaza Á.* On k-Fibonacci sequences and polynomials and their derivatives// Chaos Solitons Fractals. — 2009. — **39**, №3. — P. 1005–1019.
8. *Ramirez J.L.* Some properties of convolved k-Fibonacci numbers // ISRN Combinatorics. — 2013. — Article ID 759641. — 5p.
9. *Salas A.* About k-Fibonacci numbers and their associated numbers // Int. Math. Forum. — 2011. — **6**, №50. — P. 2473–2479.
10. *Falcón S., Plaza Á.* k–Fibonacci sequences modulo m // Chaos Solitons Fractals. — 2009. — **41**, №1. — P. 497–504.

11. *Berstel J.* Fibonacci words – a survey // The Book of L. Rosenberg G., Salomaa A. (Eds.). — Berlin: Springer, 1986. — P. 11–26.
12. *Lothaire M.* Algebraic Combinatorics on Words. — Cambridge: Cambridge University Press, 2002.
13. *Mignosi F., Pirillo G.* Repetitions in the Fibonacci infinite word // Theor. Inform. Appl. — 1992. — **26**, №3. — P. 199–204.
14. *Mignosi F., Restivo A., Salemi S.* Periodicity and the golden ratio // Theor. Comput. Sci. — 1998. — **204**, №1–2. — P. 153–167.
15. *Pirillo G.* Fibonacci numbers and words // Discrete Math. — 1997. — **173**, №1–3. — P. 197–207.
16. *Séébold P.* Fibonacci morphisms and sturmian words // Theor. Comput. Sci. — 1991. — **88**, №2. — P. 365–384.
17. *Wen Z.-X., Wen Z.-Y.* Some properties of the singular words of the Fibonacci word // Eur. J. Comb. — 1994. — **15**, №6. — P. 587–598.
18. *Ramirez J., Rubiano G.* On k-Fibonacci words // Acta Univ. Sapientiae, Inform. — 2013. — **5**, №2. — P. 212–226.
19. *Duval J.-P., Mignosi F., Restivo A.* Recurrence and periodicity in infinite words from local periods // Theor. Comput. Sci. — 2001. — **262**, №1-2. — P. 269–284.
20. *Barabash G.M., Kholyavka Ya.M., Tytar I.V.* Periodic words connected with the Fibonacci words // Carpathian Math. Publ. — 2016. — **8**, №1. — P. 11–15.

Стаття: надійшла до редколегії 05.12.2016
прийнята до друку 27.02.2017

ПЕРІОДИЧНІ СЛОВА, ЯКІ ПОВ'ЯЗАНІ З ЧИСЛАМИ k-ФІБОНАЧЧІ

Галина БАРАБАШ, Ярослав ХОЛЯВКА, Ірина ТИТАР

Львівський національний університет імені Івана Франка,
вул. Університетська, 1, Львів, 79000
e-mails: galynabarabash71@gmail.com,
ya_khol@franko.lviv.ua,
iratytar1217@gmail.com

Означено періодичні слова, які пов'язані з числами k-Фібоначчі, досліджено їхні властивості.

Ключові слова: числа k-Фібоначчі, слова k-Фібоначчі.