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ON THE MONOID OF MONOTONE INJECTIVE PARTIAL SELFMAPS OF $\mathbb{N}^2_{\leqslant}$ WITH COFINITE DOMAINS AND IMAGES

Dedicated to the memory of Professor Mykola Komarnytskyy

Oleg GUTIK, Inna POZDNIAKOVA

Ivan Franko National University of Lviv,
Universytetska Str., 1, 79000, Lviv,
e-mails: o_gutik@franko.lviv.ua,
ovgutik@yahoo.com, pozdnyakova.inna@gmail.com

Let $\mathbb{N}^2_{\leqslant}$ be the set $\mathbb{N}^2$ with the partial order defined as the product of usual order $\leqslant$ on the set of positive integers $\mathbb{N}$. We study the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ of monotone injective partial selfmaps of $\mathbb{N}^2_{\leqslant}$ having cofinite domain and image. We describe properties of elements of the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ as monotone partial bijections of $\mathbb{N}^2_{\leqslant}$ and show that the group of units of $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ is isomorphic to the cyclic group of order two. Also we describe the subsemigroup of idempotents of $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ and the Green relations on $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$. In particular, we show that $\mathscr{D} = \mathscr{J}$ in $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$.

Key words: semigroup of partial bijections, monotone partial map, idempotent, Green's relations.

1. Introduction and preliminaries

We shall follow the terminology of [1] and [9].

In this paper we shall denote the cardinality of the set A by $|A|$. We shall identify all sets X with their cardinality $|X|$. By $\mathbb{Z}_2$ we shall denote the cyclic group of order two. Also, for infinite subsets A and B of an infinite set X we shall write $A \subseteq^* B$ if and only if there exists a finite subset A_0 of A such that $A \setminus A_0 \subseteq B$.

An algebraic semigroup S is called *inverse* if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element x^{-1} is called the *inverse of* $x \in S$.

If S is a semigroup, then we shall denote the subset of idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as a *band* (or the *band of* S). If the band $E(S)$ is a non-empty subset of S, then the semigroup operation on S determines the following partial order $\leqslant$ on $E(S)$: $e \leqslant f$

if and only if $ef = fe = e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents. A semilattice E is called *linearly ordered* or a *chain* if its natural order is a linear order.

If S is a semigroup, then we shall denote the Green relations on S by $\mathscr{R}$, $\mathscr{L}$, $\mathscr{J}$, $\mathscr{D}$ and $\mathscr{H}$ (see [1, Section 2.1]):

$$a\mathscr{R}b \text{ if and only if } aS^1 = bS^1;$$
$$a\mathscr{L}b \text{ if and only if } S^1a = S^1b;$$
$$a\mathscr{J}b \text{ if and only if } S^1aS^1 = S^1bS^1;$$
$$\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L};$$
$$\mathscr{H} = \mathscr{L} \cap \mathscr{R}.$$

The $\mathscr{R}$-class (resp., $\mathscr{L}$-, $\mathscr{H}$-, $\mathscr{D}$- or $\mathscr{J}$-class) of the semigroup S which contains an element a of S will be denoted by R_a (resp., L_a, H_a, D_a or J_a).

If $\alpha\colon X \rightharpoonup Y$ is a partial map, then by $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$ we denote the domain and the range of α, respectively.

Let $\mathscr{I}_\lambda$ denote the set of all partial one-to-one transformations of an infinite set X of cardinality λ together with the following semigroup operation: $x(\alpha\beta) = (x\alpha)\beta$ if $x \in \operatorname{dom}(\alpha\beta) = \{y \in \operatorname{dom}\alpha \mid y\alpha \in \operatorname{dom}\beta\}$, for $\alpha, \beta \in \mathscr{I}_\lambda$. The semigroup $\mathscr{I}_\lambda$ is called the *symmetric inverse semigroup* over the set X (see [1, Section 1.9]). The symmetric inverse semigroup was introduced by Vagner [16] and it plays a major role in the semigroup theory. An element $\alpha \in \mathscr{I}_\lambda$ is called *cofinite*, if the sets $\lambda \setminus \operatorname{dom}\alpha$ and $\lambda \setminus \operatorname{ran}\alpha$ are finite.

Let $(X, \leqslant)$ be a partially ordered set (a poset). A non-empty subset A of $(X, \leqslant)$ is called a *chain* if the induced partial order from $(X, \leqslant)$ onto A is linear. For an arbitrary $x \in X$ and non-empty $A \subseteq X$ we denote

$$\uparrow x = \{y \in X \colon x \leqslant y\}, \quad \downarrow x = \{y \in X \colon y \leqslant x\}, \quad \uparrow A = \bigcup_{x\in A} \uparrow x \quad \text{and} \quad \downarrow A = \bigcup_{x\in A} \downarrow x.$$

We shall say that a partial map $\alpha\colon X \rightharpoonup X$ is *monotone* if $x \leqslant y$ implies $(x)\alpha \leqslant (y)\alpha$ for $x, y \in \operatorname{dom}\alpha$.

Let $\mathbb{N}$ be the set of positive integers with the usual linear order $\leq$. On the Cartesian product $\mathbb{N} \times \mathbb{N}$ we define the product partial order, i.e.,

$$(i, m) \leqslant (j, n) \qquad \text{if and only if} \qquad (i \leqslant j) \quad \text{and} \quad (m \leqslant n).$$

Later the set $\mathbb{N} \times \mathbb{N}$ with this partial order will be denoted by $\mathbb{N}^2_\leqslant$.

By $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ we denote the subsemigroup of injective partial monotone selfmaps of $\mathbb{N}^2_\leqslant$ with cofinite domains and images. Obviously, $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ is a submonoid of the semigroup $\mathscr{I}_\omega$ and $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ is a countable semigroup.

Furthermore, we shall denote the identity of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ by $\mathbb{I}$ and the group of units of $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ by $H(\mathbb{I})$.

It well known that each partial injective cofinite selfmap f of λ induces a homeomorphism $f^*\colon \lambda^* \to \lambda^*$ of the remainder $\lambda^* = \beta\lambda \setminus \lambda$ of the Stone-Čech compactification of the discrete space λ. Moreover, under some set theoretic axioms (like **PFA** or **OCA**), each homeomorphism of ω^* is induced by some partial injective cofinite selfmap of ω, where ω is a first infinite cardinal (see [10]–[15] and the corresponding sections in the book [17]). Thus, the inverse semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ of injective partial selfmaps of an infinite cardinal λ with cofinite domains and images admits a natural homomorphism $\mathfrak{h}\colon \mathscr{I}_\lambda^{\mathrm{cf}} \to \mathscr{H}(\lambda^*)$ to the homeomorphism group $\mathscr{H}(\lambda^*)$ of λ^* and this homomorphism is surjective under certain set theoretic assumptions.

In the paper [8] algebraic properties of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ are studied. It is shown that $\mathscr{I}_\lambda^{\mathrm{cf}}$ is a bisimple inverse semigroup and that for every non-empty chain L in $E(\mathscr{I}_\lambda^{\mathrm{cf}})$ there exists an inverse subsemigroup S of $\mathscr{I}_\lambda^{\mathrm{cf}}$ such that S is isomorphic to the bicyclic semigroup and $L \subseteq E(S)$, described the Green relations on $\mathscr{I}_\lambda^{\mathrm{cf}}$ and proved that every non-trivial congruence on $\mathscr{I}_\lambda^{\mathrm{cf}}$ is a group congruence. Also, the structure of the quotient semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}/\sigma$, where σ is the least group congruence on $\mathscr{I}_\lambda^{\mathrm{cf}}$, is described.

The semigroups $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ and $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ of injective isotone partial selfmaps with cofinite domains and images of positive integers and integers, respectively, are studied in [6] and [7]. It was proved that the semigroups $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ and $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ have similar properties to the bicyclic semigroup: they are bisimple and every non-trivial homomorphic image $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ and $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ is a group, and moreover the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ has $\mathbb{Z}(+)$ as a maximal group image and $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ has $\mathbb{Z}(+)\times\mathbb{Z}(+)$, respectively.

In the paper [5] we studied the semigroup $\mathscr{I\!O}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ of monotone injective partial selfmaps of the set of $L_n \times_{\mathrm{lex}} \mathbb{Z}$ having cofinite domain and image, where $L_n \times_{\mathrm{lex}} \mathbb{Z}$ is the lexicographic product of n-elements chain and the set of integers with the usual linear order. We described the Green relations on $\mathscr{I\!O}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$, showed that the semigroup $\mathscr{I\!O}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ is bisimple and established its projective congruences. Also, we proved that $\mathscr{I\!O}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ is finitely generated, every automorphism of $\mathscr{I\!O}_\infty(\mathbb{Z})$ is inner, and showed that in the case $n \geqslant 2$ the semigroup $\mathscr{I\!O}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ has non-inner automorphisms. In [5] we proved that for every positive integer n the quotient semigroup $\mathscr{I\!O}_\infty(\mathbb{Z}^n_{\mathrm{lex}})/\sigma$, where σ is a least group congruence on $\mathscr{I\!O}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$, is isomorphic to the direct power $(\mathbb{Z}(+))^{2n}$. The structure of the sublattice of congruences on $\mathscr{I\!O}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ which are contained in the least group congruence is described in [4].

In this paper we study algebraic properties of the semigroup $\mathscr{P\!O}_\infty(\mathbb{N}^2_{\leqslant})$. We describe properties of elements of the semigroup $\mathscr{P\!O}_\infty(\mathbb{N}^2_{\leqslant})$ as monotone partial bijection of $\mathbb{N}^2_{\leqslant}$ and show that the group of units of $\mathscr{P\!O}_\infty(\mathbb{N}^2_{\leqslant})$ is isomorphic to the cyclic group of the order two. Also, the subsemigroup of idempotents of $\mathscr{P\!O}_\infty(\mathbb{N}^2_{\leqslant})$ and the Green relations on $\mathscr{P\!O}_\infty(\mathbb{N}^2_{\leqslant})$ are described. In particular, we show that $\mathscr{D} = \mathscr{J}$ in $\mathscr{P\!O}_\infty(\mathbb{N}^2_{\leqslant})$.

2. Properties of elements of the semigroup $\mathscr{P\!O}_\infty(\mathbb{N}^2_{\leqslant})$ as monotone partial permutations

In this short section we describe properties of elements of the semigroup $\mathscr{P\!O}_\infty(\mathbb{N}^2_{\leqslant})$ as monotone partial transformations of the poset $\mathbb{N}^2_{\leqslant}$.

For any $n \in \mathbb{N}$ and an arbitrary $\alpha \in \mathscr{P\!O}_\infty(\mathbb{N}^2_{\leqslant})$ we denote:

$$\begin{aligned}
&\mathsf{V}^n = \{(n,j)\colon j\in\mathbb{N}\}; && \mathsf{H}^n = \{(j,n)\colon j\in\mathbb{N}\};\\
&\mathsf{V}^n_{\operatorname{dom}\alpha} = \mathsf{V}^n \cap \operatorname{dom}\alpha; && \mathsf{V}^n_{\operatorname{ran}\alpha} = \mathsf{V}^n \cap \operatorname{ran}\alpha;\\
&\mathsf{H}^n_{\operatorname{dom}\alpha} = \mathsf{H}^n \cap \operatorname{dom}\alpha; && \mathsf{H}^n_{\operatorname{ran}\alpha} = \mathsf{H}^n \cap \operatorname{ran}\alpha.
\end{aligned}$$

Remark 1. We observe that the definition of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ implies that for any $n \in \mathbb{N}$ and arbitrary $\alpha \in \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ the sets $\mathsf{V}^n_{\operatorname{dom}\alpha}$, $\mathsf{V}^n_{\operatorname{ran}\alpha}$, $\mathsf{H}^n_{\operatorname{dom}\alpha}$ and $\mathsf{H}^n_{\operatorname{ran}\alpha}$ are infinite, and moreover all of these sets with the partial order induced from $\mathbb{N}^2_\leqslant$ are order isomorphic to $(\mathbb{N}, \leqslant)$.

Lemma 1. *There exists no element α of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ such that $(m,n) < (m,n)\alpha$ for some $(m,n) \in \operatorname{dom}\alpha$.*

Proof. Suppose the contrary, i.e., that there exists an element α of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ such that $(m,n) < (m,n)\alpha$ for some $(m,n) \in \operatorname{dom}\alpha$. We denote $(m,n)\alpha = (i,j)$. Then our assumption implies that the family of subsets

$$\mathfrak{R}_\alpha = \left\{\mathsf{V}^k_{\operatorname{ran}\alpha} \colon k < i\right\} \cup \left\{\mathsf{H}^k_{\operatorname{ran}\alpha} \colon k < j\right\}$$

has more elements than the family

$$\mathfrak{D}_\alpha = \left\{\mathsf{V}^k_{\operatorname{dom}\alpha} \colon k < m\right\} \cup \left\{\mathsf{H}^k_{\operatorname{dom}\alpha} \colon k < n\right\}.$$

Then there exist $A \in \mathfrak{D}_\alpha$ and distinct $B_1, B_2 \in \mathfrak{R}_\alpha$ such that the following conditions hold:

(i) $(p,q)\alpha \in B_1$ for infinitely many $(p,q) \in A$; and
(ii) $(s,t)\alpha \in B_2$ for infinitely many $(s,t) \in A$.

We observe that A is a linearly ordered subset of the poset $\mathbb{N}^2_\leqslant$. Hence, the definition of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ implies that the image $(A)\alpha$ must be a linearly ordered subset of the poset $\mathbb{N}^2_\leqslant$ as well. This implies that one of the following conditions holds:

(a) there exist distinct elements $\mathsf{V}^{k_1}_{\operatorname{ran}\alpha}$ and $\mathsf{V}^{k_2}_{\operatorname{ran}\alpha}$ of the family $\mathfrak{R}_\alpha$ such that the sets $\mathsf{V}^{k_1}_{\operatorname{ran}\alpha} \cap (A)\alpha$ and $\mathsf{V}^{k_2}_{\operatorname{ran}\alpha} \cap (A)\alpha$ are infinite;
(b) there exist distinct elements $\mathsf{H}^{k_1}_{\operatorname{ran}\alpha}$ and $\mathsf{H}^{k_2}_{\operatorname{ran}\alpha}$ of the family $\mathfrak{R}_\alpha$ such that the sets $\mathsf{H}^{k_1}_{\operatorname{ran}\alpha} \cap (A)\alpha$ and $\mathsf{H}^{k_2}_{\operatorname{ran}\alpha} \cap (A)\alpha$ are infinite;
(c) there exist distinct elements $\mathsf{V}^{k_1}_{\operatorname{ran}\alpha}$ and $\mathsf{H}^{k_2}_{\operatorname{ran}\alpha}$ of the family $\mathfrak{R}_\alpha$ such that the sets $\mathsf{V}^{k_1}_{\operatorname{ran}\alpha} \cap (A)\alpha$ and $\mathsf{H}^{k_2}_{\operatorname{ran}\alpha} \cap (A)\alpha$ are infinite.

Each of the above conditions contradicts the fact that $(A)\alpha$ is a linearly ordered subset of the poset $\mathbb{N}^2_\leqslant$. The obtained contradiction implies the statement of the lemma.

By ϖ we denote the bijective transformation of $\mathbb{N} \times \mathbb{N}$ defined by the formula $(i,j)\varpi = (j,i)$, for any $(i,j) \in \mathbb{N} \times \mathbb{N}$. It is obvious that ϖ is an element of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ and $\varpi\varpi = \mathbb{I}$.

Lemma 2. *There exists no element α of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ such that $(n,m) < (m,n)\alpha$ for some $(m,n) \in \operatorname{dom}\alpha$.*

Proof. Suppose the contrary. Then there exists an element α of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ such that $(n,m) < (m,n)\alpha$ for some $(m,n) \in \operatorname{dom}\alpha$. Then we obtain that $(m,n) < (m,n)\alpha\varpi$, which contradicts Lemma 1. The obtained contradiction implies the statement of our lemma.

For arbitrary positive integer l we define a partial map $\alpha^l_{\mathsf{V}}\colon \mathbb{N}^2 \rightharpoonup \mathbb{N}^2$ in the following way:

$$\operatorname{dom}(\alpha^l_{\mathsf{V}}) = \mathbb{N}^2 \setminus \{(1,1),\dots,(l,1)\}, \qquad \operatorname{ran}(\alpha^l_{\mathsf{V}}) = \mathbb{N}^2 \qquad \text{and}$$

$$(i,j)\alpha^l_{\mathsf{V}} = \begin{cases} (i,j), & \text{if } i > l; \\ (i,j-1), & \text{if } i \leqslant l. \end{cases}$$

It it obvious that $\alpha^l_{\mathsf{V}} \in \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ for any positive integer l.

Lemma 3. *For any element α of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ the following assertions hold:*

(1) *either* $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{H}^1$ *or* $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{V}^1$*;*
(2) *either* $(\mathsf{V}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{V}^1$ *or* $(\mathsf{V}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{H}^1$*.*

Proof. We shall show that assertion (1) holds. The proof of (2) is similar.

First we observe that $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{H}^1$ if and only if $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha\varpi \subseteq \mathsf{V}^1$.

Suppose the contrary: there exists an element α of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ such that neither $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{H}^1$ nor $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{V}^1$. Then the definition of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$, Lemma 1 and the above observation imply that without loss of generality we may assume that $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \nsubseteq \mathsf{H}^1 \cup \mathsf{V}^1$ and there exists $(k,1) \in \operatorname{dom}\alpha$ such that $(k,1)\alpha = (i,j)$, $j \neq 1$ and $2 \leqslant i < k$. Also, by the definition of $\alpha^l_{\mathsf{V}} \in \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ we get that without loss of generality we may assume that $j = 2$, i.e., $(k,1)\alpha = (i,2)$. Then there exist disjoint infinite subsets A and B of the set $\mathsf{V}^1_{\operatorname{dom}\alpha} \cup \ldots \cup \mathsf{V}^{k-1}_{\operatorname{dom}\alpha}$ such that

$$A \cup B = \mathsf{V}^1_{\operatorname{dom}\alpha} \cup \ldots \cup \mathsf{V}^{k-1}_{\operatorname{dom}\alpha}, \qquad \mathsf{H}^1_{\operatorname{ran}\alpha} \subseteq (A)\alpha \qquad \text{and} \qquad \mathsf{V}^1_{\operatorname{ran}\alpha} \cup \ldots \cup \mathsf{V}^{k-1}_{\operatorname{ran}\alpha} \subseteq (B)\alpha.$$

If $A \cap \mathsf{V}^1_{\operatorname{dom}\alpha} \neq \varnothing$ then the definition of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ and Lemma 1 imply that there exists $(a,b) \in B$ such that $(a,b)\alpha \in \mathsf{V}^1_{\operatorname{ran}\alpha}$ and $(c,d) \leqslant (a,b)$ for some $(c,d) \in A$, which contradicts the definition of the partial order $\leqslant$ of the poset $\mathbb{N}^2_\leqslant$.

Assume that $A \subseteq \mathsf{V}^2_{\operatorname{dom}\alpha} \cup \ldots \cup \mathsf{V}^{k-1}_{\operatorname{dom}\alpha}$. Then there exist infinite subsets $A_1 \subseteq A$ and $B_1 \subseteq B$ such that $(A_1)\alpha = \mathsf{H}^1_{\operatorname{ran}\alpha} \setminus \{(1,1)\}$ and $(B_1)\alpha = \mathsf{V}^1_{\operatorname{ran}\alpha} \setminus \{(1,1)\}$. Hence the definition of the poset $\mathbb{N}^2_\leqslant$ implies that at least one of the following conditions holds: ${\uparrow}A_1 \cap {\downarrow}B_1 \neq \varnothing$ or ${\downarrow}A_1 \cap {\uparrow}B_1 \neq \varnothing$. If ${\uparrow}A_1 \cap {\downarrow}B_1 \neq \varnothing$ then $({\downarrow}B_1)\alpha \subseteq {\downarrow}\mathsf{V}^1_{\operatorname{ran}\alpha} = \mathsf{V}^1$ but $\mathsf{V}^1 \cap {\uparrow}\left(\mathsf{H}^1_{\operatorname{ran}\alpha} \setminus \{(1,1)\}\right) \subseteq \mathsf{V}^1 \cap {\uparrow}\left(\mathsf{H}^1 \setminus \{(1,1)\}\right) = \varnothing$, a contradiction. Similarly, if ${\downarrow}A_1 \cap {\uparrow}B_1 \neq \varnothing$ then $({\downarrow}A_1)\alpha \subseteq {\downarrow}\mathsf{H}^1_{\operatorname{ran}\alpha} = \mathsf{H}^1$ and we get a contradiction with

$$\mathsf{H}^1 \cap {\uparrow}\left(\mathsf{V}^1_{\operatorname{ran}\alpha} \setminus \{(1,1)\}\right) \subseteq \mathsf{H}^1 \cap {\uparrow}\left(\mathsf{V}^1 \setminus \{(1,1)\}\right) = \varnothing.$$

The obtained contradictions imply the statement of the lemma.

Proposition 1. *Let α be an arbitrary element of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$. Then the following assertions hold:*

(1) $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{H}^1$ *if and only if* $(\mathsf{V}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{V}^1$*, and moreover in this case the sets* $\mathsf{H}^1 \setminus (\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha$ *and* $\mathsf{V}^1 \setminus (\mathsf{V}^1_{\operatorname{dom}\alpha})\alpha$ *are finite;*
(2) $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{V}^1$ *if and only if* $(\mathsf{V}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{H}^1$*, and moreover in this case* $\mathsf{V}^1 \setminus (\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha$ *and* $\mathsf{H}^1 \setminus (\mathsf{V}^1_{\operatorname{dom}\alpha})\alpha$ *are finite.*

Proof. The first statements of assertions (1) and (2) follow from Lemma 3 and their second parts follow from Lemma 1.

Theorem 1. *Let α be an arbitrary element of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ and n be an arbitrary positive integer. Then the following assertions hold:*

(1) *if* $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{H}^1$ *then* $(\mathsf{H}^n_{\operatorname{dom}\alpha})\alpha \subseteq^* \mathsf{H}^n$ *and* $(\mathsf{V}^n_{\operatorname{dom}\alpha})\alpha \subseteq^* \mathsf{V}^n$*, and moreover* $(\mathsf{H}^1_{\operatorname{dom}\alpha} \cup \ldots \cup \mathsf{H}^n_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{H}^1 \cup \ldots \cup \mathsf{H}^n$ *and* $(\mathsf{V}^1_{\operatorname{dom}\alpha} \cup \ldots \cup \mathsf{V}^n_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{V}^1 \cup \ldots \cup \mathsf{V}^n$;

(2) *if* $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{V}^1$ *then* $(\mathsf{H}^n_{\operatorname{dom}\alpha})\alpha \subseteq^* \mathsf{V}^n$ *and* $(\mathsf{V}^n_{\operatorname{dom}\alpha})\alpha \subseteq^* \mathsf{H}^n$*, and moreover* $(\mathsf{H}^1_{\operatorname{dom}\alpha} \cup \ldots \cup \mathsf{H}^n_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{V}^1 \cup \ldots \cup \mathsf{V}^n$ *and* $(\mathsf{V}^1_{\operatorname{dom}\alpha} \cup \ldots \cup \mathsf{V}^n_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{H}^1 \cup \ldots \cup \mathsf{H}^n$.

Proof. (1) We shall prove this assertion by induction.

In the case when $n = 1$ our statement follows from Lemma 3 and Proposition 1. Next we shall show that the step of induction holds.

We assume that our assertion holds for arbitrary $\alpha \in \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ and for all positive integers $n \leq k$ and we shall prove that then the assertion is true in the case when $n = k + 1$.

For an arbitrary element α of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ we define a partial map $\alpha_{[k+1]} \colon \mathbb{N}^2 \rightharpoonup \mathbb{N}^2$ in the following way:

$(i,j)\alpha_{[k+1]}$ is defined if and only if $(i,j) \in \operatorname{dom}\alpha \cap {\uparrow}(k+1,k+1)$ and $(i,j)\alpha \in \operatorname{ran}\alpha \cap {\uparrow}(k+1,k+1)$, and moreover in this case we put $(i,j)\alpha_{[k+1]} = (i,j)\alpha$,

i.e., the partial map $\alpha_{[k+1]} \colon \mathbb{N}^2 \rightharpoonup \mathbb{N}^2$ is the restriction of the partial map $\alpha \colon \mathbb{N}^2 \rightharpoonup \mathbb{N}^2$ onto the set ${\uparrow}(k+1,k+1)$. Since the set ${\uparrow}(k+1,k+1)$ with the partial induced from $\mathbb{N}^2_\leqslant$ is order isomorphic to $\mathbb{N}^2_\leqslant$, the assumption of induction and Lemma 3 imply that either $(\mathsf{H}^{k+1} \cap \operatorname{dom}(\alpha_{[k+1]}))\alpha_{[k+1]} \subseteq \mathsf{H}^{k+1}$ or $(\mathsf{H}^{k+1} \cap \operatorname{dom}(\alpha_{[k+1]}))\alpha_{[k+1]} \subseteq \mathsf{V}^{k+1}$. Then the inclusion

$$\downarrow(\mathsf{H}^1_{\operatorname{dom}\alpha} \cup \ldots \cup \mathsf{H}^k_{\operatorname{dom}\alpha}) \subseteq \downarrow(\mathsf{H}^1_{\operatorname{dom}\alpha} \cup \ldots \cup \mathsf{H}^k_{\operatorname{dom}\alpha} \cup \mathsf{H}^{k+1}_{\operatorname{dom}\alpha})$$

implies that

$$(\mathsf{H}^{k+1} \cap \operatorname{dom}(\alpha_{[k+1]}))\alpha = (\mathsf{H}^{k+1} \cap \operatorname{dom}(\alpha_{[k+1]}))\alpha_{[k+1]} \subseteq \mathsf{H}^{k+1}.$$

Hence we have that $(\mathsf{H}^{k+1}_{\operatorname{dom}\alpha})\alpha \subseteq^* \mathsf{H}^{k+1}$, because the set $\operatorname{dom}\alpha \setminus \operatorname{dom}(\alpha_{[k+1]}) \cap \mathsf{H}^{k+1}$ is finite. Also, since $(i,j) \leqslant (p,q)$ for all $(i,j) \in \operatorname{dom}\alpha \setminus \operatorname{dom}(\alpha_{[k+1]}) \cap \mathsf{H}^{k+1}$ and $(p,q) \in \operatorname{dom}(\alpha_{[k+1]}) \cap \mathsf{H}^{k+1}$, the definition of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$, the assumption of induction and the inclusion $(\mathsf{H}^{k+1} \cap \operatorname{dom}(\alpha_{[k+1]}))\alpha \subseteq \mathsf{H}^{k+1}$ imply the requested inclusion

$$(\mathsf{H}^1_{\operatorname{dom}\alpha} \cup \ldots \cup \mathsf{H}^k_{\operatorname{dom}\alpha} \cup \mathsf{H}^{k+1}_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{H}^1 \cup \ldots \cup \mathsf{H}^k \cup \mathsf{H}^{k+1}.$$

Again using indiction and Proposition 1 we get that the condition $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{H}^1$ implies that $(\mathsf{H}^n_{\operatorname{dom}\alpha})\alpha \subseteq^* \mathsf{H}^n$ and $(\mathsf{V}^1_{\operatorname{dom}\alpha} \cup \ldots \cup \mathsf{V}^n_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{V}^1 \cup \ldots \cup \mathsf{V}^n$ for every positive integer n.

(2) If $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{V}^1$ then $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha\varpi \subseteq \mathsf{H}^1$. Then assertion (1) and the equality $\alpha\varpi\varpi = \alpha$ imply assertion (2).

The following theorem describes the structure of elements of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ as monotone partial permutations of the poset $\mathbb{N}^2_\leqslant$.

Theorem 2. *Let α be an arbitrary element of the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$. Then the following assertions hold:*

(1) *if* $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{H}^1$ *then*
  (i_1) $(i,j)\alpha \leqslant (i,j)$ *for each* $(i,j) \in \operatorname{dom}\alpha$*; and*
  (ii_1) *there exists a smallest positive integer* n_α *such that* $(i,j)\alpha = (i,j)$ *for each* $(i,j) \in \operatorname{dom}\alpha \cap {\uparrow}(n_\alpha, n_\alpha)$*;*

(2) *if* $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{V}^1$ *then*
  (i_2) $(i,j)\alpha \leqslant (j,i)$ *for each* $(i,j) \in \operatorname{dom}\alpha$*; and*
  (ii_2) *there exists a smallest positive integer* n_α *such that* $(i,j)\alpha = (j,i)$ *for each* $(i,j) \in \operatorname{dom}\alpha \cap {\uparrow}(n_\alpha, n_\alpha)$.

Proof. (1) Fix an arbitrary element α of the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ such that $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{H}^1$. Suppose to the contrary that there exists $(i,j) \in \operatorname{dom}\alpha$ such that $(i,j)\alpha = (k,l) \nleqslant (i,j)$. Then Lemma 1, Theorem 1(1) and the definition of the partial order of the poset $\mathbb{N}^2_{\leqslant}$ imply that $k > i$ and $l < j$. Now, by the definition of the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ we get that there exists a positive integer $m \leqslant i$ such that

$$(\mathsf{V}^1_{\operatorname{dom}\alpha} \cup \ldots \cup \mathsf{V}^m_{\operatorname{dom}\alpha})\alpha \nsubseteq \mathsf{V}^1 \cup \ldots \cup \mathsf{V}^m,$$

which contradicts Theorem 1(1). The obtained contradiction implies the requested inequality $(i,j)\alpha \leqslant (i,j)$ and this completes the proof of (i).

Next we shall prove (ii). Fix an arbitrary element α of the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ such that $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{H}^1$. Suppose to the contrary that for any positive integer n there exists $(i,j) \in \operatorname{dom}\alpha \cap {\uparrow}(n,n)$ such that $(i,j)\alpha \neq (i,j)$. We put $\mathsf{N}_{\operatorname{dom}\alpha} = \left|\mathbb{N}^2 \setminus \operatorname{dom}\alpha\right| + 1$ and

$$\mathsf{M}_{\operatorname{dom}\alpha} = \max\left\{\{i\colon (i,j) \notin \operatorname{dom}\alpha\}, \{j\colon (i,j) \notin \operatorname{dom}\alpha\}\right\} + 1.$$

The definition of the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ implies that the positive integers $\mathsf{N}_{\operatorname{dom}\alpha}$ and $\mathsf{M}_{\operatorname{dom}\alpha}$ are well defined. Put $n_0 = \max\{\mathsf{N}_{\operatorname{dom}\alpha}, \mathsf{M}_{\operatorname{dom}\alpha}\}$. Then our assumption implies that there exists $(i,j) \in \operatorname{dom}\alpha \cap {\uparrow}(n_0,n_0)$ such that $(i,j)\alpha = (i_\alpha, j_\alpha) \neq (i,j)$. By (i), we have that $(i_\alpha, j_\alpha) < (i,j)$. We consider the case when $i_\alpha < i$. In the case when $j_\alpha < j$ the proof is similar. Assume that $i \leqslant j$. By Theorem 1 the partial bijection α maps the set $S_i = \{(n,m)\colon n,m \leqslant i-1\}$ into itself. Also, by the definition of the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ the partial bijection α maps the set $\{(i,1),\ldots,(i,i)\}$ into S_i as well. Then our construction implies that

$$|S_i \setminus \operatorname{dom}\alpha| = \left|\mathbb{N}^2 \setminus \operatorname{dom}\alpha\right| = \mathsf{N}_{\operatorname{dom}\alpha} - 1 \qquad \text{and} \qquad |\{(i,1),\ldots,(i,i)\}| \geqslant \mathsf{N}_{\operatorname{dom}\alpha},$$

a contradiction. In the case when $j \leqslant i$ we get a contradiction in a similar way. This completes the proof of existence of such a positive integer n_α for any $\alpha \in \mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$. The existence of such minimal positive integer n_α follows from the fact that the set of all positive integers with the usual order $\leqslant$ is well-ordered.

(2) If $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{V}^1$ then $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha\varpi \subseteq \mathsf{H}^1$, and hence (1) and the equality $\alpha\varpi\varpi = \alpha$ imply our assertion.

Theorem 2 implies the following corollary:

Corollary 1. $\left|\mathbb{N}^2 \setminus \operatorname{ran}\alpha\right| \leqslant \left|\mathbb{N}^2 \setminus \operatorname{dom}\alpha\right|$ *for an arbitrary* $\alpha \in \mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$.

For an arbitrary non-empty subset A of $\mathbb{N}\times\mathbb{N}$ and any element $(i,j)\in\mathbb{N}\times\mathbb{N}$ we denote $\overline{A}=\{(i,j)\in\mathbb{N}\times\mathbb{N}\colon (j,i)\in A\}$ and $\overline{(i,j)}=(j,i)$.

Proposition 2. *Let α be an arbitrary element of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$. Then the following assertions hold:*

(i) $\operatorname{dom}(\varpi\alpha)=\operatorname{dom}(\varpi\alpha\varpi)=\overline{\operatorname{dom}\alpha}$ *and* $\operatorname{dom}(\alpha\varpi)=\operatorname{dom}\alpha$*;*
(ii) $\operatorname{ran}(\varpi\alpha)=\operatorname{ran}\alpha$ *and* $\operatorname{ran}(\varpi\alpha\varpi)=\operatorname{ran}(\alpha\varpi)=\overline{\operatorname{ran}\alpha}$*;*
(iii) *α is an idempotent if and only if so is $\varpi\alpha\varpi$.*

Proof. Items (i) and (ii) follow from the definition of the composition of partial maps.

(iii) Suppose that α is an idempotent of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$. By items (i) and (ii) we have that $\operatorname{dom}(\varpi\alpha\varpi)=\overline{\operatorname{dom}\alpha}=\overline{\operatorname{ran}\alpha}=\operatorname{ran}(\varpi\alpha\varpi)$. Then $(j,i)\varpi\alpha\varpi=(i,j)\alpha\varpi=(i,j)\varpi=(j,i)$ for an arbitrary $(i,j)\in\operatorname{dom}\alpha$, and hence $\varpi\alpha\varpi\in E(\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant))$. The converse statement follows from the equality $\varpi\varpi=\mathbb{I}$.

The following statement follows from the definition of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ and Lemma 3.

Proposition 3. *Let α and β be arbitrary elements of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$. Then $(\mathsf{H}^1_{\operatorname{dom}(\alpha\beta)})\alpha\beta\subseteq\mathsf{H}^1$ if and only if $(\mathsf{H}^1_{\operatorname{dom}(\beta\alpha)})\beta\alpha\subseteq\mathsf{H}^1$.*

3. Algebraic properties of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$

Theorems 1 and 2 imply the following

Proposition 4. *The group of units $H(\mathbb{I})$ of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ is isomorphic to $\mathbb{Z}_2$.*

Proposition 5. *Let α be an element of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$. Then $\alpha\in H(\mathbb{I})$ if and only if* $\operatorname{dom}\alpha=\mathbb{N}^2$.

Proof. The implication $(\Rightarrow)$ is trivial. The implication $(\Leftarrow)$ follows from Theorems 1, 2 and Corollary 1.

Proposition 6. *An element α of $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ is an idempotent if and only if α is an identity partial self-map of $\mathbb{N}^2_\leqslant$ with the cofinite domain.*

Proof. The implication $(\Leftarrow)$ is trivial.

$(\Rightarrow)$ Let an element α be an idempotent of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$. Then for every $x\in\operatorname{dom}\alpha$ we have that $(x)\alpha\alpha=(x)\alpha$ and hence we get that $\operatorname{dom}\alpha^2=\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha^2=\operatorname{ran}\alpha$. Also since α is a partial bijective self-map of $\mathbb{N}^2_\leqslant$ we conclude that the previous equalities imply that $\operatorname{dom}\alpha=\operatorname{ran}\alpha$. Fix an arbitrary $x\in\operatorname{dom}\alpha$ and suppose that $(x)\alpha=y$. Then $(x)\alpha=(x)\alpha\alpha=(y)\alpha=y$. Since α is a partial bijective self-map of $\mathbb{N}^2_\leqslant$ we have that the equality $(y)\alpha=y$ implies that the full preimage of y under the partial map α is equal to y. Similarly the equality $(x)\alpha=y$ implies that the full preimage of y under the partial map α is equal to x. Thus we get that $x=y$ and our implication holds.

Remark 2. The proof of Proposition 6 implies that the statement of the proposition holds for any semigroup of partial bijections, but in the general case of a semigroup of transformations this statement is not true.

The following theorem describes the subset of idempotents of the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$.

Theorem 3. *For an element α of the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ the following conditions are equivalent:*

(*i*) *α is an idempotent of $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$;*

(*ii*) $\operatorname{dom}\alpha = \operatorname{ran}\alpha$ *and there exists a positive integer $n > 1$ such that $(n,1) \in \operatorname{dom}\alpha$ and $(n,1)\alpha \in \mathsf{H}^1$;*

(*iii*) $\operatorname{dom}\alpha = \operatorname{ran}\alpha$ *and there exists a positive integer $m > 1$ such that $(1,m) \in \operatorname{dom}\alpha$ and $(1,m)\alpha \in \mathsf{V}^1$.*

Proof. Implications $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$ follow from Proposition 6.

We shall prove implication $(ii) \Rightarrow (i)$ by induction in two steps. The proof of implication $(iii) \Rightarrow (i)$ is similar.

First we remark that if $(1,1) \in \operatorname{dom}\alpha$ then since $(1,1) \leqslant (i,j)$ for any $(i,j) \in \operatorname{dom}\alpha$, the definition of the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ implies that $(1,1)\alpha = (1,1)$.

Now, condition (ii) and Lemma 3 imply that $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{H}^1$. Since the set $\mathsf{H}^1_{\operatorname{dom}\alpha}$ with the induced order from the poset $\mathbb{N}^2_{\leqslant}$ is order isomorphic to the set of all positive integers with the usual linear order, without loss of generality we may assume that $\mathsf{H}^1_{\operatorname{dom}\alpha} = \left\{x^1_i \colon i = 1,2,3,\ldots\right\}$ and $x^1_i \leqslant x^1_j$ in $\mathsf{H}^1_{\operatorname{dom}\alpha}$ if and only if $i \leqslant j$. Since $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{H}^1$, Theorem 2(1) implies that $(x^1_1,1)\alpha \leqslant (x^1_1,1)$, and by the equality $\mathsf{H}^1_{\operatorname{dom}\alpha} = \mathsf{H}^1_{\operatorname{ran}\alpha}$ we get that $(x^1_1,1)\alpha = (x^1_1,1)$. Suppose that we have shown that $(x^1_l,1)\alpha = (x^1_l,1)$ for every positive integer $l < t_0$, where t_0 is some positive integer $\geqslant 2$. Then the equality $\mathsf{H}^1_{\operatorname{dom}\alpha} = \mathsf{H}^1_{\operatorname{ran}\alpha}$ and Theorem 2(1) imply that $(x^1_{t_0},1)\alpha = (x^1_{t_0},1)$, because $(x^1_{t_0},1)\alpha \leqslant (x^1_{t_0},1)$ and $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{H}^1$. Therefore, we have proved that $(x^1_k,1)\alpha = (x^1_k,1)$ for every $(x_k,1) \in \operatorname{dom}\alpha$.

Now, we shall show that the equality $(p,q)\alpha = (p,q)$ for all positive integers $q < k_0$ and all positive integers p such that $(p,q) \in \operatorname{dom}\alpha$, where k_0 is some positive integer $\geqslant 2$, implies that $(p,k_0)\alpha = (p,k_0)$ for all $(p,k_0) \in \operatorname{dom}\alpha$. Since the set $\mathsf{H}^{k_0}_{\operatorname{dom}\alpha}$ with the induced order from the poset $\mathbb{N}^2_{\leqslant}$ is order isomorphic to the set of all positive integers with the usual linear order, without loss of generality we may assume that $\mathsf{H}^{k_0}_{\operatorname{dom}\alpha} = \left\{x^{k_0}_i \colon i = 1,2,3,\ldots\right\}$ and $x^{k_0}_i \leqslant x^{k_0}_j$ in $\mathsf{H}^{k_0}_{\operatorname{dom}\alpha}$ if and only if $i \leqslant j$. Then the assumption of induction and Theorem 1(1) imply that $(\mathsf{H}^{k_0}_{\operatorname{dom}\alpha})\alpha \subseteq^* \mathsf{H}^{k_0}$. Theorem 2(1) implies that $(x^{k_0}_1,k_0)\alpha \leqslant (x^{k_0}_1,k_0)$, and by the equality $\mathsf{H}^{k_0}_{\operatorname{dom}\alpha} = \mathsf{H}^{k_0}_{\operatorname{ran}\alpha}$ we get that $(x^{k_0}_1,k_0)\alpha = (x^{k_0}_1,k_0)$. Suppose that we showed that $(x^{k_0}_l,k_0)\alpha = (x^{k_0}_l,k_0)$ for every positive integer $l < s_0$, where s_0 is a some positive integer $\geqslant 2$. Then the equality $\mathsf{H}^{k_0}_{\operatorname{dom}\alpha} = \mathsf{H}^{k_0}_{\operatorname{ran}\alpha}$ and Theorem 2(1) imply that $(x^{k_0}_{s_0},k_0)\alpha = (x^{k_0}_{s_0},k_0)$, because $(x^{k_0}_{s_0},k_0)\alpha \leqslant (x^{k_0}_{s_0},k_0)$ and $(\mathsf{H}^{k_0}_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{H}^{k_0}$. Therefore, we have proved that $(x^{k_0}_k,k_0)\alpha = (x^{k_0}_k,k_0)$ for every $(x^{k_0}_k,k_0) \in \operatorname{dom}\alpha$.

The proof of implication $(ii) \Rightarrow (i)$ is complete.

Proposition 6 implies the following proposition.

Proposition 7. *The subset of idempotents $E(\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant))$ of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ is a commutative submonoid of $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ and moreover $E(\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant))$ is isomorphic to the free semilattice with unit $\big(\mathscr{P}^*(\mathbb{N}^2),\cup\big)$ over the set $\mathbb{N}^2$ under the mapping $(\varepsilon)\mathfrak{h}=\mathbb{N}^2\setminus\operatorname{dom}\varepsilon$.*

Later we shall need the following technical lemma.

Lemma 4. *Let α be an element of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$. Then the following assertions hold:*

(i) $\alpha=\gamma\alpha$ for some $\gamma\in\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ if and only if the restriction $\gamma|_{\operatorname{dom}\alpha}\colon \operatorname{dom}\alpha\to\mathbb{N}^2$ is an identity partial map;

(ii) $\alpha=\alpha\gamma$ for some $\gamma\in\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ if and only if the restriction $\gamma|_{\operatorname{ran}\alpha}\colon \operatorname{ran}\alpha\to\mathbb{N}^2$ is an identity partial map

Proof. (i) The implication $(\Leftarrow)$ is trivial.

$(\Rightarrow)$ Suppose that $\alpha=\gamma\alpha$ for some $\gamma\in\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$. Then we have that $\operatorname{dom}\alpha\subseteq\operatorname{dom}\gamma$ and $\operatorname{dom}\alpha\subseteq\operatorname{ran}\gamma$. Since $\gamma\colon\mathbb{N}^2\rightharpoonup\mathbb{N}^2$ is a partial bijection, the above arguments imply that $(i,j)\gamma=(i,j)$ for each $(i,j)\in\operatorname{dom}\alpha$. Indeed, if $(i,j)\gamma=(m,n)\neq(i,j)$ for some $(i,j)\in\operatorname{dom}\alpha$ then since $\alpha\colon\mathbb{N}^2\rightharpoonup\mathbb{N}^2$ is a partial bijection we have that either

$$(i,j)\alpha=(i,j)\gamma\alpha=(m,n)\alpha\neq(i,j)\alpha, \qquad \text{if} \quad (m,n)\in\operatorname{dom}\alpha,$$

or $(m,n)\alpha$ is undefined. This completes the proof of the implication.

The proof of (ii) is similar to that of (i).

The following theorem describes the Green relations $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$ and $\mathscr{D}$ on the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$.

Theorem 4. *Let α and β be elements of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$. Then the following assertions hold:*

(i) $\alpha\mathscr{L}\beta$ if and only if either $\alpha=\beta$ or $\alpha=\varpi\beta$;

(ii) $\alpha\mathscr{R}\beta$ if and only if either $\alpha=\beta$ or $\alpha=\beta\varpi$;

(iii) $\alpha\mathscr{H}\beta$ if and only if either $\alpha=\beta$ or $\alpha=\varpi\beta=\beta\varpi$;

(iv) $\alpha\mathscr{D}\beta$ if and only if $\alpha=\mu\beta\nu$ for some $\mu,\nu\in H(\mathbb{I})$.

Proof. (i) The implication $(\Leftarrow)$ is trivial.

$(\Rightarrow)$ Suppose that $\alpha\mathscr{L}\beta$ in the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$. Then there exist $\gamma,\delta\in\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ such that $\alpha=\gamma\beta$ and $\beta=\delta\alpha$. The last equalities imply that $\operatorname{ran}\alpha=\operatorname{ran}\beta$.

By Lemma 3 only one of the following cases holds:

(i_1) $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha\subseteq\mathsf{H}^1$ and $(\mathsf{H}^1_{\operatorname{dom}\beta})\beta\subseteq\mathsf{H}^1$;

(i_2) $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha\subseteq\mathsf{H}^1$ and $(\mathsf{H}^1_{\operatorname{dom}\beta})\beta\subseteq\mathsf{V}^1$;

(i_3) $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha\subseteq\mathsf{V}^1$ and $(\mathsf{H}^1_{\operatorname{dom}\beta})\beta\subseteq\mathsf{H}^1$;

(i_4) $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha\subseteq\mathsf{V}^1$ and $(\mathsf{H}^1_{\operatorname{dom}\beta})\beta\subseteq\mathsf{V}^1$.

Suppose that case (i_1) holds. Then the equalities $\alpha=\gamma\beta$ and $\beta=\delta\alpha$ imply that

$$(\mathsf{H}^1_{\operatorname{dom}\gamma})\gamma\subseteq\mathsf{H}^1 \qquad \text{and} \qquad (\mathsf{H}^1_{\operatorname{dom}\delta})\delta\subseteq\mathsf{H}^1, \tag{1}$$

and moreover we have that $\alpha=\gamma\delta\alpha$ and $\beta=\delta\gamma\beta$. Hence by Lemma 4 we have that the restrictions $(\gamma\delta)|_{\operatorname{dom}\alpha}\colon\operatorname{dom}\alpha\to\mathbb{N}^2$ and $(\delta\gamma)|_{\operatorname{dom}\beta}\colon\operatorname{dom}\beta\to\mathbb{N}^2$ are identity partial maps. Then by condition (1) we obtain that the restrictions $\gamma|_{\operatorname{dom}\alpha}\colon\operatorname{dom}\alpha\to\mathbb{N}^2$

and $\delta|_{\operatorname{dom}\beta}\colon \operatorname{dom}\beta \to \mathbb{N}^2$ are also identity partial maps. Indeed, other wise there exists $(i,j) \in \operatorname{dom}\alpha$ such that either $(i,j)\gamma \nleqslant (i,j)$ or $(i,j)\delta \nleqslant (i,j)$, which contradicts Theorem 2(1). Thus, the above arguments imply that in case (i_1) we have that $\alpha = \beta$.

Suppose that case (i_2) holds. Then we have that $\alpha = \gamma\beta = \gamma\mathbb{I}\beta = \gamma(\varpi\varpi)\beta = (\gamma\varpi)(\varpi\beta)$ and $\varpi\beta = (\varpi\delta)\alpha$. Hence we get that $\alpha\mathscr{L}(\varpi\beta)$, $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{H}^1$ and $(\mathsf{H}^1_{\operatorname{dom}(\varpi\beta)})\varpi\beta \subseteq \mathsf{H}^1$. Then we apply case (i_1) for elements α and $\varpi\beta$ and obtain that $\alpha = \varpi\beta$.

In case (i_3) the proof of the equality $\alpha = \varpi\beta$ is similar to case (i_2).

Suppose that case (i_4) holds. Then the equalities $\alpha = \gamma\beta$ and $\beta = \delta\alpha$ imply that $\alpha\varpi = \gamma(\beta\varpi)$ and $\beta\varpi = \delta(\alpha\varpi)$, which implies that $(\alpha\varpi)\mathscr{L}(\beta\varpi)$. Since for the elements $\alpha\varpi$ and $\beta\varpi$ of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ case (i_1) holds, $\alpha\varpi = \beta\varpi$ and hence $\alpha = \alpha\varpi\varpi = \beta\varpi\varpi = \beta$, which completes the proof of (i).

The proof of assertion (ii) is dual to that of (i).

Assertion (iii) follows from (i) (ii).

(iv) Suppose that $\alpha\mathscr{D}\beta$ in $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$. Then there exists $\gamma \in \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ such that $\alpha\mathscr{L}\gamma$ and $\gamma\mathscr{R}\beta$. By Proposition 4 the group of units $H(\mathbb{I})$ of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ has two distinct elements $\mathbb{I}$ and ϖ. By (i), (ii), there exist $\mu,\nu \in H(\mathbb{I})$ such that $\alpha = \mu\gamma$ and $\gamma = \beta\nu$ and hence $\alpha = \mu\beta\nu$. Converse, suppose that $\alpha = \mu\beta\nu$ for some $\mu,\nu \in H(\mathbb{I})$. Then by (i), (ii), we have that $\alpha\mathscr{L}(\beta\nu)$ and $\beta\mathscr{R}(\beta\nu)$, and hence $\alpha\mathscr{D}\beta$.

Theorem 4 implies Corollary 2 which gives the inner characterization of the Green relations $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$ and $\mathscr{D}$ on the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ as partial permutations of the poset $\mathbb{N}^2_\leqslant$.

Corollary 2. *(i) Every $\mathscr{L}$-class of $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ contains two distinct elements.*

(ii) Every $\mathscr{R}$-class of $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ contains two distinct elements.

(iii) Every $\mathscr{H}$-class of $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ contains at most two distinct elements.

(iv) The $\mathscr{H}$-class of $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ which contains an element α consists of two distinct elements if and only if $\operatorname{dom}\alpha = \overline{\operatorname{dom}\alpha}$, $\operatorname{ran}\alpha = \overline{\operatorname{ran}\alpha}$ and $(\overline{(i,j)})\alpha = \overline{(i,j)\alpha}$ for each $(i,j) \in \operatorname{dom}\alpha$, and the $\mathscr{H}$-class of α is a singleton in the other case.

(v) The $\mathscr{H}$-class of $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ which contains an idempotent ε consists of two distinct elements if and only if $\operatorname{dom}\varepsilon = \overline{\operatorname{dom}\varepsilon}$.

(vi) The $\mathscr{H}$-class of $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ which contains an idempotent ε is a singleton if and only if $\operatorname{dom}\varepsilon \neq \overline{\operatorname{dom}\varepsilon}$.

(vii) Every $\mathscr{D}$-class of $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ contains either two or four distinct elements.

$(viii)$ A $\mathscr{D}$-class of $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ has two distinct elements if and only if it contains only one $\mathscr{H}$-class.

(ix) A $\mathscr{D}$-class of $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ has two distinct elements if and only if it contains a non-singleton $\mathscr{H}$-class.

(x) A $\mathscr{D}$-class of $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ has four distinct elements if and only every its $\mathscr{H}$-class is singleton.

(xi) A $\mathscr{D}$-class of $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ has four distinct elements if and only it contains a singleton $\mathscr{H}$-class.

(xii) The $\mathscr{D}$-class of $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ which contains an idempotent ε consists of two distinct elements if and only if $\operatorname{dom}\varepsilon = \overline{\operatorname{dom}\varepsilon}$.

(*xiii*) *The* $\mathscr{D}$*-class of* $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ *which contains an idempotent* ε *consists of four distinct elements if and only if* $\operatorname{dom}\varepsilon \neq \overline{\operatorname{dom}\varepsilon}$.

Proof. Statements (*i*), (*ii*) and (*iii*) are trivial and they follow from the equality $\varpi\varpi = \mathbb{I}$ and the corresponding statements of Theorem 4.

(*iv*) By (*i*) and (*ii*) we have that the $\mathscr{H}$-class of $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ which contains an element α contains at most two distinct elements.

($\Rightarrow$) Assume that $\alpha\mathscr{H}\beta$ in $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ and $\alpha \neq \beta$. By Theorem 4(*iii*), $\beta = \alpha\varpi = \varpi\alpha$. Then by the definition of ϖ we get that $\operatorname{dom}\beta = \operatorname{dom}\alpha = \overline{\operatorname{dom}\alpha}$ and $\operatorname{ran}\beta = \operatorname{ran}\alpha = \overline{\operatorname{ran}\alpha}$. If $(i,j) \in \operatorname{dom}\alpha$ and $(i,j)\alpha = (m,n)$ then

$$(n,m) = (m,n)\varpi = (i,j)\alpha\varpi = (i,j)\beta = (i,j)\varpi\alpha = (j,i)\alpha.$$

This completes the proof of the implication.

The converse implication is trivial, and the last statement of item (*iv*) follows from the above part of its proof.

(*v*) If $\operatorname{dom}\varepsilon = \overline{\operatorname{dom}\varepsilon}$ then $\varepsilon\varpi = \varpi\varepsilon \neq \varepsilon$. Conversely, suppose that $\varepsilon\varpi = \varpi\varepsilon \neq \varepsilon$. Since $\operatorname{dom}\varpi = \operatorname{ran}\varpi = \mathbb{N}\times\mathbb{N}$ and $\operatorname{dom}\varepsilon = \operatorname{ran}\varepsilon$, the equality $\varepsilon\varpi = \varpi\varepsilon$ implies that $\operatorname{dom}(\varepsilon\varpi) = \operatorname{dom}\varepsilon = \operatorname{ran}\varepsilon = \operatorname{ran}(\varpi\varepsilon)$, and hence the definition of the element $\varpi \in H(\mathbb{I})$ implies that $\operatorname{dom}\varepsilon = \overline{\operatorname{dom}\varepsilon}$.

Statement (*vi*) follows from items (*iii*), (*v*).

(*vii*) Theorem 4(*iv*) and (*i*), (*ii*) imply that every $\mathscr{D}$-class of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ contains at most four and at least two distinct elements. Suppose to the contrary that there exists a $\mathscr{D}$-class D_α in $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ which contains three distinct elements such that $\alpha \in D_\alpha$ for some element α of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$. By Theorem 4(*iv*), $\varpi\alpha, \alpha\varpi, \varpi\alpha\varpi \in D_\alpha$. Since $\varpi\gamma \neq \gamma \neq \gamma\varpi$ for any $\gamma \in \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$, we have that $\varpi\alpha = \alpha\varpi$ or $\alpha = \varpi\alpha\varpi$. If $\varpi\alpha = \alpha\varpi$ then the definition of the element ϖ of $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ implies that $\alpha = \varpi\varpi\alpha = \varpi\alpha\varpi$. Similarly, if $\alpha = \varpi\alpha\varpi$ then $\varpi\alpha = \varpi\varpi\alpha\varpi = \alpha\varpi$. This completes the proof of the statement.

(*viii*) ($\Rightarrow$) Assume that a $\mathscr{D}$-class of $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ has two distinct elements and it contains α. Then the proof of item (*vii*) implies that $\varpi\alpha = \alpha\varpi$ and $\alpha = \varpi\alpha\varpi$. By Theorem 4(*iv*) we have that $D_\alpha = H_\alpha$.

Implication ($\Leftarrow$) is trivial.

(*ix*) Implication ($\Rightarrow$) follows form item (*viii*).

($\Leftarrow$) Assume that there exists a $\mathscr{D}$-class of $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ which contains a non-singleton $\mathscr{H}$-class H_α of $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ for some $\alpha \in \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$. By Theorem 4(*iii*) we have that $H_\alpha = \{\alpha, \alpha\varpi\}$ and $\alpha \neq \alpha\varpi = \varpi\alpha$. Then the last equality implies that $\alpha = \varpi\alpha\varpi$. Hence by Theorem 4(*iv*), $D_\alpha = H_\alpha$, which complete the proof of the implication.

Statement (*x*) follows from (*viii*), (*ix*).

(*xi*) By Theorem 2.3 of [1] any two $\mathscr{H}$-classes of an arbitrary $\mathscr{D}$-class are of the same cardinality. Now, we apply statement (*x*).

Statement (*xii*) follows from (*viii*), (*v*).

Items (*x*) and (*vi*) imply statement (*xiii*).

We need the following three lemmas.

Lemma 5. *Let* α, β *and* γ *be elements of the semigroup* $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ *such that* $\alpha = \beta\alpha\gamma$. *Then the following statements hold:*

(*i*) *if* $(\mathsf{H}^1_{\operatorname{dom}\beta})\beta \subseteq \mathsf{H}^1$ *then the restrictions* $\beta|_{\operatorname{dom}\alpha}\colon \operatorname{dom}\alpha \rightharpoonup \mathbb{N}\times\mathbb{N}$ *and* $\gamma|_{\operatorname{ran}\alpha}\colon \operatorname{ran}\alpha \rightharpoonup \mathbb{N}\times\mathbb{N}$ *are identity partial maps;*

(*ii*) *if* $(\mathsf{H}^1_{\operatorname{dom}\beta})\beta \subseteq \mathsf{V}^1$ *then* $(i,j)\beta=(j,i)$ *for each* $(i,j)\in\operatorname{dom}\alpha$ *and* $(m,n)\gamma=(n,m)$ *for each* $(m,n)\in\operatorname{ran}\alpha$*; and moreover in this case we have that* $\operatorname{dom}\alpha=\overline{\operatorname{dom}\alpha}$, $\operatorname{ran}\alpha=\overline{\operatorname{ran}\alpha}$ *and* $(j,i)\alpha=\overline{(i,j)\alpha}$ *for any* $(i,j)\in\operatorname{dom}\alpha$*, i.e.,* $\alpha=\varpi\alpha\varpi$.

Proof. (*i*) Assume that the inclusion $(\mathsf{H}^1_{\operatorname{dom}\beta})\beta \subseteq \mathsf{H}^1$ holds. Then one of the following cases holds:

(1) $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{H}^1$;

(2) $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{V}^1$.

If case (1) holds then the equality $\alpha=\beta\alpha\gamma$ and Lemma 3 imply that $(\mathsf{H}^1_{\operatorname{dom}\gamma})\gamma \subseteq \mathsf{H}^1$. By Theorem 2(1), $(i,j)\beta\leqslant(i,j)$ for any $(i,j)\in\operatorname{dom}\beta$ and $(m,n)\gamma\leqslant(m,n)$ for any $(m,n)\in\operatorname{dom}\gamma$. Suppose that $(i,j)\beta<(i,j)$ for some $(i,j)\in\operatorname{dom}\alpha$. Then we have that

$$(i,j)\alpha=(i,j)\beta\alpha\gamma<(i,j)\alpha\gamma\leqslant(i,j)\alpha,$$

which contradicts the equality $\alpha=\beta\alpha\gamma$. The obtained contradiction implies that the restriction $\beta|_{\operatorname{dom}\alpha}\colon \operatorname{dom}\alpha\rightharpoonup\mathbb{N}\times\mathbb{N}$ is an identity partial map. This and the equality $\alpha=\beta\alpha\gamma$ imply that the restriction $\gamma|_{\operatorname{ran}\alpha}\colon \operatorname{ran}\alpha\rightharpoonup\mathbb{N}\times\mathbb{N}$ is an identity partial map too.

Suppose that case (2) holds. Then we have that $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha\varpi \subseteq \mathsf{H}^1$. Now, the equality $\alpha=\beta\alpha\gamma$ and the definition of the element ϖ the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ imply that

$$\alpha\varpi=\beta\alpha\gamma\varpi=\beta(\alpha\varpi)(\varpi\gamma\varpi).$$

Then we apply case (1). This completes the proof of (*i*).

(*ii*) Assume that the inclusion $(\mathsf{H}^1_{\operatorname{dom}\beta})\beta \subseteq \mathsf{V}^1$ holds. Then the equality $\alpha=\beta\alpha\gamma$ implies that $\alpha=\beta\beta\alpha\gamma\gamma$ and the inclusion $(\mathsf{H}^1_{\operatorname{dom}\beta})\beta \subseteq \mathsf{V}^1$ implies that $(\mathsf{H}^1_{\operatorname{dom}(\beta\beta)})\beta\beta \subseteq \mathsf{H}^1$. Now, by (*i*), the restrictions $(\beta\beta)|_{\operatorname{dom}\alpha}\colon \operatorname{dom}\alpha\rightharpoonup\mathbb{N}\times\mathbb{N}$ and $(\gamma\gamma)|_{\operatorname{ran}\alpha}\colon \operatorname{ran}\alpha\rightharpoonup\mathbb{N}\times\mathbb{N}$ are identity partial maps. Since $(\mathsf{H}^1_{\operatorname{dom}\beta})\beta \subseteq \mathsf{V}^1$, Theorem 2(2) implies that $(i,j)\beta\leqslant(j,i)$ for any $(i,j)\in\operatorname{dom}\alpha$. Suppose that $(i,j)\beta<(j,i)$ for some $(i,j)\in\operatorname{dom}\alpha$. Again, by Theorem 2(2) we get that $(j,i)\beta\leqslant(i,j)$ and hence we have that $(i,j)=(i,j)\beta\beta<(j,i)\beta\leqslant(i,j)$, a contradiction. The obtained contradiction implies that $(i,j)\beta=(j,i)$ for each $(i,j)\in\operatorname{dom}\alpha$. Next, the inclusion $(\mathsf{H}^1_{\operatorname{dom}\beta})\beta \subseteq \mathsf{V}^1$ and the equality $\alpha=\beta\alpha\gamma$ imply that $(\mathsf{H}^1_{\operatorname{dom}\gamma})\gamma \subseteq \mathsf{V}^1$. Then the similar arguments as in the above part of the proof imply that $(m,n)\gamma=(n,m)$ for each $(m,n)\in\operatorname{ran}\alpha$.

Now, the property that $(i,j)\beta=(j,i)$ for each $(i,j)\in\operatorname{dom}\alpha$ and $(m,n)\gamma=(n,m)$ for each $(m,n)\in\operatorname{ran}\alpha$, and the equality $\alpha=\beta\alpha\gamma$ imply that $\operatorname{dom}\alpha=\overline{\operatorname{dom}\alpha}$ and $\operatorname{ran}\alpha=\overline{\operatorname{ran}\alpha}$. Fix an arbitrary $(i,j)\in\operatorname{dom}\alpha$. Put $(m,n)=(i,j)\alpha$. Then the above part of the proof of this item implies that $(m,n)=(i,j)\alpha=(i,j)\beta\alpha\gamma=(j,i)\alpha\gamma$ and hence $(n,m)=(m,n)\varpi=(j,i)\alpha\gamma\varpi=(j,i)\alpha$.

Lemma 6. *Let* α *and* β *be elements of the semigroup* $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ *and* A *be a cofinite subset of* $\mathbb{N}\times\mathbb{N}$*. If the restriction* $(\alpha\beta)|_A\colon A\rightharpoonup\mathbb{N}\times\mathbb{N}$ *is an identity partial map then one of the following conditions holds:*

(*i*) *the restrictions* $\alpha|_A\colon A\rightharpoonup\mathbb{N}\times\mathbb{N}$ *and* $\beta|_A\colon A\rightharpoonup\mathbb{N}\times\mathbb{N}$ *are identity partial maps;*

(*ii*) $(i,j)\alpha=(j,i)$ *for all* $(i,j)\in A$ *and* $(m,n)\beta=(n,m)$ *for all* $(m,n)\in\overline{A}$.

Proof. By Lemma 3 we have that either $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{H}^1$ or $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{V}^1$. Suppose that the inclusion $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{H}^1$ holds. Then the definition of the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ implies that $(\mathsf{H}^1_{\operatorname{dom}\beta})\beta \subseteq \mathsf{H}^1$. By Theorem 2(1) we have that

$$(i,j)\alpha \leqslant (i,j)$$

for any $(i,j) \in \operatorname{dom}\alpha$ and $(m,n)\beta \leqslant (m,n)$ for any $(m,n) \in \operatorname{dom}\beta$. Suppose that $(i,j)\alpha < (i,j)$ for some $(i,j) \in A$. Then we have that

$$(i,j) = (i,j)\alpha\beta < (i,j)\beta \leqslant (i,j),$$

which contradicts the assumption that the restriction $(\alpha\beta)|_A \colon A \rightharpoonup \mathbb{N}\times\mathbb{N}$ is an identity partial map. Hence the restriction $\alpha|_A \colon A \rightharpoonup \mathbb{N}\times\mathbb{N}$ is an identity partial map. Similar arguments imply that the restriction $\beta|_A \colon A \rightharpoonup \mathbb{N}\times\mathbb{N}$ is also an identity partial map. Thus, in the case when $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{H}^1$, item (i) holds.

Suppose that the inclusion $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{V}^1$ holds. By the definition of the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ we have that

$$(\mathsf{H}^1_{\operatorname{dom}\beta})\beta \subseteq \mathsf{V}^1, \;\; \alpha\beta = (\alpha\varpi)(\varpi\beta), \;\; (\mathsf{H}^1_{\operatorname{dom}(\alpha\varpi)})\alpha\varpi \subseteq \mathsf{H}^1$$

and

$$(\mathsf{H}^1_{\operatorname{dom}(\varpi\beta)})\varpi\beta \subseteq \mathsf{H}^1.$$

Then the previous part of the proof implies that the restrictions $(\alpha\varpi)|_A \colon A \rightharpoonup \mathbb{N}\times\mathbb{N}$ and $(\varpi\beta)|_A \colon A \rightharpoonup \mathbb{N}\times\mathbb{N}$ are identity partial maps. Since $(\alpha\varpi)\varpi = \alpha$ and $\varpi(\varpi\beta) = \beta$, the inclusion $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{V}^1$ implies that (ii) holds.

Lemma 7. *Let α and β be elements of the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ and A be a cofinite subset of $\mathbb{N}\times\mathbb{N}$. If $(i,j)\alpha\beta = (j,i)$ for all $(i,j) \in A$, then one of the following conditions holds:*

- (i) *the restriction $\alpha|_A \colon A \rightharpoonup \mathbb{N}\times\mathbb{N}$ is an identity partial map and $(m,n)\beta = (n,m)$ for all $(m,n) \in A$;*
- (ii) *$(i,j)\alpha = (j,i)$ for all $(i,j) \in A$ and $\beta|_{\overline{A}} \colon \overline{A} \rightharpoonup \mathbb{N}\times\mathbb{N}$ is an identity partial map.*

Proof. The assumption of the lemma implies that the restriction $\alpha(\beta\varpi)|_A \colon \rightharpoonup \mathbb{N}\times\mathbb{N}$ is an identity partial map. Hence by Lemma 6 only one of the following conditions holds:

1. the restrictions $\alpha|_A \colon A \rightharpoonup \mathbb{N}\times\mathbb{N}$ and $(\beta\varpi)|_A \colon A \rightharpoonup \mathbb{N}\times\mathbb{N}$ are identity partial maps;
2. $(i,j)\alpha = (j,i)$ for all $(i,j) \in A$ and $(m,n)\beta\varpi = (n,m)$ for all $(m,n) \in \overline{A}$.

Since $(\beta\varpi)\varpi = \beta$, the above arguments imply the statement of the lemma.

Elementary calculations and the definition of the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ imply the following proposition.

Proposition 8. *Let α and β be elements of the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$. Then the following assertions hold:*

- (i) *if the restriction $\beta|_{\operatorname{ran}\alpha} \colon \operatorname{ran}\alpha \rightharpoonup \mathbb{N}\times\mathbb{N}$ is an identity partial map then $\alpha\beta = \alpha\mathbb{I} = \alpha$;*
- (ii) *if the restriction $\beta|_{\operatorname{dom}\alpha} \colon \operatorname{dom}\alpha \rightharpoonup \mathbb{N}\times\mathbb{N}$ is an identity partial map then $\beta\alpha = \mathbb{I}\alpha = \alpha$;*

(*iii*) *if* $(m,n)\beta=(n,m)$ *for all* $(m,n)\in\operatorname{ran}\alpha$ *then* $\alpha\beta=\alpha\varpi$*;*
(*iv*) *if* $(m,n)\beta=(n,m)$ *for all* $(m,n)\in\overline{\operatorname{dom}\alpha}$ *then* $\beta\alpha=\varpi\alpha$*.*

Theorem 5. $\mathscr{D}=\mathscr{J}$ *in* $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$*.*

Proof. The inclusion $\mathscr{D}\subseteq\mathscr{J}$ is trivial.

Fix any $\alpha,\beta\in\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ such that $\alpha\mathscr{J}\beta$. Then there exist $\gamma_\alpha,\delta_\alpha,\gamma_\beta,\delta_\beta\in\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ such that $\alpha=\gamma_\alpha\beta\delta_\alpha$ and $\beta=\gamma_\beta\alpha\delta_\beta$ (see [2] or [3, Section II.1]). Hence we have that

$$\alpha=\gamma_\alpha\gamma_\beta\alpha\delta_\beta\delta_\alpha \quad\text{and}\quad \beta=\gamma_\beta\gamma_\alpha\beta\delta_\alpha\delta_\beta.$$

Suppose that

$$(\mathsf{H}^1_{\operatorname{dom}(\gamma_\alpha\gamma_\beta)})\gamma_\alpha\gamma_\beta\subseteq\mathsf{H}^1.$$

By Proposition 3,

$$(\mathsf{H}^1_{\operatorname{dom}(\gamma_\beta\gamma_\alpha)})\gamma_\beta\gamma_\alpha\subseteq\mathsf{H}^1.$$

Lemma 5(i) implies that the restrictions

$$(\gamma_\alpha\gamma_\beta)|_{\operatorname{dom}\alpha}\colon \operatorname{dom}\alpha\rightharpoonup\mathbb{N}\times\mathbb{N},\quad (\delta_\beta\delta_\alpha)|_{\operatorname{ran}\alpha}\colon \operatorname{ran}\alpha\rightharpoonup\mathbb{N}\times\mathbb{N},$$
$$(\gamma_\beta\gamma_\alpha)|_{\operatorname{dom}\beta}\colon \operatorname{dom}\beta\rightharpoonup\mathbb{N}\times\mathbb{N} \quad\text{and}\quad (\delta_\alpha\delta_\beta)|_{\operatorname{ran}\beta}\colon \operatorname{ran}\beta\rightharpoonup\mathbb{N}\times\mathbb{N}$$

are identity partial maps. Then by Lemma 6 and Proposition 8 there exist $\omega_1,\omega_2\in H(\mathbb{I})$ such that $\gamma_\beta\alpha=\omega_1\alpha$, $\alpha\delta_\beta=\alpha\omega_2$, $\gamma_\alpha\beta=\omega_1\beta$ and $\beta\delta_\alpha=\beta\omega_2$. This implies that

$$\alpha=\gamma_\alpha\beta\delta_\alpha=\omega_1\beta\delta_\alpha=\omega_1\beta\omega_2 \qquad\text{and}\qquad \beta=\gamma_\beta\alpha\delta_\beta=\omega_1\alpha\delta_\beta=\omega_1\alpha\omega_2,$$

and hence by Theorem 4 we get that $\alpha\mathscr{D}\beta$.

Suppose that

$$(\mathsf{H}^1_{\operatorname{dom}(\gamma_\alpha\gamma_\beta)})\gamma_\alpha\gamma_\beta\subseteq\mathsf{V}^1.$$

Then by Proposition 3 and Lemma 3 we have that

$$(\mathsf{H}^1_{\operatorname{dom}(\gamma_\beta\gamma_\alpha)})\gamma_\beta\gamma_\alpha\subseteq\mathsf{V}^1.$$

Now, as in the above part of the proof the statement of the theorem follows from Lemma 5(ii), Lemma 7 and Proposition 8.

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ПРО МОНОЇД МОНОТОННИХ ІН'ЄКТИВНИХ ЧАСТКОВИХ ПЕРЕТВОРЕНЬ МНОЖИНИ $\mathbb{N}^2_{\leqslant}$ З КОСКІНЧЕННИМИ ОБЛАСТЯМИ ВИЗНАЧЕНЬ І ЗНАЧЕНЬ

Олег ГУТІК, Інна ПОЗДНЯКОВА

Львівський національний університет імені Івана Франка,
вул. Університетська 1, Львів, 79000,
e-mails: o_gutik@franko.lviv.ua,
ovgutik@yahoo.com, pozdnyakova.inna@gmail.com

Нехай $\mathbb{N}^2_{\leqslant}$ – множина $\mathbb{N}^2$ з частковим порядком, визначеним як добуток звичайного лінійного порядку $\leqslant$ на множині натуральних чисел $\mathbb{N}$. Вивчено напівгрупу $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ монотонних ін'єктивних часткових перетворень частково впорядкованої множини $\mathbb{N}^2_\leqslant$, які мають коскінченні області визначення та значення. Описуємо властивості елементів напівгрупи $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ як монотонних часткових бієкцій частково впорядкованої множини $\mathbb{N}^2_\leqslant$ і доводимо, що група одиниць напівгрупи $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ ізоморфна циклічній групі другого порядку. Також описуємо піднапівгрупу ідемпотентів напівгрупи $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ та відношення Гріна $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$. Зокрема, доведено, що $\mathscr{D} = \mathscr{J}$ в $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$.

Ключові слова: напівгрупа часткових бієкцій, монотонне часткове відображення, ідемпотент, відношення Гріна.