ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2016. Випуск 81. С. 95–99
Visnyk of the Lviv Univ. Series Mech. Math. 2016. Issue 81. P. 95–99

УДК 517.956

ESTIMATES OF SOLUTIONS OF ELLIPTIC-PARABOLIC EQUATIONS

Tahir GADJIEV, Soltan ALIEV, Gunel GASANOVA

Institute of Mathematics and Mechanics,
NAS of Azerbaijan, Baku, Azerbaijan
e-mail: tgadjiev@mail.az

A class of degenerated second order elliptic-parabolic equations of non-divergent structure is considered. For solutions of the boundary value problems of these equations the coercive estimation in an appropriate Sobolev space is established.

Key words: elliptic-parabolic equations, boundary value problems, Sobolev space.

1. Introduction. Let Ω be a bounded domain in R^n with the boundary $\partial\Omega \subset C^2$, let $\Omega \times (0;T)$ be the cylinder, where $T \in (0;\infty)$. Let us consider in Q_T the boundary value problem

$$lu = \sum_{i,j=1}^{n} a_{ij}(x,t)u_{ij} + \psi(x,t)u_{tt} - u_t = f(x,t), \tag{1}$$

$$u|\Gamma(Q_T) = 0, \tag{2}$$

where for $i,j \in \{1,\ldots,n\}$, $\quad u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad u_i = \frac{\partial u}{\partial x_i}$,

$\Gamma(Q_T) = (\partial\Omega\times[0,T])\cup(\Omega\times(x,t) : t=0)$ is parabolic boundary ofQ_T, and

$$\Psi(x,t) = \omega(x)\lambda(t)\varphi(T-t), \tag{3}$$

where $\omega(x)\in A_p$ satisfies the condition of Muckenhoupt (see [8]),

$\lambda(t)\geqslant 0, \lambda(t)\in C^1[0,T], \varphi(z)\geqslant 0, \varphi^{\backprime}(z)\geqslant 0, \varphi(z)\in C^1[0,T], \varphi(0) = \varphi^{\backprime}(0) = 0, \varphi(z)\geqslant \beta z\varphi^{\backprime}(z)$,

here β is a positive constant.

Assume that for the coefficients of the operator ???? the following conditions hold:

If $\|a_{ij}(x,t)\|$ is a real symmetrical matrix with elements measurable in Q_T for every $(x,t)\in Q_T$ and $\xi\in R^n$ then the inequalities

$$\gamma\omega(x)|\xi|^2\leqslant\sum_{i,j=1}^{n}a_{ij}(x,t)\xi_i\xi_j\leqslant\gamma^{-1}\omega(x)|\xi|^2 \tag{4}$$

hold, where γ is a constant from the interval $[0,1]$.

The purpose of this paper is to obtain a coercive estimation for problem (1)–(2) in an appropriate Sobolev space.

The obtained estimation can be used when proving a unique strong (almost everywhere) solvability of the first boundary value problem (1)–(2) at every

$$f(x,t)\in L_2(Q_T).$$

The theory of degenerated elliptic-parabolic equations ascends to the classical paper by Keldysh [1] in which the correct statements of the boundary value problems for the equations of kind (1) with one space variable were found. G.Fickera [2] has established a weak solvability of the first boundary value problem for a wide class of the second order equations with the non-negative characteristic form (see also [3]). As to strong solvability of the first boundary value problem for elliptic-parabolic equations in the non-divergent form with smooth coefficients, we shall note in this connection the papers [4-6]. The similar result for the equations of kind (1) is the case when the coefficients satisfy the Cordes condition is obtained in [7].

The paper is organized as follows. In Section 2 we present some definitions and preliminary results. In Section 3 we give main results.

2. Definitions and preliminary results. For $R>0$ and $x^0\in\Omega$ we denote the ball $\{x:|x-x^0|<R\}$ by $B_R(x^0)$ and the cylinder $B_R(x^0)\times(0,T)$ by $Q_T^R(x^0)$. Let $\overline{B}_R(x^0)\subset\Omega$. We say that $u(x,t)\in A(Q_T^R(x^0))$ if $u(x,t)\in C^\infty(\overline{Q}_T^R(x^0)), u|_{t=0}=0$ and $\operatorname{supp} u\in\overline{Q}_T^\rho(x^0)$ for some $\rho\in(0,R)$.

We say that $u(x,t)\in A_1(Q_T^R(x^0))$ if $u(x,t)\in C^\infty(\overline{Q}_T^R(x^0)), u|_{t=0}=0$. Finally, $u(x,t)\in B(Q_T^R(x^0))$ if $u(x,t)\in A(\overline{Q}_T^R(x^0))$ and $u|_{t=T}=u_t|_{t=T}=0$. In the sequel, the notation $C(\cdot)$ shows that a positive constants C depends only on the contents of brackets.

Let us introduce the Banach spaces of the functions $u(x,t)$ given on Q_T with finite norms

$$\|u\|_{w^1_{2,\omega}(Q_T)}=(\textstyle\int_{Q_T}\omega(x)(u^2+\sum_{i=1}^n u_{x_i}^2)dxdt)^{\frac12},$$
$$\|u\|_{w^2_{2,\omega}(Q_T)}=(\textstyle\int_{Q_T}\omega(x)(u^2+\sum_{i=1}^n u_{x_i}^2+\sum_{i,j=1}^n u_{x_ix_i}^2)dxdt)^{\frac12},$$
$$\|u\|_{w^{2,1}_2(Q_T)}=\|u\|_{w^2_{2,\omega}(Q_T)}+\|u_t\|_{L_2(Q_T)},$$
$$\|u\|_{w^{2,2}_{2,\psi}(Q_T)}=$$
$$(\textstyle\int_{Q_T}(\omega(x)(u^2+\sum_{i=1}^n u_{x_i}^2+\sum_{i,j=1}^n u_{x_ix_i}^2)+u_t^2+\psi^2(x,t)u_{tt}^2+\psi(x,t)\sum_{i=1}^n u_{tt}^2)dxdt)^{\frac12},$$
$$\|u\|_{w^{1,1}_{2,\psi}(Q_T)}=(\textstyle\int_{Q_T}(\omega(x)(u^2+\sum_{i=1}^n u_{x_i}^2)+u_t^2+\psi^2(x,t)u_{tt}^2)dxdt)^{\frac12}.$$

Suppose that $\dot{w}^{1,1}_{2,\psi}(Q_T)$ is a subspace of the space $w^{1,1}_{2,\psi}(Q_T)$ that contains the set of all functions from $C^\infty(Q_T)$ vanishing on the parabolic boundary $\Gamma(Q_T)$.

Consider the model operator

$$Z_0 = \omega(x)\Delta + \psi(x,t)\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t},$$

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x^2}$ is the Laplace operator.

Lemma 1. *If the function $\psi(x,t)$ satisfies (3) and conditions (4) are fulfilled, then there exists $T_1(\psi(x,t),n)$ such that $T \leqslant T_1$ and for any function $u(x,t) \in A(Q_T^R(x^0))$ the estimate*

$$\int\limits_{Q_T^R(x^0)} \left[\omega^2(x)\sum_{i=1}^{n} u_{u_{ij}}^2 + u_t^2 + \psi^2(x,t)u_{it}^2 + \psi(x,t)\sum_{i=1}^{n} u_{tt}^2\right] dxdt \leqslant$$

$$\leqslant (1+DS) \int\limits_{Q_T^R(x^0)} (Z_0 u)^2 dxdt \qquad (5)$$

holds, where $S = S(\psi,n)$ is some constant, $D = D(T) = q(T) + q_1(T)$ and $q(T) = \sup\limits_{t\in[0,T]} \varphi'(t)$, $\quad q_1(T) = \sup\limits_{t\in[0,T]} \varphi(t)$.

3. Coercive estimates for weak solutions and main result.

Lemma 2. *If the conditions of the previous lemma are fulfilled, then for any function $u(x,t) \in A(Q_T^R(x^0))$, where $T \leqslant T_2(\psi,n,\delta)$ the following estimate is true:*

$$I = \int\limits_{Q_T^R(x^0)} \left(\omega^2(x)\sum_{i=1}^{n} u_{u_{ij}}^2 + u_t^2 + \psi^2(x,t)u_{it}^2 + \psi(x,t)\sum_{i=1}^{n} u_{tt}^2\right) dxdt \leqslant$$

$$\leqslant C_2 \int\limits_{Q_T^R(x^0)} (Zu)^2 dxdt,$$

where $C_2 = C_2(\psi,n,\delta)$.

Lemma 3. *If conditions (4), (5) are fulfilled for the coefficients of the operator Z, then at $T \leqslant T_2$ the following estimate is true for any function $u(x,t) \in A(Q_T^R(x^0))$*

$$\|u\|_{w_{2,\psi}^{2,2}(Q_T^R(x^0))} \leqslant C_4(\psi,\delta,n)\|Zu\|_{l_2(Q_T^R(x^0))}.$$

Lemma 4. *If conditions (4), (5) are fulfilled for the coefficients of the operator Z, then for every $T \leqslant T_2$ and $\varepsilon > 0$ the following estimate holds for any function $u(x,t) \in A_1(Q_T^R(x^0))$:*

$$\|u\|_{w_{2,\psi}^{2,2}(Q_T^{R/2}(x^0))} \leqslant C_8\|Zu\|_{l_2(Q_T^R(x^0))} + \varepsilon\|u\|_{w_{2,\psi}^{2,2}(Q_T^R(x^0))} + \frac{C_9(\psi,\delta,n)}{\varepsilon R^2}\|u\|_{l_2(Q_T^R(x^0))}.$$

Corollary 1. *If the coefficients of the operator Z satisfy conditions (4), (5), then for every $T \leqslant T_2$ and $\varepsilon > 0$ the estimate*

$$\|u\|_{w_{2,\psi}^{2,2}(Q_{T(\rho)})} \leqslant C_{13}(\psi,\delta,n,\rho,\Omega)\|Zu\|_{l_2(Q_T)} + \varepsilon\|u\|_{w_{2,\psi}^{2,2}(Q_T)} + C_{14}(\psi,\delta,n,\rho,\Omega)\|u\|_{l_2(Q_T)}$$

holds for any function $u(x,t) \in C^\infty(\overline{Q}_T(x^0))$, where $u|_{t=0} = 0$.

Lemma 5. *If the coefficients of the operator* Z *satisfy conditions (4), (5), then there exists* $\rho_1(n,\sigma,\Omega)$ *such that for every* $T\leqslant T_2$ *and* $\delta>0$ *the estimate*

$$\|u\|_{w^{2,2}_{2,\psi}(Q^1_{T(\rho_1)})}\leqslant C_{15}(\psi,\delta,n,\rho_1,\Omega)\|Zu\|_{l_2(Q_T)}+\varepsilon\|u\|_{w^{2,2}_{2,\psi}(Q_T)}+\frac{C_{16}(\psi,\delta,n,\rho_1,\Omega)}{\varepsilon}\|u\|_{l_2(Q_T)}$$

holds for any function $u(x,t)\in C^\infty(\overline{Q}_T(x^0))$, *where* $u|_{t=0}=0$ *and* $Q^1_T(\rho_1)=Q_T\backslash Q_T(\rho_1)$.

Lemma 6. *Under the conditions of Lemma 5 the estimate*

$$\|u\|_{w^{2,2}_{2,\psi}(Q_T)}\leqslant C_{27}(\psi,\delta,n,\rho,\Omega)\|Zu\|_{l_2(Q_T)}+C_{28}(\psi,\delta,n,\rho_1,\Omega)\|u\|_{l_2(Q_T)}$$

holds for any $u(x,t)\in w^{2,2}_{2,\psi}(Q_T)$ *and* $T\leqslant T_2$.

Theorem 1. *If conditions (4), (5) are fulfilled, then there exists* $T_0=T_0(\psi,\delta,n,\Omega)$ *such that for every* $T\leqslant T_2$ *the estimate*

$$\|u\|^2_{w^{2,2}_{2,\psi}(Q)}\leqslant C_{29}(\psi,\delta,n,\Omega)\|Zu\|_{l_2(Q)}$$

holds for any function $u(x,t)\in w^{2,2}_{2,\psi}(Q)$.

References

1. 1. *Keldysh M. V.* On some cases of degeneration of an equation of elliptic type on the domain boundary, Dokl. Akad. Nauk SSSR. — 1951. —- T. 77. — № 2. — C. 181–183.
2. *Fichera G.* To the unique theory of boundary-value problems for the second order elliptic-parabolic equations (The collections of the translated foreign articles) — 1963. — Vol. 7, №6. — P. 99–121.
3. *Oleynik O.A., Radkevitch J.V.* Second order quations with nonnegative characteristic form. Viniti, ser. Itogi nauki, Math. Analysis — 1971. — P. 7–252.
4. *Franciosi M.* Sul de un equazioni elliptico-parabolica a coefficienti discontinue. Boll. Un. Math. Ital. — 1983. — Vol. 6, №2. — P. 63–75.
5. *Franciosi M.* Un theoreme di esistenza ed unicita per la soluzione di un'equazione elliptico-parabolica, a coefficienti-discontinui, in forma non divergenza. Bull. Un. Mat. Ital. — 1985. — Vol. 6, №4-B. — P. 253–263.
6. *Alvino A., Trombetti G.* Second order elliptic equation whose coefficients have their first derivatives weakly-Ln????. Publ. Ist.Mat. "R.Cacciopoli", 1983, Vol. 14.
7. *Gadjiev T.S., Gasimova E.* On smoothness of solution of the first boundary-value problem for second order degenerate elliptic-parabolic equations. Ukranian Mathematical Journal — 2008. — Vol. 60, №6. — P. 723–736.
8. *Chanillo S., Wheeden R. L.* Existence and estimates of Green's function for degenerate elliptic equations //Annali della Scuola Normale Superiore di Pisa-Classe di Scienze. — 1988. —- T. 15, № 2. —- C. 3090-340.

Стаття: надійшла до редколегії 18.11.2015
доопрацьована 27.02.2016
прийнята до друку 08.06.2016

ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2016. Випуск 81

ОЦІНКИ РОЗВ'ЯЗКІВ ЕЛІПТИЧНО-ПАРАБОЛІЧНИХ РІВНЯНЬ

Тагір ГАДЖИЄВ, Солтан АЛІЄВ, Гюнель ГАСАНОВА

Інститут математики і механіки,
НАН Азербайджану, Баку, Азербайджан
e-mail: tgadjiev@mail.az

Досліджено клас вироджених еліптично-параболічних рівнянь другого порядку в недивергентній формі. Визначено оцінку розв'язків крайових задач для цих рівнянь у відповідних просторах Соболєва.

Ключові слова: еліптично-параболічні рівняння, крайові задачі, простір Соболєва.