ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2016. Випуск 81. С. 58–60
Visnyk of the Lviv Univ. Series Mech. Math. 2016. Issue 81. P. 58–60

УДК 512.552.13

BEZOUT DOMAINS WHOSE FINITE HOMOMORPHIC IMAGES ARE SEMIPOTENT RINGS

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We establish that finite homomorphic images of a commutative Bezout domain are semipotent rings.

Key words: Bezout domain, semipotent ring.

The concept of semipotent ring is important and is of particular interest in the modern research [2, 3]

Let R be a commutative Bezout domain, then for $a \in R \setminus \{0\}$ the factor-ring R/aR is *an exchange ring* if and only if a is an avoidable element i.e. if for any $b, c \in R$ such that $aR + bR + cR = R$ there exist elements $r, s \in R$ such that $a = r \cdot s$, where $rR + bR = R$, $sR + bR = R$ and $rR + sR = R$ (see [1]).

In this paper we establish that for a commutative Bezout domain R and $a \in R \setminus \{0\}$ the factor-ring R/aR is *a semipotent ring* if and only if a is a semipotent element i.e. if for any $b \in R$ there exist elements $r, s \in R$ such that $a = r \cdot s$, where $rR + bR = R$ and $rR + sR = R$.

All rings considered are commutative and have the identity, $1 \neq 0$. A ring is *a Bezout ring* if every its finite generated ideal is principal.

We denote by $U = U(R)$ the group of units of R and by $J(R)$ the Jacobson radical of R. Recall that a ring R is a semipotent ring, also called J_0-ring by Nicholson [2], if every its principal ideal not contained in $J(R)$ contains a nonzero idempotent. The exampels of such rings include the exchange rings and can be found in [2]. A ring R is *an exchange ring* if for every $a \in R$ there exists an idempotent $e \in aR$ such that $(1 - e) \in (1 - a)R$.

Definition 1. *Let R be a commutative Bezout domain. An element $a \in R \setminus \{0\}$ is said to be semipotent if for any $b \in R$ we have $a = r \cdot s$, where $rR + bR = R$ and $rR + sR = R$.*

Theorem 1. *Let R be a commutative Bezout domain. Then a is a semipotent element if and only if R/aR is a semipotent ring.*

Proof. The equality $rR + sR = R$ implies $ru + sv = 1$ for some elements u, v, $\overline{r}^2\overline{u} = \overline{r}$, $\overline{s}^2\overline{v} = \overline{s}$ for $\overline{r} = r + aR$ and $\overline{s} = s + aR$. Obviously, $\overline{ru} = \overline{e}$, $\overline{e}^2 = \overline{e}$ and $\overline{1} - \overline{e} = \overline{s}\ \overline{v}$. The equality $rR+bR=R$ implies $\overline{1}-\overline{e}\in\overline{b}\,\overline{R}$ where $\overline{b}=b+aR$. Since $\overline{1}-\overline{e}$ is an indempotent, $\overline{R}$ is a semipotent ring. Note that if $\overline{b}\notin J(aR)$ then $\overline{1}-\overline{e}$ is a proper indempotent. If for $\overline{b}=b+aR$ we have that there exist an idempotent $\overline{e}^2=\overline{e}$ such that $\overline{e}\in\overline{b}\,\overline{R}$. Since $\overline{e}^2=\overline{e}$, we have $e(1-e)=a\alpha$ for some $\alpha\in R$. And since $\overline{e}\in\overline{b}\,\overline{R}$, we have $e-bt=as$ for some elements $t,s\in R$. Let $eR+aR=dR$, then $e=de_0$, $a=da_0$ and $a_0R+e_0R=R$. The equality $e(1-e)=a\alpha$ implies $e_0(1-e)=a_0\alpha$. Since $a_0R+e_0R=R$, we have $a_0R+eR=R$. Let $r=a_o$, $s=d$, then we have $rR+bR=R$. Since $e-bt=as$, then $rR+sR=R$. Theorem is proved.

Since an exchange ring is a semipotent ring, we have that avoidable element is obviously a semipotent element.

The following result connects the concept of adequate element and the concept of semipotent ring.

Definition 2. *Let R be a commutative Bezout domain. An element $a\in R$ is called adequate for an element b if we can find elements $r,s\in R$ such that*

1) $a=r\cdot s$;
2) $rR+bR=R$;
3) *for any nonunit divisor s' of s we have $s'R+bR\neq R$.* [3]

That a is adequate for b will be denoted by ${}_aA_b$.

Theorem 2. *Let R be commutative Bezout domain. If element $a\in R$ is semipotent than for any element $b\notin J(aR)$ there exists some element $u\in R$ that ${}_aA_{bu}$.*

Proof. Let $\overline{R}=R/aR$ be a semipotent ring and $b\notin J(aR)$. Then there exists a nonzero idempotent $\overline{e}$ such that $\overline{e}\in\overline{b}\,\overline{R}$. Hence, there exist some elements $u,t\in R$ such that $e-bu=at$. Moreover, since $\overline{e}^2=\overline{e}$, we have $(1-e)=as$ for some element $s\in R$.

Let $eR+aR=dR$, where $e=de_0$, $a=da_0$ and $a_0R+e_0R=R$. Then

$$e_0(1-e)=a_0s \quad\text{and}\quad e+a_0j=1$$

for some element $j\in R$. Taking $r=a_0$, $s=d$ we obtain a decomposition

$$a=r\cdot s$$

where $rR+eR=R$ and $s'R+eR\neq R$, for some nonunit divisor s' of s. Thus, ${}_aA_e$ and hence from $bu=e+at$ we obviously conclude ${}_aA_{bu}$. Theorem is proved.

As a corollary of previous theorem we obtain the following.

Theorem 3. *Let R be a commutative Bezout domain and $a\in R\setminus\{0\}$. The factor-ring R/aR is semipotent if and only if for any element $b\notin J(aR)$ there exists an element $u\in R$ such that ${}_aA_{bu}$ and $bu\notin aR$.*

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Стаття: надійшла до редколегії 12.03.2016
прийнята до друку 08.06.2016

КОМУТАТИВНІ ОБЛАСТІ БЕЗУ, СКІНЧЕННІ ГОМОМОРФНІ ОБРАЗИ ЯКИХ Є НАПІВПОТУЖНИМИ КІЛЬЦЯМИ

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Визначено умови, коли скінченно гомоморфні образи комутативної області Безу є напівпотужними кільцями.

Ключові слова: область Безу, напівпотужні кільця.