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UNIQUE SOLVABILITY OF INITIAL-BOUNDARY VALUE PROBLEMS FOR NONLINEAR PARABOLIC EQUATIONS WITH TIME DEPENDENT DELAY

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The existence and uniqueness of a weak solutions of the mixed problems for nonlinear parabolic equations with variable delay are investigated and its a priori estimate is obtained.

Key words: initial-boundary value problem, equation with delay, nonlinear parabolic equation.

1. Introduction.

The initial-boundary value problems for the nonlinear parabolic equations with time depended delay are considered. A typical example of the equations being studied here is

$$u_t - \sum_{i=1}^{n} \Big(\widehat{a}_i(x,t) u_{x_i} \Big)_{x_i} + \widehat{a}_0(x,t) u + \int\limits_{t-\tau(t)}^{t} c_0(x,t,s) u(x,s)\, ds = f(x,t), \qquad (1)$$

$(x,t) \in Q := \Omega \times (0,T)$, where $n \in \mathbb{N}$, Ω is a domain in $\mathbb{R}^n$, $T > 0$, $\widehat{a}_0, \ldots, \widehat{a}_n$ are measurable positive functions on Q, τ is a nonnegative continuous function, c_0 is a measurable bounded function, f is an integrable function, u is an unknown function.

Equations with time delay arise in modelling population dynamics, in non-Newtonian filtration, heat flux, etc. ([5]). The equations of type (1) with constant delay were investigated in [1], [2], [3], [6], [7] and others. A good reference overview of such papers can be found in [3]. Note that in these papers the semigroup theory is used.

Partial differential equations with a variable delay are less studied, and we know only publications of Rezounenko and Chueshov (in particular, [4], [8]), where equations of type (1), with $\tau = \tau(u)$, are considered. In [4], a certain abstract parabolic problem with the state dependent delay term of a rather general structure is considered. In [8], the nonlinear partial functional differential equations with main linear elliptic operator and non-local nonlinear term is considered. For proving existence of solutions of problems considered in [4], [8] Galerkin's approximations are used.

To the best of our knowledge, the initial-boundary value problems for parabolic equations with time dependent delay is an untreated topic in the literature. These problems are considered in our paper. Existence and uniqueness of solution of the problem are proved. The methods of investigation as in [10] are used.

The paper is organized as follows. In Section 2, the main notations and functional spaces are introduced. The statement of the problem and formulation of the main result are given in Section 3. The main result is proved in Section 4.

2. Notations and auxiliary facts.

Let n be a natural number, $\mathbb{R}^n$ be the standard linear space of ordered collections $x=(x_1,...,x_n)$ of real numbers with the norm $|x|:=(|x_1|^2+\ldots+|x_n|^2)^{1/2}$. Suppose that $\Omega\subset\mathbb{R}^n$ is a bounded domain with a piecewise smooth boundary $\partial\Omega$, $\partial\Omega=\Gamma_0\cup\Gamma_1$, where Γ_0 is the closure of an open set on $\partial\Omega$ (in particular, either $\Gamma_0=\emptyset$ or $\Gamma_0=\partial\Omega$), $\Gamma_1:=\partial\Omega\setminus\Gamma_0$, $\nu=(\nu_1,\ldots,\nu_n)$ is a unit outward pointing normal vector on $\partial\Omega$. Let $T>0$ and $Q:=\Omega\times(0,T)$, $\Sigma_0:=\Gamma_0\times(0,T)$, $\Sigma_1:=\Gamma_1\times(0,T)$.

Now let us give the definitions of the following functional spaces. First, denote by $H^1(\Omega)$ the Sobolev space of the functions $v\in L_2(\Omega)$ such that $v_{x_i}\in L_2(\Omega)$ $(i=\overline{1,n})$, with the norm $||v||_{H^1(\Omega)}:=\Big(||v||^2_{L_2(\Omega)}+\sum_{i=1}^n||v_{x_i}||^2_{L_2(\Omega)}\Big)^{1/2}$. Let $\widetilde{H}^1(\Omega)$ be the closure of the space $\widetilde{C}^1(\overline{\Omega}):=\{v\in C^1(\overline{\Omega})\mid v|_{\Gamma_0}=0\}$ in $H^1(\Omega)$.

Denote by $L_2(0,T;\widetilde{H}^1(\Omega))$ the space of measurable functions $w:(0,T)\to\widetilde{H}^1(\Omega)$ such that for a.e. $t\mapsto||w(t)||_{\widetilde{H}^1(\Omega)}\in L_2(0,T)$.

Denote by $F(Q)$ the space of vector-functions $(f_0,f_1,\ldots,f_n)\in[L_2(Q)]^{1+n}$ such that $f_i\in L_2(Q)$, and for each $i\in\{1,...,n\}$, $f_i=0$ a.e. in some neighborhood of the surface Σ_1.

Finally, define $C_0^1(0,T)$ as the subset of the set $C^1(0,T)$ whose elements are of compact support in $(0,T)$. Also denote by $C([t_1,t_2];L_2(\Omega))$ $(t_1,t_2\in\mathbb{R},\ t_1<t_2)$ the space of continuous functions $w:[t_1,t_2]\to L_2(\Omega)$.

3. Statement of the problem and main result.

In this paper we consider the problem of finding a function $u:\overline{\Omega}\times[-\tau_0,T]\to\mathbb{R}$ satisfying (in some sense) the equation

$$u_t-\sum_{i=1}^n\frac{d}{dx_i}a_i(x,t,u,\nabla u)+a_0(x,t,u,\nabla u)+\int\limits_{t-\tau(t)}^{t}c(x,t,s,u(x,s))ds$$
$$=-\sum_{i=1}^n\frac{\partial}{\partial x_i}f_i(x,t)+f_0(x,t),\quad (x,t)\in Q, \tag{2}$$

the boundary conditions

$$u\Big|_{\Sigma_0}=0,\qquad \sum_{i=1}^n a_i(x,t,u,\nabla u)\,\nu_i\Big|_{\Sigma_1}=0, \tag{3}$$

and the initial condition

$$u(x,t)=u_0(x,t),\quad (x,t)\in\overline{\Omega}\times[-\tau_0,0]. \tag{4}$$

Here $\tau : [0,T] \to \mathbb{R}$ is a continuous function such that $\tau(t) \geqslant 0$ for all $t \in [0,T]$, $\tau_0 := \max\{-\inf\limits_{t\in[0,T]}(t-\tau(t)),0\}$ (assume $[-0,0]=\{0\}$), $\tau^+ := \max\limits_{t\in[0,T]}\tau(t)$, and $a_i : Q\times\mathbb{R}^{1+n} \to \mathbb{R}$, $c : Q\times(-\tau_0,T)\times\mathbb{R}\to\mathbb{R}$, $f_i : Q\to\mathbb{R}$ $(i=\overline{0,n})$, $u_0 : \overline{\Omega}\times[-\tau_0,0]\to\mathbb{R}$ are given real-valued functions from the corresponding classes of initial data.

We introduce the following classes of the initial data.

Define $\mathbb{A}$ to be the set of the collections $(a_0,a_1,\dots,a_n)$ of the functions satisfying the following assumptions:

$(\mathcal{A}_1)$ for every $i\in\{0,1,\dots,n\}$,

$$Q\times\mathbb{R}^{1+n}\ni(x,t,\rho,\xi)\mapsto a_i(x,t,\rho,\xi)\in\mathbb{R}$$

is a Caratheodory function, i.e., $a_i(x,t,\cdot,\cdot):\mathbb{R}^{1+n}\to\mathbb{R}$ is a continuous function for a.e. $(x,t)\in Q$, and $a_i(\cdot,\cdot,\rho,\xi):Q\to\mathbb{R}$ is a measurable function for every $(\rho,\xi)\in\mathbb{R}^{1+n}$;

$(\mathcal{A}_2)$ for every $i\in\{0,1,\dots,n\}$, for every $(\rho,\xi)\in\mathbb{R}^{1+n}$, and for a.e. $(x,t)\in Q$ the estimate

$$|a_i(x,t,\rho,\xi)|\le C_1\big(|\rho|+\sum_{j=1}^n|\xi_j|\big)+h_i(x,t),$$

is valid, where C_1 is a positive constant (depending on $(a_0,a_1...,a_n)$), and $h_i\in L_2(Q)$;

$(\mathcal{A}_3)$ for every (ρ_1,ξ^1), $(\rho_2,\xi^2)\in\mathbb{R}^{1+n}$ and for a.e. $(x,t)\in Q$ the inequality

$$\sum_{i=1}^n\big(a_i(x,t,\rho_1,\xi^1)-a_i(x,t,\rho_2,\xi^2)\big)(\xi_i^1-\xi_i^2)+\big(a_0(x,t,\rho_1,\xi^1)-a_0(x,t,\rho_2,\xi^2)\big)(\rho_1-\rho_2)$$
$$\ge -K_0|\rho_1-\rho_2|^2,$$

holds, where $K_0\geqslant 0$ is a constant;

$(\mathcal{A}_4)$ for every $(\rho,\xi)\in\mathbb{R}^{1+n}$ and for a.e. $(x,t)\in Q$ we have

$$\sum_{i=1}^n a_i(x,t,\rho,\xi)\xi_i+a_0(x,t,\rho,\xi)\rho\ge K_1\sum_{i=1}^n|\xi_i|^2-K_2|\rho|^2-g(x,t),$$

where K_1,K_2 are positive constants (depending on $(a_0,a_1...,a_n)$), and $0\leqslant g\in L_1(Q)$.

Define $\mathbb{C}$ to be the set of the functions $c(x,t,s,\rho)$, $(x,t,s,\rho)\in Q\times(-\tau_0,T)\times\mathbb{R}$, satisfying the following assumptions:

$(\mathcal{C}_1)$ c is a Caratheodory function, i.e., $c(x,t,s,\cdot):\mathbb{R}\to\mathbb{R}$ is a continuous function for a.e. $(x,t,s)\in Q\times(-\tau_0,T)$, and $c(\cdot,\cdot,\cdot,\rho):Q\times(-\tau_0,T)\to\mathbb{R}$ is a measurable function for every $\rho\in\mathbb{R}$; in addition, $c(x,t,s,0)=0$ for a.e. $(x,t,s)\in Q\times(-\tau_0,T)$;

$(\mathcal{C}_2)$ there exists a constant $L>0$ (depending on c) such that for every ρ_1, $\rho_2\in\mathbb{R}$ and for a.e. $(x,t,s)\in Q\times(-\tau_0,T)$ the inequality

$$\big|c(x,t,s,\rho_1)-c(x,t,s,\rho_2)\big|\le L|\rho_1-\rho_2| \qquad (5)$$

holds.

Remark. From the condition $c(x,t,s,0)=0$ and $(\mathcal{C}_2)$ it follows that for every $\rho\in\mathbb{R}$, and for a.e. $(x,t,s)\in Q\times(-\tau_0,T)$ the estimate

$$|c(x,t,s,\rho)|\le L|\rho| \qquad (6)$$

is valid.

Now we can give a definition of a weak solution to problem (2)–(4).

Definition 1. *Let* $(a_0, a_1, \dots, a_n) \in \mathbb{A}$, $c \in \mathbb{C}$, $(f_0, f_1, \dots, f_n) \in F(Q)$, $u_0 \in C([-\tau_0, 0]; L_2(\Omega))$. *The function* $u \in L_2(0,T; \widetilde{H}^1(\Omega)) \cap C([-\tau_0, T]; L_2(\Omega))$ *is called a weak solution of problem (2)–(4) if* u *satisfies the initial condition*

$$||u(\cdot,t) - u_0(\cdot,t)||_{L_2(\Omega)} = 0 \qquad \forall t \in [-\tau_0, 0], \tag{7}$$

and the integral equality

$$\iint\limits_Q \Big\{ \sum_{i=1}^n a_i(x,t,u,\nabla u) v_{x_i} \varphi + a_0(x,t,u,\nabla u) v\varphi + v\varphi \int\limits_{t-\tau(t)}^{t} c(x,t,s,u(x,s))\, ds$$
$$- uv\varphi' \Big\}\, dxdt = \iint\limits_Q \Big\{ \sum_{i=1}^n f_i v_{x_i}\varphi + f_0 v\varphi \Big\}\, dxdt \tag{8}$$

holds for every $v \in \widetilde{H}^1(\Omega)$ *and* $\varphi \in C_0^1(0,T)$.

Theorem 1. *If* $(a_0, a_1, \dots, a_n) \in \mathbb{A}$, $c \in \mathbb{C}$, $(f_0, f_1, \dots, f_n) \in F(Q)$, $u_0 \in C([-\tau_0, 0]; L_2(\Omega))$, *then problem (2)–(4) has a unique weak solution. Moreover, the weak solution* u *of this problem satisfies the estimate*

$$\max_{t\in[0,T]} \int\limits_\Omega |u(x,t)|^2\, dx + \iint\limits_Q \Big\{ \sum_{i=1}^n \big|u_{x_i}\big|^2 + |u|^2 \Big\}\, dxdt$$
$$\le C_2 \Bigg(\iint\limits_Q \Big\{ \sum_{i=1}^n \big|f_i\big|^2 + g \Big\}\, dxdt + \max_{t\in[-\tau_0,0]} \int\limits_\Omega |u_0(x,t)|^2\, dxdt \Bigg), \tag{9}$$

where C_2 *is a positive constant depending only on* K_1, K_2, L, τ_0, T.

4. Proof of the main result.

The following auxiliary result, which had been proved in [10], will be used in the sequel.

Lemma 1. *Suppose that* $w \in L_2(0,T; \widetilde{H}^1(\Omega))$ *satisfies the following identity*

$$\iint\limits_Q \Big\{ \sum_{i=1}^n g_i v_{x_i}\varphi + g_0 v\varphi - wv\varphi' \Big\}\, dxdt = 0, \quad v \in \widetilde{H}^1(\Omega),\ \varphi \in C_0^1(0,T), \tag{10}$$

for some $g_j \in L_2(Q)$ $(j = \overline{0,n})$. *Then* $w \in C([0,T]; L_2(\Omega))$ *and for every* $\theta \in C^1([0,T])$, $v \in \widetilde{H}^1(\Omega)$, *and* $t_1, t_2 \in [0,T]$ $(t_1 < t_2)$, *we have*

$$\theta(t_2) \int\limits_\Omega w(x,t_2) v(x)\, dx - \theta(t_1) \int\limits_\Omega w(x,t_1) v(x)\, dx$$
$$+ \int\limits_{t_1}^{t_2} \int\limits_\Omega \Big\{ \sum_{i=1}^n g_i v_{x_i}\theta + g_0 v\theta - wv\theta' \Big\}\, dxdt = 0, \tag{11}$$
$$\frac{1}{2}\theta(t_2) \int\limits_\Omega |w(x,t_2)|^2\, dx - \frac{1}{2}\theta(t_1) \int\limits_\Omega |w(x,t_1)|^2\, dx$$

$$-\frac{1}{2}\int_{t_1}^{t_2}\int_{\Omega}|w|^2\theta'\,dxdt+\int_{t_1}^{t_2}\int_{\Omega}\Big\{\sum_{i=1}^{n}g_iw_{x_i}+g_0w\Big\}\theta\,dxdt=0. \tag{12}$$

Proof of Theorem 1. For a function $w: Q\to\mathbb{R}$ we denote

$$a_j(w)(x,t):=a_j(x,t,w(x,t),\nabla w(x,t)),\quad (x,t)\in Q,\quad j=\overline{0,n},$$

$$c(w)(x,t,s):=c(x,t,s,w(x,s)),\quad (x,t,s)\in Q\times(-\tau_0,T).$$

Let us prove Theorem 1 in three steps: firstly we prove the uniqueness of solution of problem (2)–(4), later, its existence and, finally, we prove correctness of estimate (9).

First step (uniqueness of solution). Assume the opposite. Let u_1 and u_2 be two different weak solutions of problem (2)–(4). Consider the difference between (8) with $u=u_2$ and (8) with $u=u_1$. From the obtained integral identity by Lemma 1 with $w=u_1-u_2$, $\theta=e^{-\lambda t}$ ($\lambda=\mathrm{const}>0$), $t_1=0$, $t_2=T$, we obtain (see (12))

$$\frac{1}{2}e^{-\lambda T}\int_{\Omega}|w(x,T)|^2\,dx+\frac{\lambda}{2}\iint_{Q}|w|^2e^{-\lambda t}dxdt+\iint_{Q}\Big\{\sum_{i=1}^{n}(a_i(u_1)-a_i(u_2))(u_{1,x_i}-u_{2,x_i})$$

$$+(a_0(u_1)-a_0(u_2))(u_1-u_2)+w(x,t)\int_{t-\tau(t)}^{t}[c(u_1)(x,t,s)-c(u_2)(x,t,s)]\,ds\Big\}e^{-\lambda t}dxdt=0. \tag{13}$$

From condition $(\mathcal{A}_3)$ we have

$$\sum_{i=1}^{n}(a_i(u_1)-a_i(u_2))(u_{1,x_i}-u_{2,x_i})+(a_0(u_1)-a_0(u_2))(u_1-u_2)$$

$$\geqslant -K_0|u_1-u_2|^2. \tag{14}$$

Extend $w(x,t)$ by 0 for all $(x,t)\in\Omega\times\{(-\infty,-\tau_0)\cup(T,+\infty)\}$. Note that $w(x,t)=0$ for a.e. $(x,t)\in\Omega\times[-\tau_0,0]$. Using condition $(\mathcal{C}_2)$, the Fubini Theorem (see, e.g., [13, p.91]) and Hölder's inequality (see, e.g., [13, p.92]) we obtain

$$\Big|\iint_{Q}w(x,t)\Big(\int_{t-\tau(t)}^{t}[c(u_1)(x,t,s)-c(u_2)(x,t,s)]\,ds\Big)e^{-\lambda t}dxdt\Big|$$

$$\leqslant\iint_{Q}|w(x,t)|\Big(\int_{t-\tau(t)}^{t}|c(u_1)(x,t,s)-c(u_2)(x,t,s)|\,ds\Big)e^{-\lambda t}dxdt$$

$$\leqslant L\int_{\Omega}dx\int_{0}^{T}|w(x,t)|\Big(\int_{t-\tau(t)}^{t}|w(x,s)|ds\Big)e^{-\lambda t}dt$$

$$\leqslant L\int_{\Omega}dx\int_{0}^{T}|w(x,t)|e^{-\frac{\lambda t}{2}}\Big(e^{-\frac{\lambda t}{2}}\int_{t-\tau^+}^{t}|w(x,s)|\,ds\Big)dt$$

$$\leqslant L\sqrt{\tau^+}\int\limits_{\Omega}\Big[\Big(\int\limits_0^T |w(x,t)|^2 e^{-\lambda t}dt\Big)^{1/2}\Big(\int\limits_0^T e^{-\lambda t}dt\int\limits_{t-\tau^+}^{t}|w(x,s)|^2 ds\Big)^{1/2}\Big]dx. \qquad (15)$$

Now consider the second integral in the right side of inequality (15). Changing the order of integration, for a.e. $x\in\Omega$ we have

$$\int\limits_0^T e^{-\lambda t}dt\int\limits_{t-\tau^+}^{t}|w(x,s)|^2 ds \leqslant \int\limits_{-\tau^+}^{T}|w(x,s)|^2 ds\int\limits_{s}^{s+\tau^+}e^{-\lambda t}dt =$$

$$= \lambda^{-1}(1-e^{-\lambda\tau^+})\int\limits_0^T |w(x,s)|^2 e^{-\lambda s}ds.$$

Substituting in (15) the last term from the obtained above relations chain instead of the first one, we obtain

$$\Big|\iint\limits_{Q} w(x,t)\Big(\int\limits_{t-\tau(t)}^{t}|c(u_1)(x,t,s)-c(u_2)(x,t,s)|\ ds\Big)e^{-\lambda t}dxdt\Big|$$

$$\leqslant L\sqrt{\tau^+\lambda^{-1}(1-e^{-\lambda\tau^+})}\iint\limits_{Q}|w(x,t)|^2 e^{-\lambda t}dxdt. \qquad (16)$$

Using (14), (16), from (13) we obtain

$$\Big(\lambda/2-K_0-L\sqrt{\tau^+\lambda^{-1}(1-e^{-\lambda\tau^+})}\Big)\iint\limits_{Q}|w(x,t)|^2 e^{-\lambda t}dtdx\leqslant 0. \qquad (17)$$

Choosing λ big enough and such that $\lambda/2-K_0-L\sqrt{\lambda^{-1}\tau^+(1-e^{-\lambda\tau^+})}>0$, from (17) we obtain $u_1=u_2$ for a.e. $(x,t)\in Q$, i.e., a contradiction to our assumption. Therefore, a solution of problem (2)–(4) is unique.

Second step (existence of solution). For proving existence of a weak solution of problem (2)-(4) Galerkin's method is used. Let $\{w_j\,|\,j\in\mathbb{N}\}$ be a full linear independent set of the functions from $\widetilde{H}^1(\Omega)$, which is an orthonormalized basis in $L_2(\Omega)$. For each $k\in\mathbb{N}$, set

$$\alpha_k(t):=\int\limits_{\Omega}u_0(x,t)w_k(x)dx,\quad t\in[-\tau_0,0]. \qquad (18)$$

Obviously, $\alpha_k\in C([-\tau_0,0])$ $(k\in\mathbb{N})$.

For all $m\in\mathbb{N}$ we denote

$$u_{0,m}(x,t):=\sum_{k=1}^{m}\alpha_k(t)w_k(x),\quad (x,t)\in\overline{\Omega}\times[-\tau_0,0]. \qquad (19)$$

It is clear that

$$\max_{t\in[-\tau_0,0]}||u_0(\cdot,t)-u_{0,m}(\cdot,t)||_{L_2(\Omega)}\underset{m\to\infty}{\longrightarrow}0. \qquad (20)$$

According to Galerkin's method, for every $m \in \mathbb{N}$ we put

$$u_m(x,t) := \sum_{k=1}^{m} c_{m,k}(t) w_k(x), \quad (x,t) \in \overline{\Omega} \times [-\tau_0, T], \tag{21}$$

where $c_{m,1}, \ldots, c_{m,m}$ are continuous on $[-\tau_0, T]$ and absolutely continuous on $[0,T]$ functions, which are solutions of the Cauchy problem for the system of ordinary differential equations with delay

$$\int\limits_{\Omega} u_{m,t} w_j \, dx + \int\limits_{\Omega} \Big\{ \sum_{i=1}^{n} (a_i(u_m) - f_i) w_{j,x_i} + (a_0(u_m) - f_0) w_j$$

$$+ w_j(x) \int\limits_{t-\tau(t)}^{t} c(u_m)(x,t,s) ds \Big\} \, dx = 0, \quad t \in [0,T], \quad j = \overline{1,m}, \tag{22}$$

$$c_{m,k}(t) = \alpha_k(t), \quad t \in [-\tau_0, 0], \ k = \overline{1,m}. \tag{23}$$

Note that from (19), (21) and (23) it follows that

$$u_m(x,t) = u_{0,m}(x,t) \quad \text{for a.e. } (x,t) \in \overline{\Omega} \times [-\tau_0, 0]. \tag{24}$$

The linear independence of functions $w_1, \ldots, w_m$ yields that the matrix $\Big(\int\limits_{\Omega} w_k w_j dx \Big)_{k,j=1}^{m}$ is invertible. Thus the system of ordinary differential equations with delay (22) can be transformed to the normal form. Hence, according to the theorems of existence and extension of the solution to this problem (see [11, p. 54], [12, p. 31]), we obtain a global solution $c_{1,m}, \ldots .., c_{m,m}$ of problem (22), (23). This solution is defined on the interval $[-\tau_0, T_m\rangle$, where $0 < T_m \le T$. Here the braces "$\rangle$" means either ")" or "]". Further we will obtain the estimates that imply the equality $[-\tau_0, T_m\rangle = [-\tau_0, T]$.

Now we shall obtain estimates of u_m for each $m \in \mathbb{N}$. Multiply the equation of system (22) with a number $j \in \{1, \ldots, m\}$ by $c_{m,j} e^{-\lambda t}$, where $\lambda > 0$ is a positive number, and sum over $j \in \{1, \ldots, m\}$. Integrating the obtained equality over $t \in [0, \sigma]$, where $\sigma \in [0, T_m\rangle$, and using the integration-by-parts formula and equality (24), we have

$$\frac{1}{2} e^{-\lambda\sigma} \int\limits_{\Omega} |u_m(x,\sigma)|^2 \, dx - \frac{1}{2} \int\limits_{\Omega} |u_{0,m}(x,0)|^2 \, dx + \frac{\lambda}{2} \int\limits_{0}^{\sigma} \int\limits_{\Omega} |u_m(x,t)|^2 e^{-\lambda t} dx dt$$

$$+ \int\limits_{0}^{\sigma} \int\limits_{\Omega} \Big\{ \sum_{i=1}^{n} a_i(u_m) u_{m,x_i} + a_0(u_m) u_m + u_m(x,t) \int\limits_{t-\tau(t)}^{t} c(u_m)(x,t,s) ds \Big\} e^{-\lambda t} dx dt$$

$$= \int\limits_{0}^{\sigma} \int\limits_{\Omega} \Big\{ \sum_{i=1}^{n} f_i u_{m,x_i} + f_0 u_m \Big\} e^{-\lambda t} dx dt. \tag{25}$$

Further we need Cauchy inequality in the form

$$2ab \le \varepsilon |a|^2 + \varepsilon^{-1} |b|^2, \quad a, b \in \mathbb{R}, \quad \varepsilon > 0. \tag{26}$$

Now, extend $u_m(x,t)=0$ for all $(x,t)\in\Omega\times((-\infty,-\tau_0)\cup(T,+\infty))$. Then, using (6) and Hölder's inequality we obtain

$$\Big|\int_0^\sigma\!\!\int_\Omega u_m(x,t)\Big(\int_{t-\tau(t)}^t c(u_m)(x,t,s)\,ds\Big)e^{-\lambda t}dxdt\Big|$$

$$\leqslant \int_0^\sigma\!\!\int_\Omega |u_m(x,t)|\Big(\int_{t-\tau(t)}^t |c(u_m)(x,t,s)|\,ds\Big)e^{-\lambda t}dxdt$$

$$\leqslant L\int_\Omega dx\int_0^\sigma |u_m(x,t)|e^{-\lambda t}\Big(\int_{t-\tau(t)}^t |u_m(x,s)|ds\Big)dt$$

$$\leqslant L\int_\Omega dx\int_0^\sigma |u_m(x,t)|e^{-\frac{\lambda t}{2}}\Big(e^{-\frac{\lambda t}{2}}\int_{t-\tau^+}^t |u_m(x,s)|\,ds\Big)dt$$

$$\leqslant L\sqrt{\tau^+}\int_\Omega\Bigg[\Big(\int_0^\sigma |u_m(x,t)|^2e^{-\lambda t}dt\Big)^{1/2}\Big(\int_0^\sigma e^{-\lambda t}dt\int_{t-\tau^+}^t |u_m(x,s)|^2ds\Big)^{1/2}\Bigg]dx. \qquad (27)$$

Now, let us estimate the second integral from the right side of the inequality above, for a.e. $x\in\Omega$,

$$\int_0^\sigma e^{-\lambda t}dt\int_{t-\tau^+}^t |u_m(x,s)|^2ds\leqslant \int_{-\tau^+}^\sigma |u_m(x,s)|^2ds\int_s^{s+\tau^+} e^{-\lambda t}dt$$

$$=\lambda^{-1}(1-e^{-\lambda\tau^+})\int_{-\tau_0}^\sigma |u_m(x,s)|^2e^{-\lambda s}ds$$

$$=\lambda^{-1}(1-e^{-\lambda\tau^+})\Big(\int_0^\sigma |u_m(x,t)|^2e^{-\lambda t}dt+\int_{-\tau_0}^0 |u_{0,m}(x,t)|^2e^{-\lambda t}dt\Big).$$

Here changing order of integration for a.e. $x\in\Omega$, and (24) were used.

Substituting in the right side of (27) the last item from the obtained above chain of relations instead of the first one, and using Hölder's inequality we obtain

$$\Big|\int_0^\sigma\!\!\int_\Omega u_m(x,t)\Big(\int_{t-\tau(t)}^t c(u_m)(x,t,s)ds\Big)e^{-\lambda t}dxdt\Big|\leqslant$$

$$\leqslant L\sqrt{\tau^+\lambda^{-1}(1-e^{-\lambda\tau^+})}\Big(2\int_0^\sigma\!\!\int_\Omega |u_m(x,t)|^2e^{-\lambda t}dxdt+\int_{-\tau_0}^0\!\!\int_\Omega |u_{0,m}(x,t)|^2e^{-\lambda t}dxdt\Big). \quad (28)$$

From condition $(\mathcal{A}_4)$ we have

$$\int_0^\sigma\int_\Omega\Big\{\sum_{i=1}^n a_i(u_m)u_{m,x_i}+a_0(u_m)u_m\Big\}\,dxdt\geq\int_0^\sigma\int_\Omega\Big\{K_1\sum_{i=1}^n|u_{m,x_i}|^2-K_2|u_m|^2-g(x,t)\Big\}\,dxdt. \tag{29}$$

Applying inequality (26), we obtain the estimate

$$\int_0^\sigma\int_\Omega\Big\{\sum_{i=1}^n f_iu_{m,x_i}+f_0u_m\Big\}e^{-\lambda t}\,dxdt\leqslant\frac{\varepsilon}{2}\int_0^\sigma\int_\Omega\Big\{\sum_{i=1}^n|u_{m,x_i}(x,t)|^2$$

$$+|u_m(x,t)|^2\Big\}e^{-\lambda t}dxdt+\frac{1}{2\varepsilon}\int_0^\sigma\int_\Omega\sum_{i=0}^n|f_i(x,t)|^2e^{-\lambda t}dxdt, \tag{30}$$

where $\varepsilon>0$ is arbitrary.

From (25), using (28) – (30), for each $\sigma\in(0,T_m\rangle$ we obtain

$$e^{-\lambda\sigma}\int_\Omega|u_m(x,\sigma)|^2\,dx+(2K_1-\varepsilon)\int_0^\sigma\int_\Omega\sum_{i=1}^n|u_{m,x_i}(x,t)|^2e^{-\lambda t}dxdt$$

$$+\Big(\lambda-2K_2-\varepsilon-4L\sqrt{\tau^+\lambda^{-1}(1-e^{-\lambda\tau^+})}\Big)\int_0^\sigma\int_\Omega|u_m(x,t)|^2e^{-\lambda t}dxdt$$

$$\leq\varepsilon^{-1}\int_0^\sigma\int_\Omega\sum_{i=0}^n|f_i(x,t)|^2e^{-\lambda t}dxdt+2\int_0^\sigma\int_\Omega g(x,t)e^{-\lambda t}dxdt$$

$$+2L\tau_0\sqrt{\tau^+\lambda^{-1}(1-e^{-\lambda\tau^+})}\max_{t\in[-\tau_0,0]}\int_\Omega|u_{0,m}(x,t)|^2dx. \tag{31}$$

Taking $\varepsilon=K_1$, $\lambda=\lambda_0$, where λ_0 is a solution of the inequality

$$\lambda-2K_2-K_1-4L\sqrt{\tau^+\lambda^{-1}(1-e^{-\lambda\tau^+})}>0, \tag{32}$$

from (31) we obtain

$$\max_{t\in[0,T]}\int_\Omega|u_m(x,t)|^2\,dx+C_3\iint_Q\Big\{\sum_{i=1}^n|u_{m,x_i}(x,t)|^2+|u_m(x,t)|^2\Big\}dxdt$$

$$\leq C_4\iint_Q\Big(\sum_{i=0}^n|f_i(x,t)|^2+g(x,t)\Big)dxdt+C_5\max_{t\in[-\tau_0,0]}\int_\Omega|u_{0,m}(x,t)|^2dx, \tag{33}$$

where C_3,C_4,C_5 are positive constants depending only on K,L,τ_0,τ^+,T.

From (20) it follows that the sequence $\{\max\limits_{t\in[-\tau_0,0]}\int_\Omega|u_{0,m}(x,t)|^2dx\}_{m=1}^\infty$ is bounded. Hence from (33) we obtain for all $\sigma\in(0,T_m\rangle$ the estimates

$$\int_\Omega|u_m(x,\sigma)|^2\,dx\leq C_6, \tag{34}$$

$$\int_0^\sigma\int_\Omega \Big\{ \sum_{i=1}^n |u_{m,x_i}(x,t)|^2 + |u_m(x,t)|^2 \Big\}\, dxdt \le C_7, \tag{35}$$

where C_6, $C_7 > 0$ are independent of m, T_m, σ. Estimate (34) implies that there exists an independent of T_m constant that bounds the functions $c_{m,1}, \ldots, c_{m,m}$ on $[-\tau_0, T_m\rangle$. Thus $[-\tau_0, T_m\rangle = [-\tau_0, T]$.

Condition $(\mathcal{A}_2)$ and estimate (35) yield

$$\iint_Q |a_i(u_m)(x,t)|^2\, dxdt \le C_8, \quad i=\overline{0,n}, \tag{36}$$

where $C_8 > 0$ is independent of m.

Using (6), the Cauchy–Schwarz inequality, (20) and (34) we obtain

$$\iint_Q \Big| \int_{t-\tau(t)}^{t} c(u_m)(x,t,s)\, ds\Big|^2 dxdt \le L^2(T+\tau_0) \iint_Q \int_{-\tau_0}^{T} |u_m(x,s)|^2\, dsdxdt$$

$$\le L^2T(T+\tau_0)\Big(\iint_Q |u_m(x,t)|^2\, dxdt + \tau_0 \max_{t\in[-\tau_0,0]} \int_\Omega |u_{0,m}(x,t)|^2 dx\Big) \le C_9, \tag{37}$$

where $C_9 > 0$ is a constant independent of m.

Since the spaces $L_2(Q)$ is reflexive, and estimates (34)–(37) hold, we obtain the existence of a subsequence (we denote it $\{u_m\}_{m\in\mathbb{N}}$ again), functions $v_* \in L_2(\Omega)$, $u \in L_2(0,T;\widetilde{H}^1(\Omega))$, $\chi_i \in L_2(Q)$ $(i=\overline{0,n})$, and $\zeta \in L_2(Q)$ such that

$$u_m(\cdot,T) \underset{m\to\infty}{\longrightarrow} v_*(\cdot) \quad \text{weakly in} \quad L_2(\Omega), \tag{38}$$

$$u_m \underset{m\to\infty}{\longrightarrow} u \quad * -\text{weakly in} \quad L_\infty(0,T;L_2(\Omega)), \tag{39}$$

$$u_m \underset{m\to\infty}{\longrightarrow} u \quad \text{weakly in} \quad L_2(0,T;\widetilde{H}^1(\Omega)), \tag{40}$$

$$a_i(u_m) \underset{m\to\infty}{\longrightarrow} \chi_i \quad \text{weakly in} \quad L_2(Q) \quad (i=\overline{0,n}). \tag{41}$$

$$\int_{t-\tau(t)}^{t} c(u_m)ds \underset{m\to\infty}{\longrightarrow} \zeta \quad \text{weakly in} \quad L_2(Q). \tag{42}$$

Let us prove that u is a weak solution of problem (2)–(4).

Fix the numbers $j, m \in \mathbb{N}$ such that $m \ge j$. Multiplying the equation of system (22) with number j by the function $\theta \in C^1([0,T])$ we integrate the obtained equality over $t \in [0,T]$. Letting $m \to \infty$, and taking into account (20), (24), (38)–(42), we obtain

$$\theta(T)\int_\Omega v_*(x)w_j(x)\, dx - \theta(0)\int_\Omega u_0(x,0)w_j(x)\, dx$$

$$-\iint_Q u w_j \theta'\, dxdt + \iint_Q \Big\{ \sum_{i=1}^n (\chi_i - f_i) w_{j,x_i} + (\chi_0 + \zeta - f_0) w_j \Big\}\theta\, dxdt = 0. \tag{43}$$

This equality yields that for every $v \in \widetilde{H}^1(\Omega)$ and $\theta \in C^1([0,T])$ the equality

$$\theta(T)\int\limits_{\Omega} v_*(x)v(x)\,dx - \theta(0)\int\limits_{\Omega} u_0(x,0)v(x)\,dx$$

$$-\iint\limits_{Q} uv\theta'\,dxdt + \iint\limits_{Q}\Big\{\sum_{i=1}^{n}(\chi_i - f_i)v_{x_i} + (\chi_0 - f_0 + \zeta)v\Big\}\theta\,dxdt = 0 \tag{44}$$

holds.

Notice that if we take $\theta = \varphi \in C_0^1(0,T)$ in (44) then we have the identity

$$\iint\limits_{Q}\Big\{\sum_{i=1}^{n}(\chi_i - f_i)v_{x_i}\varphi + (\chi_0 - f_0 + \zeta)v\varphi - uv\varphi'\Big\}\,dxdt = 0 \;\; \forall\, v \in \widetilde{H}^1(\Omega),\;\; \forall \varphi \in C_0^1(0,T). \tag{45}$$

According to Lemma 1, (45) imply that

$$u \in C([0,T]; L_2(\Omega)) \tag{46}$$

and for every $v \in \widetilde{H}^1(\Omega)$ and $\theta \in C^1([0,T])$ the equality

$$\theta(T)\int\limits_{\Omega} u(x,T)v(x)\,dx - \theta(0)\int\limits_{\Omega} u(x,0)v(x)\,dx$$

$$-\iint\limits_{Q} uv\theta'\,dxdt + \iint\limits_{Q}\Big\{\sum_{i=1}^{n}(\chi_i - f_i)v_{x_i} + (\chi_0 + \zeta - f_0)v\Big\}\theta\,dxdt = 0 \tag{47}$$

holds.

From (44) and (47) we get

$$u(x,0) = u_0(x,0), \quad u(x,T) = v_*(x) \quad \text{for a.e.} \quad x \in \Omega. \tag{48}$$

Extend u by u_0 on $\overline{\Omega} \times [-\tau_0, 0)$. Let us show that this extended function belongs to the space $C([-\tau_0,T]; L_2(\Omega))$. Indeed, in view of (46) we conclude that the restriction u on $\overline{\Omega} \times [0,T]$ belongs to $C([0,T]; L_2(\Omega))$. From the conditions of the theorem we have $u_0 \in C([-\tau_0, 0]; L_2(\Omega))$. Also, $u(x,0) = u_0(x,0)$ (see (48)). These two facts imply (7). Hence, $u \in C([-\tau_0,T]; L_2(\Omega))$.

According to (45) to prove (8) it is enough to show that the equality

$$\int\limits_{\Omega}\Big\{\sum_{i=1}^{n}\chi_i v_{x_i} + (\chi_0 + \zeta)v\Big\}dx = \int\limits_{\Omega}\Big\{\sum_{i=1}^{n}a_i(u)v_{x_i} + \Big(a_0(u) + \int\limits_{t-\tau(t)}^{t} c(u)\,ds\Big)v\Big\}dx \tag{49}$$

is valid for every $v \in \widetilde{H}^1(\Omega)$ and for a.e. $t \in (0,T)$. For this we use the monotonicity method (see [15]).

Take a function $w_m \in L_2(0,T; \widetilde{H}^1(\Omega)) \cap L_2(-\tau_0,T; L_2(\Omega))$ such that $w_m(x,t) = u_{0,m}(x,t)$ for a.e. $(x,t) \in \Omega \times (-\tau_0, 0)$ and $w(x,t) = \widetilde{w}(x,t)$ for a.e. $(x,t) \in Q$, where

$\widetilde{w} \in L_2(0,T;\widetilde{H}^1(\Omega))$ is arbitrary. Denote for every $m \in \mathbb{N}$,

$$W_m := \iint_Q \Big\{ \sum_{i=1}^n (a_i(u_m) - a_i(w_m))(u_{m,x_i} - w_{m,x_i}) + (a_0(u_m) - a_0(w_m))(u_m - w_m)$$

$$+\frac{\lambda}{2}|u_m - w_m|^2 + (u_m - w_m) \int\limits_{t-\tau(t)}^{t} (c(u_m) - c(w_m))\, ds \Big\} e^{-\lambda t}\, dxdt,$$

where $\lambda > 0$ such that the inequality $\lambda/2 - K_0 - L\sqrt{\tau^+\lambda^{-1}(1-e^{-\lambda\tau^+})} > 0$ holds.

Using condition $(\mathcal{A}_3)$ for every $m \in \mathbb{N}$ we have

$$W_m \geq \iint_Q \Big\{ (\frac{\lambda}{2} - K_0)|u_m - w_m|^2 + (u_m - w_m) \int\limits_{t-\tau(t)}^{t} (c(u_m) - c(w_m))\, ds \Big\} e^{-\lambda t}\, dxdt.$$

Since the inequality

$$\Big| \iint_Q (u_m - w_m)\Big(\int\limits_{t-\tau(t)}^{t} |c(u_m) - c(w_m)|\ ds \Big) e^{-\lambda t} dxdt \Big|$$

$$\leqslant L\sqrt{\tau^+\lambda^{-1}(1-e^{-\lambda\tau^+})} \iint_Q |u_m - w_m|^2 e^{-\lambda t} dxdt$$

holds (see (16)), and because the choice of λ we obtain $W_m \geq 0$.

Hence, we have

$$W_m = \iint_Q \Big\{ \sum_{i=1}^n a_i(u_m)u_{m,x_i} + a_0(u_m)u_m + \frac{\lambda}{2}|u_m|^2 + u_m \int\limits_{t-\tau(t)}^{t} c(u_m)\, ds \Big\} e^{-\lambda t}\, dxdt$$

$$-\iint_Q \Big\{ \sum_{i=1}^n \big[a_i(u_m)w_{m,x_i} + a_i(w_m)(u_{m,x_i} - w_{m,x_i})\big] + a_0(u_m)w_m + a_0(w_m)(u_m - w_m)$$

$$+\lambda u_m w_m - \frac{\lambda}{2}|w_m|^2 + w_m \int\limits_{t-\tau(t)}^{t} c(u_m)\, ds + (u_m - w_m) \int\limits_{t-\tau(t)}^{t} c(w_m)\, ds \Big\} e^{-\lambda t}\, dxdt \geq 0, \quad m \in \mathbb{N}. \tag{50}$$

From (50), using (25) with $\sigma = T$, we obtain

$$W_m = \iint_Q \Big\{ \sum_{i=1}^n f_i u_{m,x_i} + f_0 u_m \Big\} e^{-\lambda t}\, dxdt - \frac{1}{2} e^{-\lambda T} \int_\Omega |u_m(x,T)|^2\, dx + \frac{1}{2} \int_\Omega |u_{0,m}(x,0)|^2\, dx$$

$$-\iint_Q \Big\{ \sum_{i=1}^n \big[a_i(u_m)w_{m,x_i} + a_i(w_m)(u_{m,x_i} - w_{m,x_i})\big] + \lambda u_m w - \frac{\lambda}{2}|w_m|^2 + a_0(u_m)w_m$$

$$+a_0(w)(u_m - w_m) + w_m \int\limits_{t-\tau(t)}^{t} c(u_m)\, ds + (u_m - w_m) \int\limits_{t-\tau(t)}^{t} c(w_m)\, ds\Big\} e^{-\lambda t}\, dxdt \geq 0 \tag{51}$$

for all $m \in \mathbb{N}$.

Taking into account (38) and the second equality of (48) we have

$$\lim_{m\to\infty} \inf ||u_m(\cdot, T)||_{L_2(\Omega)} \geq ||u(\cdot, T)||_{L_2(\Omega)}. \tag{52}$$

Define $w(x,t) := u_0(x,t)$ for $(x,t) \in \Omega \times (-\tau_0, 0)$, and $w(x,t) := \widetilde{w}(x,t)$ for $(x,t) \in \Omega \times (0,T)$. Then,

$$w_m \to w \text{ strongly in } L_2(\Omega \times (-\tau_0, T)). \tag{53}$$

Now, let us show that

$$\Big| \int\limits_{t-\tau(t)}^{t} c(w_m) ds - \int\limits_{t-\tau(t)}^{t} c(w) ds \Big| \to 0 \quad \text{in} \quad L_2(Q). \tag{54}$$

Extend the functions w, w_m by 0 on $\Omega \times \{(-\infty, -\tau_0) \cup (T, +\infty)\}$. Using condition $(\mathcal{C}_2)$, Cauchy–Schwarz inequality and changing order of integration, we obtain

$$\iiint\limits_{Q} \Big| \int\limits_{t-\tau(t)}^{t} c(w_m) ds - \int\limits_{t-\tau(t)}^{t} c(w) ds \Big|^2 dxdt \leqslant \tau^+ \iiint\limits_{Q} \Big(\int\limits_{t-\tau^+}^{t} |c(w_m) - c(w)|^2 ds \Big) dxdt$$

$$\leqslant L^2 \tau^+ \iiint\limits_{Q} \Big(\int\limits_{t-\tau^+}^{t} |w_m(x,s) - w(x,s)|^2 ds \Big) dxdt \leqslant$$

$$\leqslant L^2 \tau^+ \int\limits_{\Omega} dx \int\limits_{-\tau^+}^{T} |w_m(x,s) - w(x,s)|^2 ds \int\limits_{s}^{s+\tau^+} dt$$

$$= L^2 (\tau^+)^2 \int\limits_{\Omega} \int\limits_{-\tau^+}^{0} |u_{0,m}(x,t) - u_0(x,t)|^2 dxdt \underset{m\to\infty}{\longrightarrow} 0.$$

By (20), (40), (41), (42), (52), (53) from (51) we obtain

$$0 \leq \lim_{m\to\infty} \sup W_m \leq \iiint\limits_{Q} \Big\{ \sum_{i=1}^{n} f_i u_{x_i} + f_0 u \Big\} e^{-\lambda t}\, dxdt$$

$$-\frac{1}{2} e^{-\lambda T} \int\limits_{\Omega} |u(x,T)|^2\, dx + \frac{1}{2} \int\limits_{\Omega} |u_0(x,0)|^2\, dx - \iiint\limits_{Q} \Big\{ \sum_{i=1}^{n} \big[\chi_i w_{x_i} + a_i(w)(u_{x_i} - w_{x_i})\big]$$

$$+\chi_0 w + a_0(w)(u-w) + w\zeta + \lambda u w - \frac{\lambda}{2}|w|^2 + (u-w) \int\limits_{t-\tau(t)}^{t} c(w)\, ds \Big\} e^{-\lambda t} dxdt. \tag{55}$$

Using Lemma 1 with $\theta \equiv e^{-\lambda t}$ and the first equality of (48), from (45) we get

$$\iint_Q \Big\{ \sum_{i=1}^n \chi_i u_{x_i} + (\chi_0+\zeta)u + \lambda |u|^2 \Big\} e^{-\lambda t}\, dxdt = \iint_Q \Big\{ \sum_{i=1}^n f_i u_{x_i} + f_0 u \Big\} e^{-\lambda t}\, dxdt$$

$$-\frac{1}{2} e^{-\lambda T} \int_\Omega |u(x,T)|^2\, dx + \frac{1}{2}\int_\Omega |u_0(x,0)|^2\, dx. \tag{56}$$

Thus, (55) and (56) imply that

$$\iint_Q \Big\{ \sum_{i=1}^n (\chi_i - a_i(w))(u_{x_i} - w_{x_i}) + (\chi_0 - a_0(w))(u-w)$$

$$+\frac{\lambda}{2}|u-w|^2 + (u-w)\Big(\zeta - \int_{t-\tau(t)}^{t} c(w)\, ds\Big)\Big\} e^{-\lambda t}\, dxdt \geq 0. \tag{57}$$

Substituting $w = u - \mu v\varphi$ in the inequality above, where $v \in \widetilde{H}^1(\Omega)$, $\mu > 0$, $\varphi \in C_0^1(-\tau_0, T)$, such that $\operatorname{supp}\varphi \subset (0,T)$ and dividing the obtained inequality by μ we obtain

$$\iint_Q \Big\{ \sum_{i=1}^n (\chi_i - a_i(u-\mu v\varphi)) v_{x_i}\varphi + (\chi_0 - a_0(u-\mu v\varphi)) v\varphi$$

$$+\lambda\mu |v\varphi|^2 + \Big(\zeta - \int_{t-\tau(t)}^{t} c(u-\mu v\varphi)\, ds\Big) v\varphi\Big\} e^{-\lambda t}\, dxdt \geq 0. \tag{58}$$

Letting $\mu \to 0+$ in (58), using condition $(\mathcal{A}_3)$ and the Dominated Convergence Theorem (see [14, p. 648]), we have

$$\iint_Q \Big\{ \sum_{i=1}^n (\chi_i - a_i(u)) v_{x_i}\varphi + (\chi_0 - a_0(u)) v\varphi + v\varphi\Big(\zeta - \int_{t-\tau(t)}^{t} c(u)\, ds\Big)\Big\} e^{-\lambda t}\, dxdt = 0,$$

where $v \in \widetilde{H}^1(\Omega)$, $\varphi \in C_0^1(0,T)$ are arbitrary functions. Therefore we obtain (49). Thus, u is a weak solution of problem (2)–(4).

Third step. According to Lemma 1, with $w = u$, $t_1 = 0$, $t_2 = \sigma$, $\theta = e^{-\lambda t}$, $\lambda = \lambda_0$, where λ_0 is a solution of inequality (32), we obtain

$$\frac{1}{2} e^{-\lambda\sigma} \int_\Omega |u(x,\sigma)|^2\, dx - \frac{1}{2}\int_\Omega |u_0(x,0)|^2\, dx + \frac{\lambda}{2}\int_0^\sigma \int_\Omega |u(x,t)|^2 e^{-\lambda t} dxdt$$

$$+\int_0^\sigma \int_\Omega \Big\{ \sum_{i=1}^n a_i(u) u_{x_i} + a_0(u) u + u(x,t) \int_{t-\tau(t)}^{t} c(u)(x,t,s) ds\Big\} e^{-\lambda t} dxdt$$

$$= \int_0^\sigma \int_\Omega \Big\{ \sum_{i=1}^n f_i u_{x_i} + f_0 u \Big\} e^{-\lambda t} dxdt. \tag{59}$$

It is easy to show that inequalities similar to (28) – (30), with u instead of u_m, hold. Hence, the inequality, analogous to (33), can be obtained:

$$\max_{t\in[0,T]}\int\limits_{\Omega}|u(x,t)|^2\,dx + C_3\iint\limits_{Q}\Big\{\sum_{i=1}^{n}|u_{x_i}(x,t)|^2 + |u(x,t)|^2\Big\}dxdt$$

$$\le C_4\iint\limits_{Q}\Big(\sum_{i=0}^{n}|f_i(x,t)|^2 + g(x,t)\Big)dxdt + C_5\max_{t\in[-\tau_0,0]}\int\limits_{\Omega}|u_0(x,t)|^2dx. \qquad (60)$$

Therefore, inequality (9) holds.

□

References

1. *Bátkai A.* Semigroups for delay equations / A. *Bátkai*, S. Piazzera // in Resarch Notes in Mathematics, 10, A.K. Peters: Wellesley MA., 2005.
2. *Bátkai A.* Asymptotic behaviour of parabolic problems with delays in the highest order derivatives / A. *Bátkai*, R. Schnaubelt // Semigroup Forum. — 2004. — Vol. 69. №3. — P. 369–399.
3. *Khusainov D.* Strong and Mild Extrapolated L^2 -Solutions to the Heat Equation with Constant Delay / D. Khusainov, M. Pokojovy, R. Racke // SIAM Journal on Mathematical Analysis. — 2015. — Vol. 47, № 1. — P. 427–454 .
4. *Chueshov I.* Finite-dimensional global attractors for parabolic nonlinear equations with state-dependent delay / I.Chueshov, A. Rezounenko // Commun. Pure Appl. Anal. — 2015. — Vol. 14. — P. 1685–1704.
5. *Jin Ch.* Traveling wavefronts for a time delayed non-Newtonian filtration equation / Ch. Jin, J. Yin // Physica D. — 2012. — Vol. 241. — P. 1789–1803.
6. *Ezzinbi K.* Periodic solutions of non-densely defined delay evolution equations / K. Ezzinbi, J. H. Liu // Journal of Applied Mathematics and Stochastic Analysis. – 2002. — Vol. 15, №2. — P. 105–114.
7. *Di Blasio G.* L^2-regularity for parabolic partial integrodifferential equations with delay in the highest-order derivatives / G. Di Blasio, K. Kunisch, E. Sinestrari // J. Math. Anal. Appl. — 1984. — Vol.102. — P. 38–57.
8. *Rezounenko A. V.* A non-local PDE model for population dynamics with state-selective delay: Local theory and global attractors / A. V. Rezounenko, J. Wu // J. Comp. App. Math. — 2006. — Vol. 190. — P. 99—113.
9. *Bokalo M. M.* The unique solvability of a problem without initial conditions for linear and nonlinear elliptic-parabolic equations / M. M. Bokalo // J. Math. Sciences. — 2011. — Vol. 178, № 1. — P. 41–64 (Translated from Ukr. Matem. Visn. — 2011. — Vol. 8. № 1. — P. 55–86).
10. *Bokalo M.M.* Unique solvability of initial boundary value problems for anisotropic elliptic-parabolic equations with variable exponents of nonlinearity / M.M. Bokalo, O.M. Buhrii, R. A. Mashiyev // Journal of nonlinear evolution equations and applications. – 2014. — Vol.2013, № 6. — P. 67-–87.
11. *Coddington E. A.* Theory of ordinary differential equations / E. A. Coddington, N. Levinson // McGraw-Hill book company, New York, Toronto, London, 1955.
12. *Elsholts L.E.* Introduction to the theory of differential equations with deviating argument / L.E. Elsholts, S.B. Norkin // Moscow: Nauka., 1971.

13. *Brezis H.* Functional Analysis, Sobolev Spaces and Partial Differential Equations / H. Brezis // Springer, 2011.
14. *Evans L. C.* Partial differential equations / L. C. Evans // Graduate Studies in Mathematics. 2nd ed. — Vol. 19. Providence (RI): Amer. Math. Soc., 2010.
15. *Lions J.-L.* Quelques méthodes de résolution des problémes aux limites non linéaires / J.-L. Lions // Dunod Gauthier-Villars, Paris, 1969.
16. *Růžička M.* Electrorheological fluids: modeling and mathematical theory / M. Růžička // Lecture Notes in Mathematics, 1748. Springer-Verlag, Berlin, 2000.

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ОДНОЗНАЧНА РОЗВ'ЯЗНІСТЬ МІШАНИХ ЗАДАЧ ДЛЯ НЕЛІНІЙНИХ ПАРАБОЛІЧНИХ РІВНЯНЬ ЗІ ЗМІННИМ ЗАПІЗНЕННЯМ

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Досліджено питання існування та єдиності узагальнених розв'язків мішаних задач для нелінійних параболічних рівнянь зі змінним запізненням. Отримано апріорні оцінки розв'язків розглянутих задач.

Ключові слова: мішана задача, рівняння з запізненням, нелінійне параболічне рівняння.