ISSN 2078-3744. Вісник Львів. ун-ту. Серія мех.-мат. 2016. Випуск 81. С. 9–22
Visnyk of the Lviv Univ. Series Mech. Math. 2016. Issue 81. P. 9–22

УДК 512.536

ON A SEMITOPOLOGICAL α-BICYCLIC MONOID

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We consider a semitopological α-bicyclic monoid $\mathcal{B}_\alpha$ and prove that it is algebraically isomorphic to the semigroup of all order isomorphisms between the principal upper sets of ordinal ω^α. We prove that for every ordinal α and every $(a,b) \in \mathcal{B}_\alpha$ if a or b is a non-limit ordinal, then (a,b) is an isolated point in $\mathcal{B}_\alpha$. We show that for every ordinal $\alpha < \omega+1$ every locally compact semigroup topology on $\mathcal{B}_\alpha$ is discrete. On the other hand, we construct an example of a non-discrete locally compact topology τ_{lc} on $\mathcal{B}_{\omega+1}$ such that $(\mathcal{B}_{\omega+1}, \tau_{lc})$ is a topological inverse semigroup. This example shows that there is a gap in the proof of Theorem 2.9 [16] stating that for every ordinal α the semigroup $\mathcal{B}_\alpha$ does not admit non-discrete locally compact inverse semigroup topologies.

Key words: topological inverse semigroup, topological semigroup, semitopological semigroup, α-bicyclic semigroup.

In this paper all topological spaces are assumed to be Hausdorff. By $\mathbb{N}$ we shall denote the set of all positive integers. A semigroup S is called inverse if for every $x \in S$ there exists a unique $y \in S$ such that $xyx = x$ and $yxy = y$. Later such an element y will be denoted by x^{-1} and will be called the inverse of x. A map $inv : S \to S$ which assigns to every $s \in S$ its inverse is called inversion. By ω we denote the first infinite ordinal. A *topological* (*inverse*) *semigroup* is a topological space together with a continuous semigroup operation (and an inversion, respectively). Obviously, the inversion defined on a topological inverse semigroup is a homeomorphism. If S is a semigroup (an inverse semigroup) and τ is a topology on S such that (S,τ) is a topological (inverse) semigroup, then we shall call τ an (*inverse*) *semigroup topology* on S. A *semitopological semigroup* is a topological space together with a separately continuous semigroup operation. Let f be a map between two partial ordered sets $(A, \leqslant_A)$ and $(B, \leqslant_B)$, then we shall call f a monotone if for every $a,b \in A$ if $a \leqslant_A b$ then $f(a) \leqslant_B f(b)$. We shall call f an order isomorphism if f is monotone bijection and its inverse map f^{-1} is also monotone.

For a partially ordered set $(A, \leqslant)$, for an arbitrary $X \subset A$ and $x \in A$ we write:

1) $\downarrow X = \{y \in A : \text{there exists } x \in X \text{ such that } y \leqslant x\}$;
2) $\uparrow X = \{y \in A : \text{there exists } x \in X \text{ such that } x \leqslant y\}$;

3) ${\downarrow}x = {\downarrow}\{x\}$;
4) ${\uparrow}x = {\uparrow}\{x\}$;
5) X is a lower set if $X = {\downarrow}X$;
6) X is an upper set if $X = {\uparrow}X$;
7) X is a principal lower set if $X = {\downarrow}x$ for some $x \in X$;
7) X is a principal upper set if $X = {\uparrow}x$ for some $x \in X$.

The bicyclic monoid $\mathscr{B}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq = 1$. The distinct elements of $\mathscr{B}(p,q)$ are exhibited in the following useful array

$$\begin{array}{ccccc}
1 & p & p^2 & p^3 & \cdots \\
q & qp & qp^2 & qp^3 & \cdots \\
q^2 & q^2p & q^2p^2 & q^2p^3 & \cdots \\
q^3 & q^3p & q^3p^2 & q^3p^3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

and the semigroup operation on $\mathscr{B}(p,q)$ is determined as follows:

$$q^kp^l \cdot q^mp^n = q^{k+m-\min\{l,m\}}p^{l+n-\min\{l,m\}}.$$

It is well known that the bicyclic monoid $\mathscr{B}(p,q)$ is a bisimple (and hence simple) combinatorial E-unitary inverse semigroup and every non-trivial congruence on $\mathscr{B}(p,q)$ is a group congruence [8]. Also classic Andersen Theorem states that *a simple semigroup S with an idempotent is completely simple if and only if S does not contains an isomorphic copy of the bicyclic semigroup* (see [1] and [8, Theorem 2.54]). Observe that the bicyclic monoid can be represented as a semigroup of isomorphisms between principal upper sets of partially ordered set $(\mathbb{N}, \leqslant)$ (see [21]). The bicyclic semigroup admits only the discrete semigroup topology and if a topological semigroup S contains it as a dense subsemigroup then $\mathscr{B}(p,q)$ is an open subset of S [9]. In [6] and [10] this result was extended for the case of semitopological semigroups and generalized bicyclic semigroup respectively. The problem of an embedding of the bicyclic monoid into compact-like topological semigroups was discussed in [2], [3], [12]. Also, in [9] was described the closure of the bicyclic monoid $\mathscr{B}(p,q)$ in a locally compact topological inverse semigroup. In [11] was proved that a Hausdorff locally compact semitopological bicyclic semigroup with adjoined zero 0 is either compact or discrete.

Among the other natural generalization of bicyclic semigroup polycyclic monoid and α-bicyclic monoid play the major role.

Polycyclic monoid was introduced in [23]. In [4] there was introduced a notion of the λ-polycyclic monoid, which is a generalization of the polycyclic monoid and there was proved that for every cardinal $\lambda > 1$ the λ-polycyclic monoid $\mathcal{P}_\lambda$ can be represented as a subsemigroup of the semigroup of all order isomorphisms between principal lower sets of the λ-ary tree with adjoined zero. In [18] and [19] were studied algebraic properties of the polycyclic monoid. In [22] were studied topological properties of the graph inverse semigroups which are the generalization of polycyclic monoids and proved that for every finite graph E every locally compact semigroup topology on the graph inverse semigroup over E is discrete, which implies that for every positive integer n every locally compact semigroup topology on the n-Polycyclic monoid is discrete. In [4] were studied algebraic and topological properties of the λ-polycyclic monoid and was proved that for every non-zero cardinal λ every locally compact semigroup topology on the λ-Polycyclic monoid is discrete. In [5] authors investigated the closure of λ-polycyclic monoid in topological inverse semigroups.

However in this paper we are mostly concerned on the α-bicyclic monoid. This monoid was introduced in [17]. Let α be an arbitrary ordinal and $<$ be the usual order on α such that $a < b$ iff $a \in b$ for every $a, b \in \alpha$. For every $a, b \in \alpha$ we write $a \leqslant b$ iff $a = b$ or $a \in b$. Clearly that $\leqslant$ is a partial order on α. By $+$ we will denote the usual ordinal addition. Ordinal α is said to be prime if it can't be represented as a sum of two ordinals which are contained in α. For every ordinals a, b such that $a > b$ we will put $c = a - b$ if $a = b + c$. Clearly that for every ordinals $a > b$ there exists a unique ordinal c such that $a = b + c$. For more about ordinals see [20], [25] or [26]. By the α-bicyclic monoid $\mathcal{B}_\alpha$ we mean the set $\omega^\alpha \times \omega^\alpha$ endowed with the following binary operation:

$$(a,b)\cdot(c,d) = \begin{cases} (a+(c-b),d), & \text{if } \ b \leqslant c; \\ (a,d+(b-c)), & \text{if } \ b > c; \end{cases}$$

Later on we will write $(a,b)(c,d)$ instead of $(a,b)\cdot(c,d)$.

In [17] were considered algebraic properties of bisimple semigroups with well-ordered idempotents. In [24] was built a non-discrete inverse semigroup topology on $\mathcal{B}_2$. In [15] were investigated inverse semigroup topologies on $\mathcal{B}_\alpha$.

Observe that every upper set of arbitrary ordinal α is principal.

By $\mathscr{I}_{\omega^\alpha}^{\nearrow}$ we shall denote the semigroup of all order isomorphisms between the principal upper sets of the ordinal ω^α endowed with multiplication of composition of partial maps.

Topological semigroups of partial monotone bijections of linearly ordered sets were investigated in [7], [13], [14]. In [13] it was proved that every locally compact topology on the semigroup of all partial cofinite monotone injective transformations of the set of positive integers is discrete. In [7] authors proved that every Baire topology on the semigroup of almost monotone injective co-finite partial selfmaps of positive integers is discrete. In [14] was proved that every Baire topology on the semigroup of all monotone injective partial selfmaps of the set of integers having cofinite domain and image is discrete. We observe that in [7] and [14] there constructed non-discrete non-Baire inverse semigroup topologies on the corresponding semigroups.

In this paper we consider a semitopological α-bicyclic monoid $\mathcal{B}_\alpha$ and prove that it is algebraically isomorphic to a semigroup of all order isomorphisms between the principal upper sets of ordinal ω^α. We prove that for every ordinal α for every $(a,b) \in \mathcal{B}_\alpha$ if a or b is a non-limit ordinal then (a,b) is an isolated point in $\mathcal{B}_\alpha$. We show that for every ordinal $\alpha < \omega + 1$ every locally compact semigroup topology on $\mathcal{B}_\alpha$ is discrete. However we construct an example of a non-discrete locally compact topology τ_{lc} on $\mathcal{B}_{\omega+1}$ such that $(\mathcal{B}_{\omega+1}, \tau_{lc})$ is a topological inverse semigroup. This example shows that there is a gap in [16, Theorem 2.9], where is stated that for every ordinal α there is only discrete locally compact inverse semigroup topology on $\mathcal{B}_\alpha$.

Proposition 1. *For every ordinal α the semigroup $\mathscr{I}_{\omega^\alpha}^{\nearrow}$ of all order isomorphisms between principle upper sets of ordinal ω^α is isomorphic to the α-bicyclic monoid $\mathcal{B}_\alpha$.*

Proof. Observe that every principal upper set of ω^α is an interval $[a,\omega^\alpha)$ for some $a\in\omega^\alpha$. Define a map $h:\mathscr{I}^{\nearrow}_{\omega^\alpha}\to\mathcal{B}_\alpha$ in the following way: for an arbitrary order isomorphism $f:[a,\omega^\alpha)\to[b,\omega^\alpha)$ put $h(f)=(a,b)$. Clearly that if there exists an order isomorphism between two intervals $[a,\omega^\alpha)$ and $[b,\omega^\alpha)$ then it is unique. Thus the map h is injective. Since ω^α is a prime ordinal for every $a,b<\omega^\alpha$ upper sets $[a,\omega^\alpha)$ and $[b,\omega^\alpha)$ of ω^α are order isomorphic and hence the map h is surjective. Simple verifications show that the map h is a homomorphism.

Lemma 1. *Let $(\mathcal{B}_\alpha,\tau)$ be a semitopological semigroup. Then for every ordinal $a\in\omega^\alpha$ the elements $(0,a)$ and $(a,0)$ are isolated points in $(\mathcal{B}_\alpha,\tau)$.*

Proof. Observe that if e is an idempotent of semitopological semigroup S both eS and Se are retracts of the space S and hence are closed subsets of S. Since $\{(0,0)\}=\mathcal{B}_\alpha\setminus((1,1)\mathcal{B}_\alpha\cup\mathcal{B}_\alpha(1,1))$, $(0,0)$ is an isolated point in $(\mathcal{B}_\alpha,\tau)$.

Since $(0,a)(a,0)=(0,0)$ the separate continuity of multiplication implies that there exists an open neighborhood $V((0,a))$ such that $V((0,a))(a,0)=\{(0,0)\}$. Fix any point $(c,d)\in V((0,a))$. Then $(c,d)(a,0)=(0,0)$. Hence $d=a$ and $c=0$ which implies that $V((0,a))=\{(0,a)\}$. Thus $(0,a)$ is an isolated point in $(\mathcal{B}_\alpha,\tau)$. Proof of the statement that $(a,0)$ is an isolated point is similar.

Lemma 2. *Let $(\mathcal{B}_\alpha,\tau)$ be a semitopological semigroup. Then for every element $(a,b)\in\mathcal{B}_\alpha$ there exists a clopen neighborhood $V((a,b))$ of (a,b) such that the following conditions hold:*

1) *for every $(c,d)\in V((a,b))$ $c\leqslant a$ and $d\leqslant b$;*
2) *for every $(c,d)\in V((a,b))$ $a=c$ if and only if $b=d$.*

Proof. Put $V((a,b))=\{x\in S:(0,a)x=(0,b)\}$. Observe that by Lemma 1 $(0,b)$ is an isolated point of the space $\mathcal{B}_\alpha$. Since $(0,a)(a,b)=(0,b)$ the separate continuity of the multiplication in $\mathcal{B}_\alpha$ implies that $V(a,b)$ is a clopen neighborhood of the point (a,b). Fix any element $(c,d)\in V((a,b))$. Then we have that $(0,a)(c,d)=(0,b)$. Clearly that above equality holds in the only case when $c\leqslant a$. Hence $(0,a)(c,d)=(0,d+(a-c))=(0,b)$ and since $d+(a-c)=b$ we get that $d\leqslant b$. Moreover if $a=c$ then $(0,a)(c,d)=(0,d)=(0,b)$ and hence $b=d$.

Lemma 2 implies the following corollary:

Corollary 1. *Let $(\mathcal{B}_\alpha,\tau)$ be a semitopological semigroup. Then for every finite ordinals n,m element (n,m) is an isolated point in the space $(\mathcal{B}_\alpha,\tau)$.*

Lemma 3. *Let $(\mathcal{B}_\alpha,\tau)$ is a semitopological semigroup. For every distinct ordinals $a,b<\alpha$ (ω^a,ω^b) is an isolated point in $(\mathcal{B}_\alpha,\tau)$.*

Proof. First we consider the case when $a<b$. Since $(0,\omega^a)(\omega^a,\omega^b)=(0,\omega^b)$, Lemma 1 and separate continuity of the semigroup operation in $(\mathcal{B}_\alpha,\tau)$ imply that there exists an open neighborhood $V((\omega^a,\omega^b))$ of (ω^a,ω^b) in $(\mathcal{B}_\alpha,\tau)$ such that $(0,\omega^a)V((\omega^a,\omega^b))=\{(0,\omega^b)\}$ and for the neighborhood $V((\omega^a,\omega^b))$ conditions of Lemma 2 hold. Then $(0,\omega^a)(c,d)=(0,d+(\omega^a-c))=(0,\omega^b)$ for every $(c,d)\in V((a,b))$. Hence $d+(\omega^a-c)=d+\omega^a=\omega^b$, but this equation is true only if $\omega^a=\omega^b$, a contradiction.

In the case when $a > b$ the proof is similar.

By [26, Theorem 17] each ordinal α has Cantor's normal form, that is $\alpha = n_1\omega^{\beta_1} + n_2\omega^{\beta_2}+...+n_k\omega^{\beta_k}$, where n_i are positive integers and $\beta_1, \beta_2, ..., \beta_3$ is a decreasing sequence of ordinals.

Lemma 4. *Let $(\mathcal{B}_\alpha, \tau)$ be a semitopological semigroup, $a = n_1\omega^{\beta_1} + n_2\omega^{\beta_2} + ... + n_k\omega^{\beta_k}$, $b = m_1\omega^{\gamma_1} + m_2\omega^{\gamma_2}+...+m_t\omega^{\gamma_t}$ be Cantor's normal forms of ordinals a and b respectively. If $(a, b) \in \mathcal{B}_\alpha$ and $(\omega^{\beta_k}, \omega^{\gamma_t})$ is an isolated point in $(\mathcal{B}_\alpha, \tau)$ then (a, b) is an isolated point in the space $(\mathcal{B}_\alpha, \tau)$.*

Proof. Suppose $(\omega^{\beta_k}, \omega^{\gamma_t})$ is an isolated point in $(\mathcal{B}_\alpha, \tau)$. Put $a^* = n_1\omega^{\beta_1} + n_2\omega^{\beta_2} + ... + (n_k - 1)\omega^{\beta_k}$ and $b^* = m_1\omega^{\gamma_1} + n_2\omega^{\gamma_2} + ... + (n_t - 1)\omega^{\gamma_t}$. Here we agree that $a^* = 0(b^* = 0)$ in the case when $n_k = 1(n_t = 1)$. Then $(0, a^*)(a, b)(b^*, 0) = (\omega^{\beta_k}, \omega^{\gamma_t})$. The separate continuity of multiplication in $(\mathcal{B}_\alpha, \tau)$ implies that there exists an open neighborhood $V((a, b))$ of (a, b) such that $(0, a^*)V((a, b))(b^*, 0) = \{(\omega^{\beta_k}, \omega^{\gamma_t})\}$. Fix any element $(c, d) \in V((a, b))$. Obviously $(0, a^*)(c, d)(b^*, 0)$ can be equal to $(\omega^{\beta_k}, \omega^{\gamma_t})$ only if $a^* \leqslant c$ and $b^* \leqslant d$. Then $(0, a^*)(c, d)(b^*, 0) = (c - a^*, d - b^*) = (\omega^{\beta_k}, \omega^{\gamma_t})$, which implies that $c = a$ and $d = b$. Hence $V((a, b)) = \{(a, b)\}$.

Proposition 2. *Let $(\mathcal{B}_\alpha, \tau)$ be a semitopological semigroup and a be a non-limit ordinal. Then for every ordinal $b \in \omega^\alpha$ both (a, b) and (b, a) are isolated points in the space $(\mathcal{B}_\alpha, \tau)$.*

Proof. Corollary 1 and Lemma 4 imply that both points (a, b) and (b, a) are isolated in $(\mathcal{B}_\alpha, \tau)$ if b is a non-limit ordinal. Hence it is sufficient to consider the case when b is a limit ordinal. Suppose to the contrary that (a, b) is a non-isolated point in $(\mathcal{B}_\alpha, \tau)$. Since $(a, b)(b, 0) = (a, 0)$, the separate continuity of multiplication in $(\mathcal{B}_\alpha, \tau)$, Lemmas 1 and 2 imply that there exists an open neighborhood $V((a, b))$ of (a, b) satisfying conditions of Lemma 2 such that $V((a, b))(b, 0) = \{(a, 0)\}$. Fix an arbitrary element $(c, d) \in V(a, b) \setminus \{(a, b)\}$. Then $(c, d)(b, 0) = (c + (b - d), 0) = (a, 0)$. Hence $a = c + (b - d)$, but since b is a limit ordinal we have that $b - d$ is also a limit ordinal. Then $c + (b - d)$ is a limit ordinal which contradicts to the assumption that a is a non-limit ordinal. Proof of the statement that (b, a) is an isolated point in $(\mathcal{B}_\alpha, \tau)$ is similar.

Theorem 1. *For each $\alpha < \omega + 1$ every locally compact topological α-bicyclic semigroup $(\mathcal{B}_\alpha, \tau)$ is discrete.*

Proof. Lemmas 3 and 4 imply that if each idempotent (ω^a, ω^a) of the $(\mathcal{B}_\alpha, \tau)$ is an isolated point then τ is discrete.

It is obvious that the subset $\{(n, m) : n, m < \omega\} \cup \{\omega, \omega\}$ with the semigroup operation induced from $\mathcal{B}_\alpha$ is isomorphic to the bicyclic semigroup with adjoined zero. Then by Lemma 2 and [11, Corollary 1], (ω, ω) is an isolated point in $(\mathcal{B}_\alpha, \tau)$.

Suppose $(\mathcal{B}_\alpha, \tau)$ is a non-discrete semigroup. Let m be the smallest positive integer such that (ω^m, ω^m) is a non isolated idempotent of $(\mathcal{B}_\alpha, \tau)$. We remark that by the our assumption Lemmas 3 and 4 imply that $\{(a, b) : a, b < \omega^m\}$ is discrete subsemigroup of $(\mathcal{B}_\alpha, \tau)$ which is algebraically isomorphic to $\mathcal{B}_m$. By Lemma 2 there exists a clopen compact neighborhood $W((\omega^m, \omega^m))$ of (ω^m, ω^m) such that $W((\omega^m, \omega^m)) \subset \mathcal{B}_m \cup \{(\omega^m, \omega^m)\}$. The continuity of the multiplication in the $(\mathcal{B}_\alpha, \tau)$ implies that there exists a compact open neighborhood $V((\omega^m,\omega^m)) \subseteq W((\omega^m,\omega^m))$ of (ω^m,ω^m) such that $(0,0) \notin V^2((\omega^m,\omega^m))$. Since $V((\omega^m,\omega^m))$ is compact and (ω^m,ω^m) is the only non-isolated point in $V((\omega^m,\omega^m))$, we have that (ω^m,ω^m) is a limit point of every infinite sequence $x_n \in V((\omega^m,\omega^m))$ consisting of mutually distinct elements.

For an arbitrary element $(y,0) \in \mathcal{B}_m$ put

$$X_{(y,0)} = \{(0,x) : (y,0)(0,x) = (y,x) \in V\}.$$

Suppose that there exists an element $(y,0) \in \mathcal{B}_m$ for which the set $X_{(y,0)}$ is infinite. Let $X_{(y,0)} = \{(0,x_k) : k \in \mathbb{N}\}$ be an enumeration of the set $X_{(y,0)}$. Observe that

$$(\omega^m,\omega^m) = \lim_{k\to\infty}((y,0)(0,x_k)) = \lim_{k\to\infty}(y,x_k).$$

Then for every ordinal $z < \omega^m$

$$\lim_{k\to\infty}(z,x_k) = (\omega^m,\omega^m),$$

because

$$(\omega^m,\omega^m) = (z,y)(\omega^m,\omega^m) = (z,y)\lim_{k\to\infty}(y,x_k) = \lim_{k\to\infty}((z,y)(y,x_k)) = \lim_{k\to\infty}(z,x_k)$$

For every $k \in \mathbb{N}$ let c_k be the smallest ordinal such that $(x_k,c_k) \in V((\omega^m,\omega^m))$. Since $(0,0) \notin V^2((\omega^m,\omega^m))$ we have that there exists $k_0 \in \mathbb{N}$ such that for every $k > k_0$ $c_k \neq 0$. Observe that (ω^m,ω^m) is a zero of $\mathcal{B}_m$. The continuity of the multiplication in $(\mathcal{B}_\alpha,\tau)$ implies that there exists an open compact neighborhood $O((\omega^m,\omega^m)) \subseteq V((\omega^m,\omega^m))$ such that

$$O((\omega^m,\omega^m))(1,0) \cup O((\omega^m,\omega^m))(\omega,0) \cup .. \cup\ O((\omega^m,\omega^m))(\omega^{m-1},0) \subseteq V((\omega^m,\omega^m)).$$

But then an infinite discrete space $\{(x_k,c_k) : k \in \mathbb{N}\} \subset V((\omega^m,\omega^m)) \setminus O((\omega^m,\omega^m))$, which contradicts the compactness of $V((\omega^m,\omega^m))$. The obtained contradiction implies that $X_{(y,0)}$ is a finite set for every element $(y,0) \in \mathcal{B}_m$.

Since $V((\omega^m,\omega^m))$ is infinite there exist an infinite set $A = \{(y_n,0) : n \in \mathbb{N}\} \subset \mathcal{B}_m$ such that $X_{(y_n,0)} \neq \emptyset$. For an arbitrary element $(y_n,0) \in A$ by $(0,z_{y_n})$ we denote an element of $X_{(y_n,0)}$ which has the greatest second coordinate. Clearly that $\{(y_n,0)(0,z_{y_n}) = (y_n,z_{y_n})\}$ is infinite sequence of $V((\omega^m,\omega^m))$. Then we have that

$$\lim_{n\to\infty}(y_n,z_{y_n}) = (\omega^m,\omega^m).$$

But

$$\begin{aligned}(\omega^m,\omega^m) &= (\omega^m,\omega^m)(0,\omega^{m-1}) = \lim_{n\to\infty}(y_n,z_{y_n})(0,\omega^{m-1}) = \\ &= \lim_{n\to\infty}(y_n,\omega^{m-1}+z_{y_n}) \neq (\omega^m,\omega^m),\end{aligned}$$

because $\omega^{m-1}+z_{y_n} > z_{y_n}$ which contradicts the continuity of the multiplication in $(\mathcal{B}_\alpha,\tau)$. Hence $(\mathcal{B}_\alpha,\tau)$ is a discrete semigroup.

However the following example shows that there exists a non-discrete locally compact topology τ_{lc} on the $\mathcal{B}_{\omega+1}$ such that $(\mathcal{B}_{\omega+1},\tau_{lc})$ is a topological inverse semigroup.

Example 1. We define the topology τ_{lc} in the following way: all points distinct from $(n\omega^\omega, m\omega^\omega)$ for some positive integers n, m are isolated, and the family $\mathscr{B}((n\omega^\omega, m\omega^\omega)) = \{U_k((n\omega^\omega, m\omega^\omega)) : k \in \mathbb{N}\}$ forms a base of the topology τ_{lc} at the point $(n\omega^\omega, m\omega^\omega)$, where

$$U_k((n\omega^\omega, m\omega^\omega)) = \{((n-1)\omega^\omega + \omega^t, (m-1)\omega^\omega + \omega^t) : t > k\} \cup \{(n\omega^\omega, m\omega^\omega)\}.$$

Clearly that τ_{lc} is a Hausdorff topology on $\mathcal{B}_{\omega+1}$. Since every open basic neighborhood of an arbitrary non isolated point $(n\omega^\omega, m\omega^\omega)$ is compact we have that τ_{lc} is locally compact.

It is obvious that for every positive integer k

$$inv(U_k(n\omega^\omega, m\omega^\omega)) = U_k(m\omega^\omega, n\omega^\omega).$$

Hence the inversion is continuous in $(\mathcal{B}_{\omega+1}, \tau_{lc})$.

Let $a < \omega^{\omega+1}$ be an arbitrary ordinal. Below for us will be useful the following trivial modification of the Cantor's normal form of the ordinal a:

$$a = n_1\omega^\omega + n_2\omega^{t_2} + .. + n_p\omega^{t_p},$$

where n_1 is a non negative integer, $n_2, .., n_p$ are positive integers and $\omega, t_2, t_3, .., t_p$ is a decreasing sequence of ordinals.

It is sufficient to check the continuity of the multiplication at a point $((a, b), (c, d)) \in \mathcal{B}_{\omega+1} \times \mathcal{B}_{\omega+1}$ when at least one of the points (a, b) or (c, d) is non-isolated in $(\mathcal{B}_{\omega+1}, \tau_{lc})$. Hence there are three cases to consider:

(1) $(n_1\omega^\omega, m_1\omega^\omega)(n\omega^\omega, m\omega^\omega)$, where $n, m, n_1, m_1 \in \mathbb{N}$;
(2) $(a, b)(n\omega^\omega, m\omega^\omega)$, where (a, b) is an isolated point in $(\mathcal{B}_{\omega+1}, \tau_{lc})$, $n, m \in \mathbb{N}$;
(3) $(n\omega^\omega, m\omega^\omega)(a, b)$, where (a, b) is an isolated point in $(\mathcal{B}_{\omega+1}, \tau_{lc})$, $n, m \in \mathbb{N}$;

Suppose the first case holds, then we have the multiplication of the form

$$(n_1\omega^\omega, m_1\omega^\omega)(n\omega^\omega, m\omega^\omega).$$

It has the following three subcases:

(1.1) if $m_1 < n$ then $(n_1\omega^\omega, m_1\omega^\omega)(n\omega^\omega, m\omega^\omega) = ((n_1 + n - m_1)\omega^\omega, m\omega^\omega)$;
(1.2) if $m_1 = n$ then $(n_1\omega^\omega, m_1\omega^\omega)(n\omega^\omega, m\omega^\omega) = (n_1\omega^\omega, m\omega^\omega)$;
(1.3) if $m_1 > n$ then $(n_1\omega^\omega, m_1\omega^\omega)(n\omega^\omega, m\omega^\omega) = (n_1\omega^\omega, (m + m_1 - n)\omega^\omega)$.

Let's consider the subcase (1.1). Let $U_k(((n_1 + n - m_1)\omega^\omega, m\omega^\omega))$ be a basic open neighborhood of $((n_1 + n - m_1)\omega^\omega, m\omega^\omega)$. Then we state that

$$U_k((n_1\omega^\omega, m_1\omega^\omega))U_k((n\omega^\omega, m\omega^\omega)) \subseteq U_k(((n_1 + n - m_1)\omega^\omega, m\omega^\omega)).$$

Indeed, fix any elements

$$((n_1 - 1)\omega^\omega + \omega^t, (m_1 - 1)\omega^\omega + \omega^t) \in U_k((n_1\omega^\omega, m_1\omega^\omega))$$

and

$$((n - 1)\omega^\omega + \omega^p, (m - 1)\omega^\omega + \omega^p) \in U_k((n\omega^\omega, m\omega^\omega)).$$

Then

$$\begin{aligned}
&((n_1-1)\omega^\omega+\omega^t,(m_1-1)\omega^\omega+\omega^t)((n-1)\omega^\omega+\omega^p,(m-1)\omega^\omega+\omega^p)=\\
&((n_1-1)\omega^\omega+\omega^t+((n-1)\omega^\omega+\omega^p-((m_1-1)\omega^\omega+\omega^t)),(m-1)\omega^\omega+\omega^p)=\\
&((n_1-1)\omega^\omega+\omega^t+((n-1-m_1+1)\omega^\omega+\omega^p),(m-1)\omega^\omega+\omega^p)=\\
&((n_1-1+n-1-m_1+1)\omega^\omega+\omega^p,(m-1)\omega^\omega+\omega^p)=\\
&((n_1+n-m_1-1)\omega^\omega+\omega^p,(m-1)\omega^\omega+\omega^p)\in U_k(((n_1+n-m_1)\omega^\omega,m\omega^\omega)).
\end{aligned}$$

Let's consider the subcase (1.2). We have that $m_1=n$. Let $U_k((n_1\omega^\omega,m\omega^\omega))$ be a basic open neighborhood of $(n_1\omega^\omega,m\omega^\omega)$. Then we state that

$$U_k((n_1\omega^\omega,n\omega^\omega))U_k((n\omega^\omega,m\omega^\omega))\subseteq U_k((n_1\omega^\omega,m\omega^\omega)).$$

Indeed, fix any elements

$$((n_1-1)\omega^\omega+\omega^t,(n-1)\omega^\omega+\omega^t)\in U_k((n_1\omega^\omega,n\omega^\omega))$$

and

$$((n-1)\omega^\omega+\omega^p,(m-1)\omega^\omega+\omega^p)\in U_k((n\omega^\omega,m\omega^\omega)).$$

If $p>t$ then

$$\begin{aligned}
&((n_1-1)\omega^\omega+\omega^t,(n-1)\omega^\omega+\omega^t)((n-1)\omega^\omega+\omega^p,(m-1)\omega^\omega+\omega^p)=\\
&((n_1-1)\omega^\omega+\omega^t+((n-1)\omega^\omega+\omega^p-((n-1)\omega^\omega+\omega^t)),(m-1)\omega^\omega+\omega^p)=\\
&((n_1-1)\omega^\omega+\omega^t+(\omega^p-\omega^t),(m-1)\omega^\omega+\omega^p)=\\
&((n_1-1)\omega^\omega+\omega^p,(m-1)\omega^\omega+\omega^p)\in U_k((n_1\omega^\omega,m\omega^\omega)).
\end{aligned}$$

If $p=t$ then we have the following:

$$\begin{aligned}
&((n_1-1)\omega^\omega+\omega^p,(n-1)\omega^\omega+\omega^p)((n-1)\omega^\omega+\omega^p,(m-1)\omega^\omega+\omega^p)=\\
&((n_1-1)\omega^\omega+\omega^p,(m-1)\omega^\omega+\omega^p)\in U_k((n_1\omega^\omega,m\omega^\omega)).
\end{aligned}$$

If $p<t$ then

$$\begin{aligned}
&((n_1-1)\omega^\omega+\omega^t,(n-1)\omega^\omega+\omega^t)((n-1)\omega^\omega+\omega^p,(m-1)\omega^\omega+\omega^p)=\\
&((n_1-1)\omega^\omega+\omega^t,(m-1)\omega^\omega+\omega^p+((n-1)\omega^\omega+\omega^t-((n-1)\omega^\omega+\omega^p)))=\\
&((n_1-1)\omega^\omega+\omega^t,(m-1)\omega^\omega+\omega^p+(\omega^t-\omega^p))=\\
&((n_1-1)\omega^\omega+\omega^t,(m-1)\omega^\omega+\omega^t)\in U_k((n_1\omega^\omega,m\omega^\omega)).
\end{aligned}$$

Let's consider the subcase (1.3). In this subcase we have that $m_1>n$.
Let $U_k((n_1\omega^\omega,(m+m_1-n)\omega^\omega))$ be a basic open neighborhood of $(n_1\omega^\omega,(m+m_1-n)\omega^\omega)$. Then we state that

$$U_k((n_1\omega^\omega,m_1\omega^\omega))U_k((n\omega^\omega,m\omega^\omega))\subseteq U_k((n_1\omega^\omega,(m+m_1-n)\omega^\omega)).$$

Indeed, fix any elements

$$((n_1-1)\omega^\omega+\omega^t,(m_1-1)\omega^\omega+\omega^t)\in U_k((n_1\omega^\omega,m_1\omega^\omega))$$

and

$$((n-1)\omega^\omega+\omega^p,(m-1)\omega^\omega+\omega^p)\in U_k((n\omega^\omega,m\omega^\omega)).$$

Then

$$\begin{aligned}
&((n_1-1)\omega^\omega+\omega^t,(m_1-1)\omega^\omega+\omega^t)((n-1)\omega^\omega+\omega^p,(m-1)\omega^\omega+\omega^p)=\\
&((n_1-1)\omega^\omega+\omega^t,(m-1)\omega^\omega+\omega^p+((m_1-1)\omega^\omega+\omega^t-((n-1)\omega^\omega+\omega^p)))=\\
&((n_1-1)\omega^\omega+\omega^t,(m-1)\omega^\omega+\omega^p+((m_1-1-n+1)\omega^\omega+\omega^t))=\\
&((n_1-1)\omega^\omega+\omega^t,(m+m_1-n-1)\omega^\omega+\omega^t)\in U_k((n_1\omega^\omega,(m+m_1-n)\omega^\omega)).
\end{aligned}$$

Hence the continuity of the multiplication in $(\mathcal{B}_{\omega+1},\tau_{lc})$ holds in the case (1).

Let's consider the case (2). It has the following three subcases:

(2.1) $a\neq n_1\omega^\omega$ and $b\neq m_1\omega^\omega$;
(2.2) $a\neq n_1\omega^\omega$ and $b=m_1\omega^\omega$;
(2.3) $a=n_1\omega^\omega$ and $b\neq m_1\omega^\omega$.

Let's consider the subcase (2.1) Let $a=n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p}$ and $b=m_1\omega^\omega+m_2\omega^{r_2}+..+m_c\omega^{r_c}$, (note that n_1 and m_1 could be equal to 0). Then we have the following two subcases:

(2.1.1) if $m_1<n$ then

$$(n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p},m_1\omega^\omega+m_2\omega^{r_2}+..+m_c\omega^{r_c})(n\omega^\omega,m\omega^\omega)=((n_1+n-m_1)\omega^\omega,m\omega^\omega);$$

(2.1.2) if $m_1\geqslant n$ then

$$\begin{aligned}
&(n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p},m_1\omega^\omega+m_2\omega^{r_2}+..+m_c\omega^{r_c})(n\omega^\omega,m\omega^\omega)=\\
&=(n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p},(m+m_1-n)\omega^\omega+m_2\omega^{r_2}+..+m_c\omega^{r_c});
\end{aligned}$$

Let's prove the continuity in the subcase (2.1.1). Let $U_k(((n_1+n-m_1)\omega^\omega,m\omega^\omega))$ be a basic open neighborhood of $((n_1+n-m_1)\omega^\omega,m\omega^\omega)$. Note that

$$(n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p},m_1\omega^\omega+m_2\omega^{r_2}+..+m_c\omega^{r_c})$$

is an isolated point in $(\mathcal{B}_{\omega+1},\tau_{lc})$. Then we state that

$$\begin{aligned}
&(n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p},m_1\omega^\omega+m_2\omega^{r_2}+..+m_c\omega^{r_c})U_{r_2+t_2+k}((n\omega^\omega,m\omega^\omega))\subseteq\\
&\subseteq U_k(((n_1+n-m_1)\omega^\omega,m\omega^\omega)).
\end{aligned}$$

Indeed, fix any element

$$((n-1)\omega^\omega+\omega^t,(m-1)\omega^\omega+\omega^t)\in U_{r_2+t_2+k}((n\omega^\omega,m\omega^\omega)).$$

Then

$$\begin{aligned}
&(n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p},m_1\omega^\omega+m_2\omega^{r_2}+..+m_c\omega^{r_c})((n-1)\omega^\omega+\omega^t,(m-1)\omega^\omega+\\
&+\omega^t)=(n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p}+((n-1)\omega^\omega+\omega^t-(m_1\omega^\omega+m_2\omega^{r_2}+..\\
&+m_c\omega^{r_c})),(m-1)\omega^\omega+\omega^t)=\\
&=(n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p}+((n-1-m_1)\omega^\omega+\omega^t,(m-1)\omega^\omega+\omega^t)=\\
&=((n_1+n-m_1-1)\omega^\omega+\omega^t,(m-1)\omega^\omega+\omega^t)\in U_k(((n_1+n-m_1)\omega^\omega,m\omega^\omega)).
\end{aligned}$$

Hence the continuity of the multiplication in the subcase (2.1.1) holds.

Let's consider the subcase (2.1.2). Note that both $(n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p}, m_1\omega^\omega+m_2\omega^{r_2}+..+m_c\omega^{r_c})$ and $(n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p}, (m+m_1-n)\omega^\omega+m_2\omega^{r_2}+..+m_c\omega^{r_c})$ are isolated points in the $(\mathcal{B}_{\omega+1}, \tau_{lc})$. Then we state that

$$(n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p}, m_1\omega^\omega+m_2\omega^{r_2}+..+m_c\omega^{r_c})U_0((n\omega^\omega, m\omega^\omega)) =$$
$$= \{(n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p}, (m+m_1-n)\omega^\omega+m_2\omega^{r_2}+..+m_c\omega^{r_c})\}.$$

Indeed, fix any element

$$((n-1)\omega^\omega+\omega^t, (m-1)\omega^\omega+\omega^t) \in U_0((n\omega^\omega, m\omega^\omega)).$$

Then

$$(n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p}, m_1\omega^\omega+m_2\omega^{r_2}+..+m_c\omega^{r_c})((n-1)\omega^\omega+\omega^t, (m-1)\omega^\omega+$$
$$+\omega^t) = (n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p}, (m-1)\omega^\omega+\omega^t+(m_1\omega^\omega+m_2\omega^{r_2}+..+m_c\omega^{r_c}-$$
$$-((n-1)\omega^\omega+\omega^t))) =$$
$$(n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p}, (m-1)\omega^\omega+\omega^t+((m_1-n+1)\omega^\omega+m_2\omega^{r_2}+..+m_c\omega^{r_c}) =$$
$$= (n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p}, (m+m_1-n)\omega^\omega+m_2\omega^{r_2}+..+m_c\omega^{r_c}).$$

Hence the continuity of the multiplication in the subcase (2.1) holds.

Let's consider the subcase (2.2). Let $a = n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p}$ and $b = m_1\omega^\omega$. Then we have the following two subcases:

(2.2.1) if $m_1 < n$ then

$$(n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p}, m_1\omega^\omega)(n\omega^\omega, m\omega^\omega) = ((n_1+n-m_1)\omega^\omega, m\omega^\omega);$$

(2.2.2) if $m_1 \geqslant n$ then

$$(n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p}, m_1\omega^\omega)(n\omega^\omega, m\omega^\omega) =$$
$$= (n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p}, (m+m_1-n)\omega^\omega);$$

Let's consider the subcase (2.2.1). Let $U_k(((n_1+n-m_1)\omega^\omega, m\omega^\omega))$ be a basic open neighborhood of $((n_1+n-m_1)\omega^\omega, m\omega^\omega)$. Note that $(n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p}, m_1\omega^\omega)$ is an isolated point in the $(\mathcal{B}_{\omega+1}, \tau_{lc})$. Then we state that

$$(n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p}, m_1\omega^\omega)U_{t_2+k}((n\omega^\omega, m\omega^\omega)) \subseteq$$
$$\subseteq U_k(((n_1+n-m_1)\omega^\omega, m\omega^\omega)).$$

Indeed, fix any element

$$((n-1)\omega^\omega+\omega^t, (m-1)\omega^\omega+\omega^t) \in U_{t_2+k}((n\omega^\omega, m\omega^\omega)).$$

Then

$$(n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p}, m_1\omega^\omega)((n-1)\omega^\omega+\omega^t, (m-1)\omega^\omega+\omega^t) =$$
$$= (n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p}+((n-1)\omega^\omega+\omega^t-m_1\omega^\omega), (m-1)\omega^\omega+\omega^t) =$$
$$= (n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p}+((n-1-m_1)\omega^\omega+\omega^t), (m-1)\omega^\omega+\omega^t) =$$
$$= ((n_1+n-m_1-1)\omega^\omega+\omega^t, (m-1)\omega^\omega+\omega^t) \in U_k(((n_1+n-m_1)\omega^\omega, m\omega^\omega)).$$

Hence the continuity of the multiplication in the subcase (2.2.1) holds.

Let's consider the subcase (2.2.2). Note that both $(n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p},m_1\omega^\omega)$ and $(n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p},(m+m_1-n)\omega^\omega)$ are isolated points in the $(\mathcal{B}_{\omega+1},\tau_{lc})$. Then we state that

$$\begin{aligned}&(n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p},m_1\omega^\omega)U_0((n\omega^\omega,m\omega^\omega))=\\&=\{(n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p},(m+m_1-n)\omega^\omega)\}.\end{aligned}$$

Indeed, fix any element

$$((n-1)\omega^\omega+\omega^t,(m-1)\omega^\omega+\omega^t)\in U_0((n\omega^\omega,m\omega^\omega)).$$

Then

$$\begin{aligned}&(n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p},m_1\omega^\omega)((n-1)\omega^\omega+\omega^t,(m-1)\omega^\omega+\omega^t)=\\&=(n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p},(m-1)\omega^\omega+\omega^t+(m_1\omega^\omega-((n-1)\omega^\omega+\omega^t)))=\\&=(n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p},(m-1)\omega^\omega+\omega^t+((m_1-n+1)\omega^\omega)=\\&=(n_1\omega^\omega+n_2\omega^{t_2}+..+n_p\omega^{t_p},(m+m_1-n)\omega^\omega).\end{aligned}$$

Hence the continuity of the multiplication in the subcase (2.2) holds.

Let's consider the subcase (2.3). Let $a=n_1\omega^\omega$ and $b=m_1\omega^\omega+m_2\omega^{r_2}+..+m_c\omega^{r_c}$.

Then we have the following two subcases:

(2.3.1) if $m_1<n$ then

$$(n_1\omega^\omega,m_1\omega^\omega+m_2\omega^{r_2}+..+m_c\omega^{r_c})(n\omega^\omega,m\omega^\omega)=((n_1+n-m_1)\omega^\omega,m\omega^\omega);$$

(2.3.2) if $m_1\geqslant n$ then

$$\begin{aligned}&(n_1\omega^\omega,m_1\omega^\omega+m_2\omega^{r_2}+..+m_c\omega^{r_c})(n\omega^\omega,m\omega^\omega)=\\&=(n_1\omega^\omega,(m+m_1-n)\omega^\omega+m_2\omega^{r_2}+..+m_c\omega^{r_c});\end{aligned}$$

Let's consider the subcase (2.3.1). Let $U_k(((n_1+n-m_1)\omega^\omega,m\omega^\omega))$ be a basic open neighborhood of $((n_1+n-m_1)\omega^\omega,m\omega^\omega)$. Then we state that

$$\begin{aligned}&(n_1\omega^\omega,m_1\omega^\omega+m_2\omega^{r_2}+..+m_c\omega^{r_c})U_{r_2+k}((n\omega^\omega,m\omega^\omega))\subseteq\\&\subseteq U_k(((n_1+n-m_1)\omega^\omega,m\omega^\omega)).\end{aligned}$$

Indeed, fix any element

$$((n-1)\omega^\omega+\omega^t,(m-1)\omega^\omega+\omega^t)\in U_{r_2+k}((n\omega^\omega,m\omega^\omega)).$$

Then

$$\begin{aligned}&(n_1\omega^\omega,m_1\omega^\omega+m_2\omega^{r_2}+..+m_c\omega^{r_c})((n-1)\omega^\omega+\omega^t,(m-1)\omega^\omega+\omega^t)=\\&=(n_1\omega^\omega+((n-1)\omega^\omega+\omega^t-(m_1\omega^\omega+m_2\omega^{r_2}+..+m_c\omega^{r_c})),(m-1)\omega^\omega+\omega^t)=\\&=(n_1\omega^\omega+((n-1-m_1)\omega^\omega+\omega^t),(m-1)\omega^\omega+\omega^t)=\\&=((n_1+n-m_1-1)\omega^\omega+\omega^t,(m-1)\omega^\omega+\omega^t)\in U_k(((n_1+n-m_1)\omega^\omega,m\omega^\omega)).\end{aligned}$$

Hence the continuity of the multiplication in the subcase (2.3.1) holds.

Now let's consider the subcase (2.3.2). Note that both $(n_1\omega^\omega,m_1\omega^\omega+m_2\omega^{r_2}+..+m_c\omega^{r_c})$ and $(n_1\omega^\omega,(m+m_1-n)\omega^\omega+m_2\omega^{r_2}+..+m_c\omega^{r_c})$ are isolated points in the

$(\mathcal{B}_{\omega+1}, \tau_{lc})$. Then we state that

$$(n_1\omega^\omega, m_1\omega^\omega + m_2\omega^{r_2} + .. + m_c\omega^{r_c})U_0((n\omega^\omega, m\omega^\omega)) =$$
$$= \{(n_1\omega^\omega, (m + m_1 - n)\omega^\omega + m_2\omega^{r_2} + .. + m_c\omega^{r_c})\}.$$

Indeed, fix any element

$$((n-1)\omega^\omega + \omega^t, (m-1)\omega^\omega + \omega^t) \in U_0((n\omega^\omega, m\omega^\omega)).$$

Then

$$(n_1\omega^\omega, m_1\omega^\omega + m_2\omega^{r_2} + .. + m_c\omega^{r_c})((n-1)\omega^\omega + \omega^t, (m-1)\omega^\omega + \omega^t) =$$
$$= (n_1\omega^\omega, (m-1)\omega^\omega + \omega^t + (m_1\omega^\omega + m_2\omega^{r_2} + .. + m_c\omega^{r_c} - ((n-1)\omega^\omega + \omega^t))) =$$
$$= (n_1\omega^\omega, (m-1)\omega^\omega + \omega^t + ((m_1 - n + 1)\omega^\omega + m_2\omega^{r_2} + ... + m_c\omega^{r_c}) =$$
$$= (n_1\omega^\omega, (m + m_1 - n)\omega^\omega + m_2\omega^{r_2} + .. + m_c\omega^{r_c}).$$

Hence the continuity of the multiplication holds in the case (2).

Since the inversion is continuous in $(\mathcal{B}_{\omega+1}, \tau_{lc})$ and

$$((a, b)(n\omega^\omega, m\omega^\omega))^{-1} = (m\omega^\omega, n\omega^\omega)(b, a)$$

the case (2) implies that the semigroup operation in the case (3) is continuous too.

Hence $(\mathcal{B}_{\omega+1}, \tau_{lc})$ is a topological inverse semigroup.

Acknowledgements

The author acknowledges Oleg Gutik for his comments and suggestions.

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Стаття: надійшла до редколегії 31.05.2016
прийнята до друку 08.06.2016

ПРО НАПІВТОПОЛОГІЧНИЙ α-БІЦИКЛІЧНИЙ МОНОЇД

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Досліджено напівтопологічний α-біциклічний моноїд $\mathcal{B}_\alpha$. Доведено, що $\mathcal{B}_\alpha$ — алгебрично ізоморфний напівгрупі усіх порядкових ізоморфізмів між верхніми множинами ординалу ω^α. Також доведено, що для довільного ординалу α, для довільного елемента $(a, b) \in \mathcal{B}_\alpha$ з того, що a або b не є граничним ординалом, випливає, що (a, b) є ізольованою точкою в $\mathcal{B}_\alpha$. З'ясовано, що для довільно ординалу $\alpha < \omega+1$ кожна локально компактна напівгрупова топологія на $\mathcal{B}_\alpha$ є дискретною і побудований приклад недискретної локально компактної топології τ_{lc} на $\mathcal{B}_{\omega+1}$ такої, що $(\mathcal{B}_{\omega+1}, \tau_{lc})$ є топологічною інверсною напівгрупою. Цей приклад засвідчує, що є помилка в [16, Теоремі 2.9], де стверджується, що для довільного ординалу α існує лише дискретна локально компактна топологія на $\mathcal{B}_\alpha$, яка перетворює $\mathcal{B}_\alpha$ в топологічну інверсну напівгрупу.

Ключові слова: топологічна інверсна напівгрупа, топологічна напівгрупа, напівтопологічна напівгрупа, α-біциклічний моноїд.