ISSN 2078-3744. Вiсник Львiв. ун-ту. Серiя мех.-мат. 2016. Випуск 82. С. 75–93
Visnyk of the Lviv Univ. Series Mech. Math. 2016. Issue 82. P. 75–93

УДК 517.956.4; 517.977.5

OPTIMAL CONTROL IN PROBLEMS WITHOUT INITIAL CONDITIONS FOR WEAKLY NONLINEAR EVOLUTION VARIATIONAL INEQUALITIES

Mykola BOKALO, Andrii TSEBENKO

Ivan Franko National University of Lviv,
1, Universytetska St., 79000, Lviv
e-mail: mm.bokalo@gmail.com, amtseb@gmail.com

An optimal control problem for systems described by Fourier problem (problem without initial conditions) for weakly nonlinear evolution variational inequalities is studied. A control function occurs in the coefficients of the variational inequality which describes the state of control system. Different types of observation are considered. The existence of the optimal control is proved.

Key words: optimal control, problem without initial conditions, variational inequality.

1. Introduction

Optimal control problems for systems governed by variational inequalities are quite popular nowadays. A large number of such problems were considered in the monograph [3] and other publications (see, e.g., [1, 10, 16, 17]).

In particular, in [1] an optimal control problem for a parabolic variational inequality is considered. Existence and necessary conditions for the optimal control are established.

In [16] the optimal control of parabolic variational inequalities is studied in the case where the spatial domain is not necessarily bounded. An optimal control problem with the control appearing in the coefficient of the leading term is investigated and a first order optimality system in a Lagrangian framework is derived. In [17] the author proves an existence result for optimal control problem in coefficients of a nonlinear elliptic variational inequality using the direct method of calculus of variation and the compensated compactness lemma.

In this paper, we study an optimal control problem for systems whose states are described by problems without initial conditions for evolutionary variational inequalities. A particular case of the problem for the evolution variational inequalities is a problem for evolutionary equations. The research of the problem without initial conditions for the evolution equations and variational inequalities were conducted in the papers [9, 13, 15,

---

2010 Mathematics Subject Classification: 26D10, 49J40, 49J21, 47J20, 47J22

18, 19, 20, 22, 26] and others. In particular, R.E. Showalter [25] proved the existence of a unique solution $u \in \mathrm{e}^{2\omega\cdot}W^{1,2}(-\infty,0;H)$, where H is a Hilbert space, of the problem without initial condition

$$u'(t) + \mu u(t) + A\big(u(t)\big) \ni f(t), \quad t \in (-\infty, 0),$$

for every $\omega+\mu > 0$ and $f \in \mathrm{e}^{2\omega\cdot}W^{1,2}(-\infty,0;H)$, in case when $A : H \to 2^H$ is maximal monotone operator such that $0 \in A(0)$. Moreover, if $A = \partial\varphi$, where $\varphi : H \to (-\infty,+\infty]$ is proper, convex, and lower-semi-continuous functional such that $\varphi(0) = 0 = \min\{\varphi(v) : v \in H\}$, then this problem is uniquely solvable for each $\mu > 0$, $f \in L^2(-\infty,0;H)$ and $\omega = 0$.

Note that the uniqueness of the solutions of such problem for linear parabolic equations and variational inequalities is possible only under some restrictions on the behavior of solutions when $t \to -\infty$. For the first time in the case of heat equation it was strictly justified by A.N. Tikhonov [27]. However, as it was shown by M.M. Bokalo [9], the problem without initial conditions for some nonlinear parabolic equations has a unique solution in the class of functions with arbitrary behavior when $t \to -\infty$. Similar results were also obtained for evolutionary variational inequalities in [9].

Previously, optimal control problems of evolution equations without initial conditions were studied by the authors (see., e.g., [8, 7]). But as far as we know, optimal control problems for variational inequalities without initial conditions were not considered yet, which serves as one of the motivations for the study of such problems.

The outline of this paper is as follows. In Section 1, we provide notations, definitions of function spaces and auxiliary results. In Section 2, we formulate the optimal control problem. In Section 3, we prove existence and uniqueness of the solutions of problem without initial conditions which describe the state of control system. Furthermore, we obtain estimates for the solutions of the state equations. Finally, the existence of the optimal control is presented in Section 4.

2. Preliminaries

Set $S := (-\infty, 0]$. Let V and H be separable Hilbert spaces with the scalar products $(\cdot,\cdot)_V$, $(\cdot,\cdot)$ and norms $\|\cdot\|$, $|\cdot|$, respectively. Suppose that $V \subset H$ with continuous injection and V is dense and compact in H, i.e., the closure of V in H coincides with H, and there exists a constant $\lambda > 0$ such that

$$\lambda|v|^2 \leqslant \|v\|^2 \quad \forall v \in V, \tag{1}$$

and for every bounded sequence $\{w_k\}_{k=1}^{\infty}$ in V there exist an element $w \in H$ and a subsequence $\{w_{k_j}\}_{j=1}^{\infty}$ of sequence $\{w_k\}_{k=1}^{\infty}$ such that $w_{k_j} \underset{j\to\infty}{\longrightarrow} w$ strongly in H.

Let V' and H' be the dual spaces to V and H, respectively. We suppose (after appropriate identification of functionals), that the space H' is a subspace of V'. Identifying (by the Riesz–Fréchet representation theorem) spaces H and H', we obtain continuous and dense embeddings

$$V \subset H \subset V'. \tag{2}$$

Note, that in this case $\langle g, v\rangle_V = (g, v)$ for every $v \in V, g \in H$, where $\langle\cdot,\cdot\rangle_V$ is the scalar product for the duality V', V. Therefore, further we use the notation $(\cdot,\cdot)$ instead of

$\langle\cdot,\cdot\rangle_V$. Also we use the notation $\|\cdot\|_*$ for the norm in V'. Note that

$$\lambda\|h\|_*^2 \leqslant |h|^2 \quad \forall h \in H, \tag{3}$$

where λ is the constant from the equality (1). Indeed, under (1) we have

$$\|h\|_* = \sup_{v\in V, \|v\|=1} |(h,v)| \leqslant \sup_{v\in V, \|v\|=1} |h||v| \leqslant \lambda^{-1/2}|h|.$$

We introduce some spaces of functions and distributions. Let X be an arbitrary Hilbert space with the scalar product $(\cdot,\cdot)_X$ and the norm $\|\cdot\|_X$. Under $C(S;X)$ we mean the linear space of continuous functions defined on S with values in X. We say that $z_m \underset{m\to\infty}{\longrightarrow} z$ in $C(S;X)$ if for each $t_1, t_2 \in S$ $(t_1 < t_2)$ we have $\|z - z_m\|_{C([t_1,t_2];X)} \underset{m\to\infty}{\longrightarrow} 0$.

Let $q \in [1,\infty]$, q' be dual to q, i.e., $1/q + 1/q' = 1$. Denote by $L^2_{\mathrm{loc}}(S;X)$ the linear space of measurable functions defined on S with values in X, whose restrictions to any segment $[t_1,t_2] \subset S$ belong to the space $L^q(t_1,t_2;X)$. We say that a sequence $\{z_m\}$ is bounded (respectively, strongly, weakly or $*$-weakly convergent to z) in $L^q_{\mathrm{loc}}(S;X)$, if for each $t_1, t_2 \in S$ $(t_1 < t_2)$ the sequence of restrictions of $\{z_m\}$ to the segment $[t_1,t_2]$ is bounded (respectively, strongly, weakly or $*$-weakly convergent to the restrictions of z to this segment) in $L^q(t_1,t_2;X)$.

Let $\nu \in \mathbb{R}$. Put by definition

$$L^2_\nu(S;X) := \Big\{ f \in L^2_{\mathrm{loc}}(S;X) \;\Big|\; \int_S e^{-2\nu t}\|f(t)\|_X^2 dt < \infty \Big\}.$$

This space is a Hilbert space with the scalar product

$$(f,g)_{L^2_\nu(S;X)} = \int_S e^{-2\nu t}(f(t),g(t))_X \, dt$$

and the corresponding norm

$$\|f\|_{L^2_\nu(S;X)} := \Big(\int_S e^{-2\nu t}\|f(t)\|_X^2 \, dt \Big)^{1/2}.$$

Also we introduce the space $L^\infty_\nu(S;X) := \{ f \in L^\infty(S;X) \mid \operatorname*{ess\,sup}_{t\in S} \big[e^{-\nu t}\|f(t)\|_X \big] < \infty \}$.

Under $D'(-\infty,0;V'_w)$ we mean the space of defined on $D(-\infty,0)$ with values in V' distributions, i.e., the space of continuous linear functionals on $D(-\infty,0)$ with values in V'_w (hereafter $D(-\infty,0)$ is the space of test functions, that is, the space of infinitely differentiable on $(-\infty,0)$ functions with compact support, equipped with corresponding topology, and V_w is the linear space V' equipped with weak topology). It is easy to see (using (2)), that the spaces $L^2_{\mathrm{loc}}(S;V)$, $L^2_{\mathrm{loc}}(S;H)$, $L^2_{\mathrm{loc}}(S;V')$ can be identified with the corresponding subspaces of $D'(-\infty,0;V'_w)$. This, in particular, allows us to talk about the derivatives z' of the functions z from $L^2_{\mathrm{loc}}(S;V)$ or $L^2_{\mathrm{loc}}(S;H)$ in the sense of distributions $D'(-\infty,0;V'_w)$ and belonging of such derivatives to $L^2_{\mathrm{loc}}(S;H)$ or $L^2_{\mathrm{loc}}(S;V')$.

Denote by $H^1_{\mathrm{loc}}(S;H)$ the space of functions $z \in L^2_{\mathrm{loc}}(S;H)$ such that $z' \in L^2_{\mathrm{loc}}(S;H)$. Let us define the space

$$W_{2,\mathrm{loc}}(S) := \{ z \in L^2_{\mathrm{loc}}(S;V) \,|\, z' \in L^2_{\mathrm{loc}}(S;V') \}. \tag{4}$$

From known results (see., for example, [14, P. 177-179]) it follows that $H^1_{\rm loc}(S;H) \subset C(S;H)$ and $W_{2,\rm loc}(S) \subset C(S;H)$. Moreover, for every z in $W_{2,\rm loc}(S)$ or in $H^1_{\rm loc}(S;H)$ function $t \to |z(t)|^2$ is absolutely continuous on any segment of the ray S and the following equality holds

$$\frac{d}{dt}|z(t)|^2 = 2(z'(t), z(t)) \quad \text{for a.e.} \quad t \in S. \tag{5}$$

Denote

$$H^1_\nu(S) := \{z \in L^2_\nu(S;H) \mid z' \in L^2_\nu(S;H)\}, \quad \nu \in \mathbb{R}. \tag{6}$$

In this paper we use the following well-known facts.

Proposition 1 (Cauchy-Schwarz inequality [14, p. 158]). *Suppose that $t_1, t_2 \in \mathbb{R}$ $(t_1 < t_2)$, and X is a Hilbert space with the scalar product $(\cdot,\cdot)_X$. Then, if $v \in L^2(t_1,t_2;X)$ and $w \in L^2(t_1,t_2;X)$, we have $(w(\cdot),v(\cdot))_X \in L^1\big(t_1,t_2\big)$ and*

$$\int_{t_1}^{t_2} (w(t),v(t))_X\, dt \leqslant \|w\|_{L^2(t_1,t_2;X)} \|v\|_{L^2(t_1,t_2;X)}.$$

Proposition 2 ([28, p. 173,179]). *Let X be a Banach space with the norm $\|\cdot\|_X$, and $\{v_k\}_{k=1}^\infty$ be the sequence of elements of X which is weakly or $*$-weakly convergent to v in X. Then $\varliminf_{k\to\infty} \|v_k\|_X \geqslant \|v\|_X$.*

Proposition 3 ([2, Aubin theorem], [4, p. 393]). *Suppose that $q > 1, r > 1$, $t_1,t_2 \in \mathbb{R}$ $(t_1 < t_2)$, and $\mathcal{W},\mathcal{L},\mathcal{B}$ are Banach spaces such that $\mathcal{W} \overset{c}{\subset} \mathcal{L} \circlearrowleft \mathcal{B}$ (here $\overset{c}{\subset}$ means compact embedding and $\circlearrowleft$ means continuous embedding). Then*

$$\{z \in L^q(t_1,t_2;\mathcal{W}) \mid z' \in L^r(t_1,t_2;\mathcal{B})\} \overset{c}{\subset} \Big(L^q(t_1,t_2;\mathcal{L}) \cap C([t_1,t_2];\mathcal{B})\Big). \tag{7}$$

Remark 1. We understand embedding (7) as follows: if a sequence $\{z_m\}$ is bounded in the space $L^q(t_1,t_2;\mathcal{W})$ and the sequence $\{z'_m\}_{m\in\mathbb{N}}$ is bounded in the space $L^r(t_1,t_2;\mathcal{B})$, then there exist a function $z \in C([t_1,t_2];\mathcal{B}) \cap L^q(t_1,t_2;\mathcal{L})$ and a subsequence $\{z_{m_j}\}$ of the sequence $\{z_m\}$ such that $z_{m_j} \underset{j\to\infty}{\longrightarrow} z$ in $C([t_1,t_2];\mathcal{B})$ and strongly in $L^q(t_1,t_2;\mathcal{L})$.

Proposition 4. *If a sequence $\{z_m\}$ is bounded in the space $L^2_{\rm loc}(S;V)$ and the sequence $\{z'_m\}$ is bounded in the space $L^2_{\rm loc}(S;H)$, then there exist a function $z \in L^2_{\rm loc}(S;V)$, $z' \in L^2_{\rm loc}(S;H)$ and a subsequence $\{z_{m_j}\}$ of the sequence $\{z_m\}$ such that $z_{m_j} \underset{j\to\infty}{\longrightarrow} z$ in $C(S;H)$ and weakly in $L^2_{\rm loc}(S;V)$, and $z'_{m_j} \underset{j\to\infty}{\longrightarrow} z'$ weakly in $L^2_{\rm loc}(S;H)$.*

Proof. Proposition 3 when $q = 2$, $r = 2$, $\mathcal{W} = V$, $\mathcal{L} = \mathcal{B} = H$ yields, for every $t_1,t_2 \in S$ $(t_1 < t_2)$ from the sequence of restrictions of the elements of $\{z_m\}$ to the segment $[t_1,t_2]$ one can choose subsequence which is convergent in $C([t_1,t_2];H)$ and weakly in $L^2(t_1,t_2;V)$, and the sequence of derivatives of elements of this subsequence is weakly convergent in $L^2(t_1,t_2;H)$. For each $k \in \mathbb{N}$ we choose a subsequence $\{z_{m(k,j)}\}_{j=1}^\infty$ of a given sequence, which is convergent in $C([-k,0];H)$ and weakly in $L^2(-k,0;V)$ to some function $\widehat{z}_k \in C([-k,0];H) \cap L^2(-k,0;V)$, and the sequence $\{z'_{m(k,j)}\}_{j=1}^\infty$ is weakly convergent to the derivative $\widehat{z}'_k$ in $L^2(-k,0;H)$. Making this choice we ensure that the sequence $\{z_{m(k+1,j)}\}_{j=1}^{\infty}$ be a subsequence of the sequence $\{z_{m(k,j)}\}_{j=1}^{\infty}$. Now, according to the diagonal process we select the desired subsequence as $\{z_{m(j,j)}\}_{j=1}^{\infty}$, and we define the function z as follows: for each $k \in \mathbb{N}$ we take $z(t) := \widehat{z}_k(t)$ for $t \in (-k, -k+1]$. □

Let $\Phi : V \to (-\infty, +\infty]$ be a proper functional, which satisfies the conditions:

$(\mathcal{A}_1)$**:** $\Phi\big(\alpha v + (1-\alpha)w\big) \leqslant \alpha\Phi(v) + (1-\alpha)\Phi(w) \quad \forall\, v, w \in V,\ \forall\, \alpha \in [0,1],$

i.e., the functional Φ is *convex*,

$(\mathcal{A}_2)$**:** $v_k \underset{k\to\infty}{\longrightarrow} v$ in $V \quad \Longrightarrow \quad \varliminf_{k\to\infty} \Phi(v_k) \geqslant \Phi(v),$

i.e., the functional Φ is *lower semicontinuous*.

Denote by $\mathrm{dom}(\Phi) := \{v \in V : \ \Phi(v) < +\infty\}$ the *effective domain* of the functional Φ.

Recall that the *subdifferential* of a functional Φ is a mapping $\partial\Phi : V \to 2^{V'}$, defined as follows

$$\partial\Phi(v) := \{v^* \in V' \mid \Phi(w) \geqslant \Phi(v) + (v^*, w - v) \quad \forall\, w \in V\}, \quad v \in V,$$

and the *domain* of the subdifferential $\partial\Phi$ is the set $D(\partial\Phi) := \{v \in V \,|\, \partial\Phi(v) \neq \varnothing\}$. We identify the subdifferential $\partial\Phi$ with its graph assuming that $[v, v^*] \in \partial\Phi$ if and only if $v^* \in \partial\Phi(v)$, i.e., $\partial\Phi = \{[v, v^*] \mid v \in D(\partial\Phi),\ v^* \in \partial\Phi(v))\}$. Rockafellar in [23, Theorem A] proves that the subdifferential $\partial\Phi$ is a *maximal monotone operator*, that is,

$$(v_1^* - v_2^*, v_1 - v_2) \geqslant 0 \quad \forall\, [v_1, v_1^*],\ [v_2, v_2^*] \in \partial\Phi$$

and for every element $[v_1, v_1^*] \in V \times V'$ we have the implication

$$(v_1^* - v_2^*, v_1 - v_2) \geqslant 0 \quad \forall\, [v_2, v_2^*] \in \partial\Phi \quad \Longrightarrow \quad [v_1, v_1^*] \in \partial\Phi.$$

Additionally, assume that the following conditions hold:

$(\mathcal{A}_3)$**:** there exist constant $K_1 > 0$ such that

$$\Phi(v) \geqslant K_1\|v\|^2 \quad \forall\, v \in \mathrm{dom}(\Phi);$$

moreover, $\Phi(0) = 0$;

$(\mathcal{A}_4)$**:** there exists a constant $K_2 > 0$ such that

$$(v_1^* - v_2^*, v_1 - v_2) \geqslant K_2|v_1 - v_2|^2 \quad \forall\, [v_1, v_1^*],\ [v_2, v_2^*] \in \partial\Phi.$$

Remark 2. Condition $(\mathcal{A}_3)$ implies that $\Phi(v) \geqslant \Phi(0) + (0, v-0)\ \forall v \in V$, hence $0 \in \partial\Phi(0)$. From this and condition $(\mathcal{A}_4)$ we have

$$(v^*, v) \geqslant K_2|v|^2 \quad \forall\, [v, v^*] \in \partial\Phi. \tag{8}$$

Let us consider the evolutionary variational inequality

$$y'(t) + \partial\Phi\big(y(t)\big) + u(t)y(t) \ni f(t), \quad t \in S, \tag{9}$$

where $f : S \to V'$ and $u : S \to \mathbb{R}$ are given measurable functions.

Definition 1. *Let conditions* $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ *hold and* $u \in L^\infty_{\mathrm{loc}}(S)$, $f \in L^2_{\mathrm{loc}}(S; V')$. *A function* y *is called a* solution *of variational inequality* (9), *if it satisfies the following conditions:*

1) $y \in W_{2,\mathrm{loc}}(S)$;

2) $y(t) \in D(\partial\Phi)$ *for a.e.* $t \in S$;

3) *there exists a function* $g \in L^2_{\rm loc}(S;V')$ *such that for a.e.* $t \in S$ *we have* $g(t) \in \partial\Phi\big(y(t)\big)$ *and*

$$y'(t) + g(t) + u(t)y(t) = f(t) \quad \text{in } V'.$$

For variational inequality (9) consider the problem: find its solution which satisfies the condition

$$\lim_{t\to-\infty} e^{-\gamma t}|y(t)| = 0, \tag{10}$$

where $\gamma \in \mathbb{R}$ is given.

The problem of finding a solution of variational inequality (9) for given Φ, u, f, satisfying the condition (10) for given γ, is called the problem without initial conditions for the evolution variational inequality (9) or, in short, the problem $\mathbf{P}(\Phi,u,f,\gamma)$, and the function y is called its solution.

Remark 3. The problem $\mathbf{P}(\Phi,u,f,\gamma)$ can be replaced by the following problem. Let K be a convex and closed set in V, $A : V \to V'$ be a monotone, bounded and semi-continuous operator such that $(A(v),v) \geqslant \widetilde{K}_1\|v\|^2 \quad \forall v \in V$, where $\widetilde{K}_1 = \text{const} > 0$. The problem is to find a function $y \in W_{2,\rm loc}(S)$, satisfying the condition (10) and for a.e. $t \in S$

$$y(t) \in K \quad \text{and} \quad (y'(t) + A(y(t)) + u(t)y(t), v - y(t)) \geqslant (f(t), v - y(t)) \quad \forall\, v \in K.$$

Theorem 1. *Let conditions* $(\mathcal{A}_1)$ – $(\mathcal{A}_4)$ *hold. Suppose that*

$$(\mathcal{F}): \quad -\infty < \widetilde{m} := \operatorname*{ess\,inf}_{t\in S} u(t) \leqslant \operatorname*{ess\,sup}_{t\in S} u(t) =: \widetilde{M} < +\infty, \quad f \in L^2_\gamma(S;H),$$

where $\gamma \in \mathbb{R}$ *is a constant which satisfies the inequality*

$$K_2 + \widetilde{m} + \gamma > 0. \tag{11}$$

Then the problem $\boldsymbol{P}(\Phi,u,f,\gamma)$ *has a unique solution, it belongs to the space* $L^\infty_\gamma(S;V) \cap L^2_\gamma(S;V) \cap H^1_\gamma(S;H)$ *and satisfies the estimate:*

$$e^{-2\gamma\tau}\|y(\tau)\|^2 + \int_{-\infty}^{\tau} e^{-2\gamma t}\|y(t)\|^2\,dt + \int_{-\infty}^{\tau} e^{-2\gamma t}|y'(t)|^2\,dt$$
$$\leqslant C_1 \int_{-\infty}^{\tau} e^{-2\gamma t}|f(t)|^2\,dt, \quad \tau \in S \tag{12}$$

where C_1 *is a positive constant which depends on* K_1, K_2, γ, λ *and* $\widetilde{m}$, $\widetilde{M}$ *only.*

The proof of this theorem is given in Section 3.

3. Statement of the main problem and results

Let U be a closed linear subspace of $L^\infty(S)$, for example, $U := L^\infty(S)$ or $U := \{u \in L^\infty(S) \mid u(t) = 0 \text{ for a.e. } t \in S \setminus [t^*, 0]\}$, where $t^* < 0$ is arbitrary fixed. Assume that U is the space of controls and for given constants $m, M \in \mathbb{R}$ the set $U_\partial := \Big\{u \in U \Bigm| m \leqslant u(t) \leqslant M \quad \text{for a.e.} \quad t \in S\Big\}$ is the set of admissible controls.

We assume that the state of the investigated evolutionary system $y(u) = y(\cdot;u)$ for a given control $u \in U_\partial$ is described by a solution of a problem $\mathbf{P}(\Phi,u,f,\gamma)$, when the following condition holds:

$(\mathcal{P})$ Φ satisfies conditions $(\mathcal{A}_1)$ – $(\mathcal{A}_4)$, $f \in L^2_\gamma(S;H)$ and

$$K_2 + m + \gamma > 0. \tag{13}$$

From Theorem 1 we infer that there exists a unique function $y(u) = y(t;u)$, $t \in S$ which is the solution of problem $\mathbf{P}(\Phi, u, f, \gamma)$, and this function belongs to the space $L^\infty_\gamma(S;V) \cap L^2_\gamma(S;V) \cap H^1_\gamma(S;H)$.

Let $G\colon C(S;H) \to \mathbb{R}$ be a functional which satisfies condition:

$(\mathcal{G})$ G is lower semi-continuous in $C(S;H)$ and, moreover, $\inf\limits_{z \in C(S;H)} G(z) > -\infty$.

We assume that the cost functional $J : U \to \mathbb{R}$ has the form

$$J(u) := G(y(u)) + \mu \|u\|^2_U, \quad u \in U, \tag{14}$$

where $\mu > 0$ is a constant.

We consider the following **optimal control problem**: find a control $u^* \in U_\partial$ such that

$$J(u^*) = \inf_{u \in U_\partial} J(u). \tag{15}$$

We briefly call this problem (15), and its solutions will be called the *optimal controls*.

The main result of this paper is the following theorem.

Theorem 2. *Let conditions* $(\mathcal{P})$ *and* $(\mathcal{G})$ *hold. Then problem* (15) *has a solution.*

The proof of this theorem is given in Section 4.

4. Well-posedness of the problem without initial conditions for weakly nonlinear variational inequality

We now turn to the question of existence and uniqueness of the solution of the problem $\mathbf{P}(\Phi, u, f, \gamma)$.

First, we define the functional $\Phi_H : H \to \mathbb{R}_\infty$ by the rule: $\Phi_H(v) := \Phi(v)$, if $v \in V$, and $\Phi_H(v) := +\infty$ otherwise. Note that conditions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, Lemma IV.5.2 and Proposition IV.5.2 of the monograph [24] imply that Φ_H is proper, convex, and lower-semi-continuous functional on H, $\mathrm{dom}(\Phi_H) = \mathrm{dom}(\Phi) \subset V$ and $\partial\Phi_H = \partial\Phi \cap (V \times H)$, where $\partial\Phi_H : H \to 2^H$ is the subdifferential of the functional Φ_H. Moreover, condition $(\mathcal{A}_3)$ yields $0 \in \partial\Phi_H(0)$.

Proposition 5 ([24, Lemma IV.4.3])**.** *Assume that* $z \in H^1(a,b;H)$ $(-\infty < a < b < +\infty)$*, and there exists* $g \in L^2(a,b;H)$ *such that* $g(t) \in \partial\Phi_H\big(z(t)\big)$ *for a.e.* $t \in (a,b)$*. Then the function* $\Phi_H\big(z(\cdot)\big)$ *is absolutely continuous on the interval* $[a,b]$ *and for any function* $h : [a,b] \to H$ *such that* $h(t) \in \partial\Phi_H\big(z(t)\big)$ *the following equality holds*

$$\frac{d}{dt}\Phi_H\big(z(t)\big) = (h(t), z'(t)) \quad \text{for a.e. } t \in (a,b).$$

Proposition 6 ([12, Proposition 3.12], [24, Proposition IV.5.2])**.** *Suppose that* $T > 0$, $\widetilde{f} \in L^2(0,T;H)$ *and* $z_0 \in \mathrm{dom}(\Phi)$*. Then there exists a unique function* $z \in H^1(0,T;H)$ *such that* $z(0) = z_0$ *and for a.e.* $t \in (0,T)$ *we have* $z(t) \in D(\partial\Phi_H)$ *and*

$$z'(t) + \partial\Phi_H\big(z(t)\big) \ni \widetilde{f}(t) \quad \text{in } H. \tag{16}$$

Proposition 7. *Suppose that* $T > 0$, $\widetilde{f} \in L^2(0,T;H), \widetilde{u} \in L^\infty(0,T)$ *and* $z_0 \in \mathrm{dom}(\Phi)$. *Then there exists a unique function* $z \in H^1(0,T;H)$ *such that* $z(0) = z_0$ *and for a.e.* $t \in (0,T)$ *we have* $z(t) \in D(\partial\Phi_H)$ *and*

$$z'(t) + \partial\Phi_H\big(z(t)\big) + \widetilde{u}(t)z(t) \ni \widetilde{f}(t) \quad \text{in } H. \tag{17}$$

Proof. Let $\alpha > 0$ be an arbitrary fixed number and let

$$\rho(z_1, z_2) = \max_{t\in[0,T]} \big[e^{-\alpha t}|z_1(t) - z_2(t)|\big], \quad z_1, z_2 \in C([0,T];H),$$

be a metric on $C([0,T];H)$. It is obvious that the space $C([0,T];H)$ with this metric is complete. Now let us consider an operator $A : C([0,T];H) \to C([0,T];H)$ defined as follows: to any given function $\widetilde{z} \in C([0,T];H)$, it assigns a function $\widehat{z} \in H^1(0,T;H) \subset C([0,T];H)$ such that $\widehat{z}(0) = z_0$ and for a.e. $t \in (0,T)$ the following inclusions hold: $\widehat{z}(t) \in D(\Phi_H)$ and

$$\widehat{z}'(t) + \partial\Phi_H(\widehat{z}(t)) \ni \widetilde{f}(t) - \widetilde{u}(t)\widetilde{z}(t) \quad \text{in} \quad H. \tag{18}$$

Clearly, variational inequality (18) coincides with variational inequality (16) after replacing $\widetilde{f}$ by $\widetilde{f} - \widetilde{u}\widetilde{z}$, thus using Proposition 6 we get that the operator A is well-defined. Let us show that the operator A is a contraction. Indeed, let $\widetilde{z}_1, \widetilde{z}_2$ be arbitrary function from $C([0,T];H)$ and $\widehat{z}_1 := A\widetilde{z}_1$, $\widehat{z}_2 = A\widetilde{z}_2$. According to (18) there exist functions $\widetilde{g}_1$ and $\widetilde{g}_2$ from $L^2(0,T;H)$ such that for every $k \in \{1,2\}$ and for a.e. $t \in (0,T)$ we have $\widetilde{g}_k(t) \in \partial\Phi_H(\widehat{z}_k(t))$ and

$$\widehat{z}_k'(t) + \widetilde{g}_k(t) = \widetilde{f}(t) - \widetilde{u}(t)\widetilde{z}_k(t), \tag{19}$$

while $\widehat{z}_k(0) = z_0$.

Subtracting identity (19) with $k = 2$ from identity (19) with $k = 1$, and, for a.e. $t \in (0,T)$, multiplying the obtained identity by $\widehat{z}_1(t) - \widehat{z}_2(t)$, we get

$$\begin{aligned} &\big((\widehat{z}_1(t) - \widehat{z}_2(t))', \widehat{z}_1(t) - \widehat{z}_2(t)\big) + (\widetilde{g}_1(t) - \widetilde{g}_2(t), \widehat{z}_1(t) - \widehat{z}_2(t)) \\ &\quad = -\widetilde{u}(t)(\widetilde{z}_1(t) - \widetilde{z}_2(t), \widehat{z}_1(t) - \widehat{z}_2(t)) \quad \text{for a.e.} \quad t \in (0,T), \\ &\qquad \widehat{z}_1(0) - \widehat{z}_2(0) = 0. \end{aligned} \tag{20}$$

We integrate equality (20) by t from 0 to $\tau \in (0,T]$, taking into account that for a.e. $t \in (0,T)$ we have

$$\big((\widehat{z}_1(t) - \widehat{z}_2(t))', \widehat{z}_1(t) - \widehat{z}_2(t)\big) = \frac{1}{2}\frac{d}{dt}|\widehat{z}_1(t) - \widehat{z}_2(t)|^2.$$

As a result we get the equality

$$\begin{aligned} &\frac{1}{2}|\widehat{z}_1(t) - \widehat{z}_2(t)|^2 + \int_0^\tau (\widetilde{g}_1(t) - \widetilde{g}_2(t), \widehat{z}_1(t) - \widehat{z}_2(t))\, dt \\ &\qquad = -\int_0^\tau \widetilde{u}(t)(\widetilde{z}_1(t) - \widetilde{z}_2(t), \widehat{z}_1(t) - \widehat{z}_2(t))\, dt. \end{aligned} \tag{21}$$

Taking into account condition $(\mathcal{A}_4)$, for a.e. $t \in (0,T)$ we have the inequality

$$(\widetilde{g}_1(t) - \widetilde{g}_2(t), \widehat{z}_1(t) - \widehat{z}_2(t)) \geqslant K_2|\widehat{z}_1(t) - \widehat{z}_2(t))|^2. \tag{22}$$

Since $\widetilde{u} \in L^\infty(0,T)$ then there exists a constant $\widetilde{M} \geqslant 0$ such that $|\widetilde{u}(t)| \leqslant \widetilde{M}$ for a.e. $t \in (0,T)$. From this, taking into account the Cauchy inequality, for a.e. $t \in (0,T)$ we obtain

$$\begin{aligned}\big|\widetilde{u}(t)\big(\widetilde{z}_1(t)-\widetilde{z}_2(t),\widehat{z}_1(t)-\widehat{z}_2(t)\big)\big| &\leqslant \widetilde{M}|\widetilde{z}_1(t)-\widetilde{z}_2(t)||\widehat{z}_1(t)-\widehat{z}_2(t)|\\ &\leqslant \frac{\varepsilon\widetilde{M}}{2}|\widehat{z}_1(t)-\widehat{z}_2(t)|^2+\frac{\widetilde{M}}{2\varepsilon}|\widetilde{z}_1(t)-\widetilde{z}_2(t)|^2, \end{aligned}\qquad(23)$$

where $\varepsilon > 0$ is arbitrary.

From (21), according to (22) and (23), we have

$$|\widehat{z}_1(\tau)-\widehat{z}_2(\tau)|^2+(2K_2-\varepsilon\widetilde{M})\int_0^\tau|\widehat{z}_1(t)-\widehat{z}_2(t)|^2\,dt\leqslant\widetilde{M}\varepsilon^{-1}\int_0^\tau|\widetilde{z}_1(t)-\widetilde{z}_2(t)|^2\,dt.\quad(24)$$

Choosing $\varepsilon > 0$ such that $2K_2-\varepsilon\widetilde{M}\geqslant 0$, from (24) we obtain

$$|\widehat{z}_1(\tau)-\widehat{z}_2(\tau)|^2\leqslant C_2\int_0^\tau|\widetilde{z}_1(t)-\widetilde{z}_2(t)|^2\,dt,\quad \tau\in(0,T],\qquad(25)$$

where $C_2 > 0$ is a constant.

After multiplying inequality (25) by $e^{-2\alpha\tau}$ we obtain

$$\begin{aligned}e^{-2\alpha\tau}|\widehat{z}_1(\tau)-\widehat{z}_2(\tau)|^2 &\leqslant C_2e^{-2\alpha\tau}\int_0^\tau e^{2\alpha t}e^{-2\alpha t}|\widetilde{z}_1(t)-\widetilde{z}_2(t)|^2\,dt\\ &\leqslant C_2e^{-2\alpha\tau}\max_{t\in[0,T]}\big[e^{-2\alpha t}|\widetilde{z}_1(t)-\widetilde{z}_2(t)|^2\big]\int_0^\tau e^{2\alpha t}\,dt\\ =\frac{C_2}{2\alpha}(1-e^{-2\alpha\tau})\big(\rho(\widetilde{z}_1,\widetilde{z}_2)\big)^2 &\leqslant \frac{C_2}{2\alpha}\big(\rho(\widetilde{z}_1,\widetilde{z}_2)\big)^2,\quad \tau\in[0,T].\end{aligned}\qquad(26)$$

From (26) it easily follows that

$$\rho(\widehat{z}_1,\widehat{z}_2)\leqslant\sqrt{C_2/(2\alpha)}\rho(\widetilde{z}_1,\widetilde{z}_2).$$

From this, choosing $\alpha > 0$ such that the inequality $C_2/(2\alpha) < 1$ holds, we obtain that the operator A is a contraction. Hence, we may apply the Banach fixed-point theorem (the contraction mapping principle) [11, Theorem 5.7] and deduce that there exists a unique function $z \in C([0,T];H)$ such that $Az = z$. Thus, Proposition 7 is proved. □

Now let us prove Theorem 1.

Proof. The uniqueness of the solution. Assume the opposite. Let y_1, y_2 be two solutions of the problem $\mathbf{P}(\Phi,u,f,\gamma)$. Then for every $i \in \{1,2\}$ there exists a function $g_i \in L^2_{\mathrm{loc}}(S;V')$ such that for a.e. $t \in S$ we have $g_i(t) \in \partial\Phi\big(y_i(t)\big)$ and

$$y_i'(t)+g_i(t)+u(t)y_i(t)=f(t)\quad\text{in } V'.\qquad(27)$$

Denote $z := y_1 - y_2$. From equalities (27) for a.e. $t \in S$ we obtain

$$z'(t)+g_1(t)-g_2(t)+u(t)z(t)=0\quad\text{in } V'.\qquad(28)$$

From (10) it follows that the following condition holds

$$e^{-2\gamma t}|z(t)|^2\to 0\quad\text{as}\quad t\to-\infty.\qquad(29)$$

Multiplying equality (28) for almost every $t \in S$ on $z(t)$, we obtain

$$(z'(t), z(t)) + (g_1(t) - g_2(t), y_1(t) - y_2(t)) + u(t)|z(t)|^2 = 0. \tag{30}$$

According to equality (5), condition $(\mathcal{A}_4)$ and the fact that $g_i(t) \in \partial\Phi(y_i(t))$ $(i = 1, 2)$ for a.e. $t \in S$, we obtain the differential inequality

$$\frac{1}{2}\frac{d|z(t)|^2}{dt} + (K_2 + \widetilde{m})|z(t)|^2 \leqslant 0 \quad \text{for a.e.} \quad t \in S. \tag{31}$$

Let us take arbitrary numbers $\tau_1, \tau_2 \in S$ $(\tau_1 < \tau_2)$. Multiplying inequality (31) by $e^{-2\gamma t}$, integrating from τ_1 to τ_2 and using the integration-by-parts formula, we obtain

$$\frac{1}{2}e^{-2\gamma t}|z(t)|^2\Big|_{\tau_1}^{\tau_2} + (K_2 + \widetilde{m} + \gamma)\int\limits_{\tau_1}^{\tau_2} e^{-2\gamma t}|z(t)|^2\,dt \leqslant 0. \tag{32}$$

Since condition (11) hold, then from (32) we obtain

$$e^{-2\gamma\tau_2}|z(\tau_2)|^2 \leqslant e^{-2\gamma\tau_1}|z(\tau_1)|^2. \tag{33}$$

In (33) we fix τ_2 and pass to the limit as $\tau_1 \to -\infty$. According to condition (29) we obtain the equality $e^{-2\gamma\tau_2}|z(\tau_2)|^2 = 0$. Since $\tau_2 \in S$ is an arbitrary number, we have $z(t) = 0$ for a. e. $t \in S$, that is, $y_1(t) = y_2(t)$ for a. e. $t \in S$. The resulting contradiction proves the uniqueness of the solution of problem (15).

The existence of the solution. We divide the proof into three steps.

Step 1 (*Solution approximation*). We construct a sequence of functions which, in some sense, approximate the solution of the problem $\mathbf{P}(\Phi, u, f, \gamma)$.

Let $\widehat{f}_k(t) := f(t)$ for $t \in S_k := [-k, 0]$, where $k \in \mathbb{N}$. For each $k \in \mathbb{N}$ let us consider the problem of finding a function $\widehat{y}_k \in H^1(S_k; H) := \{z \in L^2(S_k; H) \mid z' \in L^2(S_k; H)\}$ such that for a.e. $t \in S_k$ we have $\widehat{y}_k(t) \in D(\partial\Phi_H)$ and

$$\widehat{y}_k'(t) + \partial\Phi_H(\widehat{y}_k(t)) + u(t)\widehat{y}_k(t) \ni \widehat{f}_k(t) \quad \text{in} \quad H, \tag{34a}$$

$$\widehat{y}_k(-k) = 0. \tag{34b}$$

Variational inequality (34a) means that there exists a function $\widehat{g}_k \in L^2(S_k; H)$ such that for a.e. $t \in S_k$ we have $\widehat{g}_k(t) \in \partial\Phi_H(\widehat{u}_k(t))$ and

$$\widehat{y}_k'(t) + \widehat{g}_k(t) + u(t)\widehat{y}_k(t) = \widehat{f}_k(t) \quad \text{in} \quad H. \tag{35}$$

Note that $D(\partial\Phi_H) \subset \mathrm{dom}(\Phi_H)$, therefore $\widehat{y}_k(t) \in V$ for a.e. $t \in S_k$. According to the definition of the subdifferential of a functional and the fact that $\widehat{g}_k(t) \in \partial\Phi(\widehat{y}(t))$ for a.e. $t \in S_k$, we have

$$\Phi(0) \geqslant \Phi(\widehat{y}_k(t)) + (\widehat{g}_k(t), 0 - \widehat{y}_k(t)) \quad \text{for a.e.} \quad t \in S_k.$$

Using this and condition $(\mathcal{A}_3)$ we obtain

$$(\widehat{g}_k(t), \widehat{y}_k(t)) \geqslant \Phi(\widehat{y}_k(t)) \geqslant K_1\|\widehat{y}_k(t)\|^2 \quad \text{for a.e.} \quad t \in S_k. \tag{36}$$

Since the left side of this chain of inequalities belongs to $L^1(S_k)$, then $\widehat{y_k}$ belongs to $L^2(S_k; V)$.

For each $k \in \mathbb{N}$ we extend the functions $\widehat{f}_k, \widehat{y}_k$ and $\widehat{g}_k$ by zero over the entire interval S, and denote these extensions by f_k, y_k and g_k respectively. From the above it follows that for each $k \in \mathbb{N}$ the function y_k belongs to $L^2(S; V)$, its derivative y_k' belongs to

$L^2(S;H)$ and for a.e. $t \in S$ the inclusion $g_k(t) \in \partial\Phi_H\big(y_k(t)\big)$ and the following equality hold (see (35)

$$y_k' + g_k(t) + u(t)y_k = f_k(t) \quad \text{in} \quad H. \tag{37}$$

In order to show the convergence of $\{y_k\}_{k=1}^{+\infty}$ to the solution of the problem $\mathbf{P}(\Phi, u, f, \gamma)$ we need some estimates of the functions y_k $(k \in \mathbb{N})$.

Step 2 (*Estimates of approximating solutions*).

Multiplying identity (37), for a.e. $t \in S$, by $e^{-2\gamma t}y_k(t)$ and integrating if from τ_1 to τ_2 ($\tau_1, \tau_2 \in S$ are arbitrary numbers, $\tau_1 < \tau_2$), we obtain

$$\int_{\tau_1}^{\tau_2} e^{-2\gamma t}(y_k'(t), y_k(t))\,dt + \int_{\tau_1}^{\tau_2} e^{-2\gamma t}(g_k(t), y_k(t))\,dt$$
$$+ \int_{\tau_1}^{\tau_2} e^{-2\gamma t}u(t)|y_k(t)|^2\,dt = \int_{\tau_1}^{\tau_2} e^{-2\gamma t}(f_k(t), y_k(t))\,dt.$$

From this, taking into account (5) and using the integration-by-parts formula, we obtain

$$e^{-2\gamma t}|y_k(t)|^2\Big|_{\tau_1}^{\tau_2} + 2\gamma\int_{\tau_1}^{\tau_2} e^{-2\gamma t}|y_k(t)|^2\,dt + 2\int_{\tau_1}^{\tau_2} e^{-2\gamma t}(g_k(t), y_k(t))\,dt \tag{38}$$
$$+ 2\int_{\tau_1}^{\tau_2} e^{-2\gamma t}u(t)|y_k(t)|^2\,dt = 2\int_{\tau_1}^{\tau_2} e^{-2\gamma t}(f_k(t), y_k(t))\,dt.$$

According to the definition of y_k and (36), we obtain

$$(g_k(t), y_k(t)) \geqslant \Phi\big(y_k(t)\big) \geqslant K_1\|y_k(t)\|^2 \quad \text{for a.e. } t \in S. \tag{39}$$

Let us estimate the third term on the left-hand side of inequality (38). From (8) and (39) for arbitrary $\delta \in (0,1)$, we obtain

$$2\int_{\tau_1}^{\tau_2} e^{-2\gamma t}(g_k(t), y_k(t))\,dt = 2(\delta + (1-\delta))\int_{\tau_1}^{\tau_2} e^{-2\gamma t}(g_k(t), y_k(t))\,dt$$
$$\geqslant 2\delta K_2\int_{\tau_1}^{\tau_2} e^{-2\gamma t}|y_k(t)|^2\,dt + (1-\delta)K_1\int_{\tau_1}^{\tau_2} e^{-2\gamma t}\|y_k(t)\|^2\,dt$$
$$+(1-\delta)\int_{\tau_1}^{\tau_2} e^{-2\gamma t}\Phi\big(y_k(t)\big)\,dt. \tag{40}$$

Using the Cauchy inequality we estimate the right-hand side of (38), as follows

$$2\int_{\tau_1}^{\tau_2} e^{-2\gamma t}(f_k(t), y_k(t))\,dt \leqslant \varepsilon\int_{\tau_1}^{\tau_2} e^{-2\gamma t}|y_k(t)|^2\,dt + \varepsilon^{-1}\int_{\tau_1}^{\tau_2} e^{-2\gamma t}|f_k(t)|^2\,dt, \tag{41}$$

where $\varepsilon > 0$ is arbitrary.

From (38), taking into account (40), (41) and the notation $\widetilde{m} := \inf\limits_{t\in S} u(t)$, we obtain

$$e^{-2\gamma t}|y_k(t)|^2\Big|_{\tau_1}^{\tau_2} + [2(\delta K_2 + \widetilde{m} + \gamma) - \varepsilon]\int_{\tau_1}^{\tau_2} e^{-2\gamma t}|y_k(t)|^2\,dt$$
$$+ (1-\delta)K_1\int_{\tau_1}^{\tau_2} e^{-2\gamma t}\|y_k(t)\|^2\,dt + (1-\delta)\int_{\tau_1}^{\tau_2} e^{-2\gamma t}\Phi\big(y_k(t)\big)\,dt$$
$$\leqslant \varepsilon^{-1}\int_{\tau_1}^{\tau_2} e^{-2\gamma t}|f_k(t)|^2\,dt, \quad \delta \in (0,1),\ \varepsilon > 0. \tag{42}$$

Since $K_1 > 0$, $K_2+\widetilde{m}+\gamma > 0$ and $\delta \in (0,1), \varepsilon > 0$ are arbitrary, then we first choose δ such that $\delta K_2 + m + \gamma > 0$, and then we choose ε such that $2(\delta K_2 + m + \gamma) - \varepsilon > 0$. As a result we obtain the estimate

$$e^{-2\gamma t}|y_k(t)|^2\Big|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} e^{-2\gamma t}\|y_k(t)\|^2\,dt + \int_{\tau_1}^{\tau_2} e^{-2\gamma t}\Phi\big(y_k(t)\big)\,dt \leqslant C_3\int_{\tau_1}^{\tau_2} e^{-2\gamma t}|f_k(t)|^2\,dt, \tag{43}$$

where C_3 is a positive constant depending on $K_1, K_2, \widetilde{m}$ and γ only.

We take $\tau_2 = \tau$, when $\tau \in S$ is arbitrary, and pass to the limit in (43) as $\tau_1 \to -\infty$. Taking into account $(\mathcal{F})$ and the definition of y_k and f_k, we obtain

$$e^{-2\gamma\tau}|y_k(\tau)|^2 + \int_{-\infty}^{\tau} e^{-2\gamma t}\|y_k(t)\|^2\,dt + \int_{-\infty}^{\tau} e^{-2\gamma t}\Phi\big(y_k(t)\big)\,dt \leqslant C_3\int_{-\infty}^{\tau} e^{-2\gamma t}|f_k(t)|^2\,dt, \quad \tau \in S. \tag{44}$$

Since $\tau \in S$ is arbitrary, from (44) it follows that

$$\text{sequence } \{e^{-\gamma\cdot}y_k(\cdot)\}_{k=1}^{+\infty} \text{ is bounded in } L^\infty(S;H) \text{ and in } L^2(S;V), \tag{45}$$

$$\text{sequence } \big\{e^{-2\gamma\cdot}\Phi\big(y_k(\cdot)\big)\big\}_{k=1}^{+\infty} \text{ is bounded in } L^1(S). \tag{46}$$

Now let us find estimates of $y_k'(t)$. For almost every $t \in S$ we multiply equality (37) by $e^{-2\gamma t}y_k'(t)$ and integrate the resulting equality from τ_1 to τ_2 ($\tau_1, \tau_2 \in S$ are arbitrary numbers, $\tau_1 < \tau_2$). Then we obtain

$$\int_{\tau_1}^{\tau_2} e^{-2\gamma t}|y_k'(t)|^2\,dt + \int_{\tau_1}^{\tau_2} e^{-2\gamma t}(g_k(t), y_k'(t))\,dt = \int_{\tau_1}^{\tau_2} e^{-2\gamma t}(f_k(t), y_k'(t))\,dt - \int_{\tau_1}^{\tau_2} e^{-2\gamma t}u(t)(y_k(t), y_k'(t))\,dt. \tag{47}$$

From (47) using the Cauchy-Schwarz inequality and the fact that $\sup\limits_{t\in S} u(t) =: \widetilde{M} < \infty$ we obtain

$$\int_{\tau_1}^{\tau_2} e^{-2\gamma t}|y_k'(t)|^2\,dt + \int_{\tau_1}^{\tau_2} e^{-2\gamma t}(g_k(t), y_k'(t))\,dt \leqslant \widetilde{M}\int_{\tau_1}^{\tau_2} e^{-2\gamma t}|y_k(t)||y_k'(t)|\,dt + \int_{\tau_1}^{\tau_2} e^{-2\gamma t}|f_k(t)||y_k'(t)|\,dt. \tag{48}$$

Since $g_k \in L^2(\tau_1, \tau_2; H)$, Statement 5 implies that the function $\Phi_H\big(y_k(\cdot)\big)$ is absolutely continuous on $[\tau_1, \tau_2]$ and

$$\frac{d}{dt}\Phi_H\big(y_k(t)\big) = (g_k(t), y_k'(t)) \quad \text{for a.e. } t \in (\tau_1, \tau_2). \tag{49}$$

Taking into account (49), we estimate the second term on the left side of (48) as follows

$$\int_{\tau_1}^{\tau_2} e^{-2\gamma t}(g_k(t), y_k'(t))\,dt = \int_{\tau_1}^{\tau_2} e^{-2\gamma t}\frac{d}{dt}\Phi_H\big(y_k(t)\big)\,dt = e^{-2\gamma t}\Phi_H\big(y_k(t)\big)\Big|_{\tau_1}^{\tau_2} + 2\gamma\int_{\tau_1}^{\tau_2} e^{-2\gamma t}\Phi_H\big(y_k(t)\big)\,dt. \tag{50}$$

Using the Cauchy inequality to the right-hand side of (48) and estimate (44), we obtain

$$\begin{aligned}
&\widetilde{M}\int_{\tau_1}^{\tau_2} e^{-2\gamma t}|y_k(t)||y_k'(t)|\,dt+\int_{\tau_1}^{\tau_2} e^{-2\gamma t}|f_k(t)||y_k'(t)|\,dt\\
&\leqslant \widetilde{M}^2\int_{\tau_1}^{\tau_2} e^{-2\gamma t}|y_k(t)|^2\,dt+\frac14\int_{\tau_1}^{\tau_2} e^{-2\gamma t}|y_k'(t)|^2\,dt\\
&+\int_{\tau_1}^{\tau_2} e^{-2\gamma t}|f_k(t)|^2\,dt+\frac14\int_{\tau_1}^{\tau_2} e^{-2\gamma t}|y_k'(t)|^2\,dt\\
&\leqslant \widetilde{M}^2\int_{\tau_1}^{\tau_2} e^{-2\gamma t}|y_k(t)|^2\,dt+\frac12\int_{\tau_1}^{\tau_2} e^{-2\gamma t}|y_k'(t)|^2\,dt+\int_{\tau_1}^{\tau_2} e^{-2\gamma t}|f_k(t)|^2\,dt. \qquad (51)
\end{aligned}$$

From (48), taking into account (50), (51), we obtain

$$\begin{aligned}
&\frac12\int_{\tau_1}^{\tau_2} e^{-2\gamma t}|y_k'(t)|^2\,dt+e^{-2\gamma t}\Phi_H\big(y_k(t)\big)\Big|_{\tau_1}^{\tau_2}\\
&\leqslant \widetilde{M}^2\int_{\tau_1}^{\tau_2} e^{-2\gamma t}|y_k(t)|^2\,dt+2|\gamma|\int_{\tau_1}^{\tau_2} e^{-2\gamma t}\Phi_H\big(y_k(t)\big)\,dt+\int_{\tau_1}^{\tau_2} e^{-2\gamma t}|f_k(t)|^2\,dt. \qquad (52)
\end{aligned}$$

Taking into account the definitions of y_k and f_k, condition $(\mathcal{A}_3)$, (1) and (44), we pass to the limit as $\tau_1\to-\infty$ in (52). As a result, taking $\tau_2=\tau\in S$, we obtain

$$e^{-2\gamma\tau}\Phi_H\big(y_k(\tau)\big)+\int_{-\infty}^{\tau} e^{-2\gamma t}|y_k'(t)|^2\,dt\leqslant C_4\int_{-\infty}^{\tau} e^{-2\gamma t}|f_k(t)|^2\,dt, \qquad (53)$$

where C_4 is a positive constant depending on K_1,γ,λ and $\widetilde{m},\widetilde{M}$ only.

According to the definitions of the functional Φ_H and the function f_k, and condition $(\mathcal{A}_3)$ (recall that $y_k(t)\in V$ for a.e. $t\in S$), we obtain

$$e^{-2\gamma\tau}\|y_k(\tau)\|^2+\int_{-\infty}^{\tau} e^{-2\gamma t}|y_k'(t)|^2\,dt\leqslant C_5\int_{-\infty}^{\tau} e^{-2\gamma t}|f_k(t)|^2\,dt, \qquad (54)$$

where $C_5>0$ is a constant depending on K_1,γ,λ and $\widetilde{m},\widetilde{M}$ only.

Estimate (54) and the definition of f_k imply that

$$\text{the sequence } \big\{y_k\big\}_{k=1}^{+\infty} \text{ is bounded in } L^\infty_\gamma(S;V), \qquad (55)$$

$$\text{the sequence } \big\{y_k'\big\}_{k=1}^{+\infty} \text{ is bounded in } L^2_\gamma(S;H). \qquad (56)$$

From (37), (44), (56), $(\mathcal{F})$ and the definition of f_k we obtain

$$\text{the sequence } \{g_k\}_{k=1}^{+\infty} \text{ is bounded in } L^2_\gamma(S;H). \qquad (57)$$

Step 3 (*Passing to the limit*). Since V and H are Hilbert spaces, and V embeds in H with compact injection, then (45), (55)–(57) and Statement 4 imply that there exist functions $y\in L^\infty_\gamma(S;V)\cap L^2_\gamma(S;V)\cap H^1_\gamma(S;H)\subset C(S;H)$, $g\in L^2_\gamma(S;H)$ and a subsequence of sequence $\{y_k, g_k\}_{k=1}^{+\infty}$ (still denoted by $\{y_k, g_k\}_{k=1}^{+\infty}$) such that

$$y_k \underset{k\to\infty}{\longrightarrow} y \quad *\text{-weakly in } L^{\infty}_{\mathrm{loc}}(S;V), \text{ weakly in } L^2_{\gamma}(S;V) \text{ and weakly in } H^1_{\gamma}(S;H), \tag{58}$$

$$y_k \underset{k\to\infty}{\longrightarrow} y \quad \text{in } C(S;H), \tag{59}$$

$$g_k \underset{k\to\infty}{\longrightarrow} g \quad \text{weakly in } L^2_{\gamma}(S;H). \tag{60}$$

Note that (58) and (60) imply

$$y_k \underset{k\to\infty}{\longrightarrow} y, \quad y_k' \underset{k\to\infty}{\longrightarrow} y', \quad g_k \underset{k\to\infty}{\longrightarrow} g \quad \text{weakly in} \quad L^2_{\mathrm{loc}}(S;H). \tag{61}$$

Let $v \in H, \varphi \in D(-\infty, 0)$ be arbitrary. For a.e. $t \in S$ we multiply equality (37) by v, and then we multiply the obtained equality by φ and integrate in t on S. As a result, we obtain the equality

$$\int_S (y_k'(t), v\varphi(t))\,dt + \int_S (g_k(t), v\varphi(t))\,dt + \int_S u(t)(y_k(t), v\varphi(t))\,dt$$
$$= \int_S (f_k(t), v\varphi(t))\,dt, \quad k \in \mathbb{N}. \tag{62}$$

We pass to the limit in (62) as $k \to \infty$, taking into account (61) and convergence $\{f_k\}$ to f in $L^2_{\mathrm{loc}}(S;H)$. As a result, since $v \in H, \varphi \in D(-\infty, 0)$ are arbitrary, for a.e. $t \in S$, we obtain the equality

$$y'(t) + g(t) + u(t)y(t) = f(t) \quad \text{in } H.$$

In order to complete the proof of the theorem it remains only to show that $y(t) \in D(\partial\Phi)$ and $g(t) \in \partial\Phi\big(y(t)\big)$ for a.e. $t \in S$.

Let $k \in \mathbb{N}$ be an arbitrary number. Since $g_k(t) \in \partial\Phi_H\big(y_k(t)\big)$ for every $t \in S \setminus \widetilde{S}_k$, where $\widetilde{S}_k \subset S$ is a set of measure zero, applying the monotonicity of subdifferential $\partial\Phi_H$ we obtain that for every $t \in S \setminus \widetilde{S}_k$ the following equality holds:

$$(g_k(t) - v^*, y_k(t) - v) \geqslant 0 \quad \forall\,[v, v^*] \in \partial\Phi_H. \tag{63}$$

Let $\tau \in S$, $h > 0$ be arbitrary numbers. We integrate (63) on $(\tau - h; \tau)$:

$$\int_{\tau-h}^{\tau} (g_k(t) - v^*, y_k(t) - v)\,dt \geqslant 0 \quad \forall\,[v, v^*] \in \partial\Phi_H. \tag{64}$$

Now according to (59) and (60) we pass to the limit in (64) as $k \to \infty$. As a result we obtain

$$0 \leqslant \int_{\tau-h}^{\tau} (g_k(\tau) - v^*, y_k(t) - v)\,dt \underset{k\to\infty}{\longrightarrow} \int_{\tau-h}^{\tau} (g(t) - v^*, y(t) - v)\,dt \geqslant 0 \quad \forall\,[v, v^*] \in \partial\Phi_H. \tag{65}$$

The monograph [28, Theorem 2, P. 192] and (65) imply that for every $[v, v^*] \in \partial\Phi_H$ there exists a set $R_{[v,v_*]} \subset S$ of measure zero such that for all $\tau \in S \setminus R_{[v,v_*]}$ we have

$$0 \leqslant \lim_{h\to+0} \frac{1}{h} \int_{\tau-h}^{\tau} \big(g(t) - v^*, y(t) - v\big)\,dt = \big(g(\tau) - v^*, y(\tau) - v\big). \tag{66}$$

Let us show that there exists a set of measure zero $R \subset S$ such that for every $\tau \in S \setminus R$ the following inequality holds

$$\big(g(\tau) - v^*, y(\tau) - v\big) \geqslant 0 \quad \forall [v, v^*] \in \partial\Phi_H. \tag{67}$$

Since V and H are separable spaces, there exists a countable set $F \subset \partial\Phi_H$, which is dense in $\partial\Phi_H$. Let us denote $R := \bigcup_{[v,v^*]\in F} R_{[v,v_*]}$. Since the set F is countable, and countable union of sets of measure zero is a set of measure zero, R is a set of measure zero. Therefore, for any $\tau \in S \setminus R$ inequality (67) holds for every $[v, v^*] \in F$. Let $[\widehat{v}, \widehat{v}^*]$ be an arbitrary element from $\partial\Phi_H$. Since F is dense in $\partial\Phi_H$, we have the existence of the sequence $\{[v_l, v_l^*]\}_{l=1}^{\infty}$ such that $v_l \to \widehat{v}$ in V, $v_l^* \to \widehat{v}^*$ in H and for every $\tau \in S \setminus R$ we have

$$(g(\tau) - v_l^*, y(\tau) - v_l) \geqslant 0 \quad \forall\, l \in \mathbb{N}. \tag{68}$$

So, passing to the limit in this equality as $l \to \infty$, we get $(g(\tau) - \widehat{v}^*, y(\tau) - \widehat{v}) \geqslant 0$. Thus, for a.e. $\tau \in S$ inequality (67) holds. From this, according to maximal monotonicity of $\partial\Phi_H$, we obtain that $[y(t), g(t)] \in \partial\Phi_H$ for a.e. $t \in S$.

Estimate (12) of the solution of the problem $\mathbf{P}(\Phi, u^*, f, \gamma)$ follows directly from (44), (54), (58) and (59), and Proposition 2. From (44), (59), (12) according to (1) we have

$$e^{-2\gamma\tau}|y_k(\tau)|^2 \leqslant C_1 \int_{-\infty}^{\tau} e^{-2\gamma t}|f(t)|^2\, dt.$$

From this we obtain that y satisfies condition (10). □

5. Proof of the main result

Proof of Theorem 2. Let $\{u_k\}$ be a minimizing sequence for functional J in U_∂: $J(u_k) \underset{k\to\infty}{\longrightarrow} \inf_{u\in U_\partial} J(u)$. According to the definition of U_∂ we obtain that

$$\text{the sequence } \{u_k\}_{k=1}^{\infty} \text{ is bounded in } L^\infty(S). \tag{69}$$

Assume that for every $k \in \mathbb{N}$ the function $y_k := y(u_k)$ is a solution of the problem $\mathbf{P}(\Phi, u_k, f, \gamma)$, that is, the following variational inequality and condition at the infinity hold

$$y_k'(t) + \partial\Phi\big(y_k(t)\big) + u_k(t)y_k(t) \ni f(t), \quad t \in S, \tag{70}$$

$$\lim_{t\to-\infty} e^{-\gamma t}|y_k(t)| = 0. \tag{71}$$

According to Definition 1 and Theorem 1, taking into account $(\mathcal{F})$, for every $k \in \mathbb{N}$ we have $y_k \in L^\infty_\gamma(S; V) \cap L^2_\gamma(S; V) \cap H^1_\gamma(S; H) \subset C(S; H)$, $y_k(t) \in D(\partial\Phi)$ for a.e. $t \in S$, and the existence of a function $g_k \in L^2_\gamma(S; H)$ such that for a.e. $t \in S$, $g_k(t) \in \partial\Phi(y_k(t))$, and

$$y_k'(t) + g_k(t) + u_k(t)y_k(t) = f(t) \quad \text{in} \quad H \tag{72}$$

and condition (71) holds.

Moreover, for arbitrary $k \in \mathbb{N}$ and $\tau \in S$ the following estimate holds

$$e^{-2\gamma\tau}\|y_k(\tau)\|^2 + \int_{-\infty}^{\tau} e^{-2\gamma t}\|y_k(t)\|^2\, dt + \int_{-\infty}^{\tau} e^{-2\gamma t}|y_k'(t)|^2\, dt \leqslant C_1 \int_{-\infty}^{\tau} e^{-2\gamma t}|f(t)|^2\, dt, \tag{73}$$

where C_1 is a positive constant depending on K_1, K_2, γ, λ and m, M only.

Estimate (73) implies that

$$\text{the sequence } \{e^{-\gamma\cdot} y_k(\cdot)\}_{k=1}^{\infty} \text{ is bounded in } L^{\infty}(S;V), \tag{74}$$

$$\text{the sequence } \{y_k\}_{k=1}^{\infty} \text{ is bounded in } L^2_{\gamma}(S;V), \tag{75}$$

$$\text{the sequence } \{y_k'\}_{k=1}^{\infty} \text{ is bounded in } L^2_{\gamma}(S;H). \tag{76}$$

From (72), taking into account (69), (75), (76) we obtain that

$$\text{the sequence } \{g_k\}_{k=1}^{\infty} \text{ is bounded in } L^2_{\gamma}(S;H). \tag{77}$$

Since V and H are reflexive spaces, and V embeds in H densely, continuously and compactly, then (69), (74)–(77), taking into account Statement 3, imply that there exist a subsequence of the sequence $\{u_k, y_k, g_k\}_{k=1}^{\infty}$ (still denoted by $\{u_k, y_k, g_k\}_{k=1}^{\infty}$) and functions $u^* \in U_\partial$, $y \in L^{\infty}_{\gamma}(S;V) \cap L^2_{\gamma}(S;V) \cap H^1_{\gamma}(S;H) \subset C(S;H)$ and $g \in L^2_{\gamma}(S;H)$ such that

$$u_k \underset{k\to\infty}{\longrightarrow} u^* \quad *\text{-weakly in} \quad L^{\infty}(S), \tag{78}$$

$$y_k \underset{k\to\infty}{\longrightarrow} y \ *\text{-weakly in } L^{\infty}_{\mathrm{loc}}(S;V), \text{ weakly in } L^2_{\gamma}(S;V) \text{ and weakly in } H^1_{\gamma}(S;H), \tag{79}$$

$$y_k \underset{k\to\infty}{\longrightarrow} y \quad \text{in} \quad C(S;H), \tag{80}$$

$$g_k \underset{k\to\infty}{\longrightarrow} g \quad \text{weakly in} \quad L^2_{\gamma}(S;H). \tag{81}$$

Similarly, as in the proof of Theorem 1, for a.e. $t \in S$ we multiply equality (72) by v, and then multiply the resulting equality by φ and integrate on S, where $v \in H, \varphi \in D(-\infty,0)$ are arbitrary. As a result, we obtain the equality

$$\int_S (y_k'(t), v\varphi(t))\,dt + \int_S (g_k(t), v\varphi(t))\,dt + \int_S u_k(t)(y_k(t), v\varphi(t))\,dt = \int_S (f(t), v\varphi(t))\,dt, \quad k \in \mathbb{N}. \tag{82}$$

Let us show that (78) and (80) yield

$$\int_S u_k(t)(y_k(t), v\varphi(t))\,dt \underset{k\to\infty}{\longrightarrow} \int_S u^*(t)(y(t), v\varphi(t))\,dt \quad \forall v \in H,\ \forall \varphi \in D(-\infty,0). \tag{83}$$

Indeed, let $t_1, t_2 \in S$ be such that $\operatorname{supp}\varphi \subset [t_1, t_2]$. Then we have

$$\int_S u_k(t)(y_k(t), v\varphi(t))\,dt = \int_{t_1}^{t_2} u_k(t)(y_k(t) - y(t) + y(t), v\varphi(t))\,dt$$

$$= \int_{t_1}^{t_2} u_k(t)(y(t), v\varphi(t))\,dt + \int_{t_1}^{t_2} u_k(t)(y_k(t) - y(t), v\varphi(t))\,dt. \tag{84}$$

From (69), (80) and the Cauchy-Schwarz inequality it follows

$$\Big|\int_{t_1}^{t_2} u_k(t)(y_k(t)-y(t), v\varphi(t))\,dt\Big| \leqslant \leqslant M\Big(\int_{t_1}^{t_2} |\varphi(t)v|^2\,dt\Big)^{1/2}\Big(\int_{t_1}^{t_2} |y_k(t)-y(t)|^2\,dt\Big)^{1/2} \underset{k\to\infty}{\longrightarrow} 0. \tag{85}$$

Since $(y(\cdot), v)\varphi(\cdot) \in L^1_{\mathrm{loc}}(S)$, (78) implies

$$\int_{t_1}^{t_2} u_k(t)(y(t), v\varphi(t))\,dt \underset{k\to\infty}{\longrightarrow} \int_{t_1}^{t_2} u^*(t)(y(t), v\varphi(t))\,dt. \tag{86}$$

From (84), taking into account (85) and (86), we obtain (83).

Taking into account (79), (81) and (83) we pass to the limit in (82) as $k\to\infty$. As a result, for a.e. $t\in S$ we obtain the equality

$$y'(t) + g(t) + u^*(t)y(t) = f(t) \quad \text{in } H.$$

Similarly, as in the proof of Theorem 1, we show that $y(t) \in D(\partial\Phi)$ and $g(t) \in \partial\Phi\big(y(t)\big)$ for a.e. $t\in S$. From (1), (73) and (80) we have $e^{-2\gamma\tau|y(\tau)|^2} \leqslant C_1\lambda^{-1}\int_{-\infty}^{\tau} e^{-2\gamma t}|f(t)|^2\,dt$, $\tau\in S$. This means that condition (10) holds. Thus, the function y is a solution of the problem $\mathbf{P}(\Phi, u^*, f, \gamma)$.

It remains to show that u^* is a minimizing element of the functional J. Indeed, since the functional G is lower semicontinuous in $C(S;H)$, then (80) implies that

$$\varliminf_{k\to\infty} G(y_k) \geqslant G(y). \tag{87}$$

Also (78) and Proposition 2 yield

$$\varliminf_{k\to\infty} \|u_k\|_U \geqslant \|u^*\|_U. \tag{88}$$

From (14), (87), (88) we obtain that $\inf\limits_{u\in U_\partial} J(u) = \lim\limits_{k\to\infty} J(u_k) \geqslant \varliminf\limits_{k\to\infty} G(y_k) + \mu \varliminf\limits_{k\to\infty} \|u_k\|_U \geqslant J(u^*)$. Thus, we have shown that u^* is a solution of problem (15), i.e., the optimal control. □

References

1. *Adams D.R., Lenhart S.* Optimal control of the obstacle for a parabolic variational inequality // J. Math. Anal. Appl. – 2002. – **268**, №2. — P. 602–614.
2. *Aubin J.-P.* Un théorème de compacité // C. R. Acad. Sci., Paris — 1963. — **256**, №24. — P. 5042–5044.
3. *Barbu V.* Optimal Control of Variational Inequalities. — London: Pitman, 1983.
4. *Bernis F.* Existence results for doubly nonlinear higher order parabolic equations on unbounded domains // Math. Ann. — 1988. — **279**, №3. — P. 373–394.
5. *Bokalo M.* Well-posedness of problems without initial conditions for nonlinear parabolic variational inequalities // Nonlinear Bound. Value Probl. — 1998. — **8**. — P. 58–63.

6. *Bokalo M., Lorenzi A.* Linear evolution first-order problems without initial conditions // Milan J. Math — 2009. — **77**. — P. 437–494.
7. *Bokalo M., Tsebenko A.* Existence of optimal control in the coefficients for problem without initial condition for strongly nonlinear parabolic equations // Mat. Stud. — 2016. — **45**, №1. — P. 40–56.
8. *Bokalo M.* Optimal control of evolution systems without initial conditions // Visn. L'viv. Univ., Ser. Mekh.-Mat. — 2010. — **73**. — P. 85–113.
9. *Bokalo M.M.* Problem without initial conditions for classes of nonlinear parabolic equations // J. Sov. Math. — 1990. — **51**, №3. — P. 2291–2322.
10. *Boukrouche M., Tarzia D.A.* Existence, uniqueness, and convergence of optimal control problems associated with parabolic variational inequalities of the second kind // Nonlinear Anal., Real World Appl. — 2011. — **12**, №4. — P. 2211–2224
11. *Brezis H.* Functional analysis, Sobolev spaces and partial differential equations. — New York: Springer, 2011.
12. *Brezis H.* Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert. — Amsterdam, London: North-Holland Publishing Comp., 1973.
13. *Buhrii O.M.* Some parabolic variational inequalities without initial conditions // Visn. L'viv. Univ., Ser. Mekh.-Mat. — 1998. — **49**. — P. 113–121.
14. *Gayevskyy H., Greger K., Zaharias K.* Nonlinear operator equations and operator differential equations. — Mjscow: Mir, 1978 (in Russian).
15. *Ivasishen S.D.* Parabolic boundary-value problems without initial conditions // Ukr. Math. J. — 1982. — **34**, №5. — P. 439—443.
16. *Kazufumi I., Kunisch K.* Optimal control of parabolic variational inequalities // J. Math. Pures Appl. — 2010. — **93**, №4. — P. 329–360.
17. *Kogut O.* On optimal control problem in coefficients for nonlinear elliptic variational inequalities // Visnik Dnipropetrovsk. Univ.. Ser. Modelyuvannya — 2011. — **19**, №8. — P. 86–98 (in Russian).
18. *Lavrenyuk S., Ptashnyk M.* Problem without initial conditions for a nonlinear pseudoparabolic system // Differ. Equ. — 2000. — **36**, №5. — P. 739–748.
19. *Lions J.-L.* Quelques méthodes de résolution des problémes aux limites non linéaires. — Paris: Dunod Gauthier-Villars, 1969.
20. *Oleinik O., Iosifjan G.* An analogue of Saint-Venant's principle and the uniqueness of solutions of boundary value problems for parabolic equations in unbounded domains // Usp. Mat. Nauk. — 1976. — **31**, №6. — P. 142–166 (in Russian).
21. *Pankov A.* Bounded and almost periodic solutions of nonlinear operator differential equations. — Dordrecht: Kluwer, 1990.
22. *Pukach P.Ya.* On problem without initial conditions for some nonlinear degenarated parabolic system // Ukr. Math. J. — 1994. — **46**, №4. — P. 484–487.
23. *Rockafellar R.* On the maximal monotonicity of subdifferential mappings // Pacific J. Math. — 1970. — **33**, №1. — P. 209–216.
24. *Showalter R.* Monotone operators in Banach space and nonlinear partial differential equations. — Providence: Amer. Math. Soc., 1997. — xiv+278 p.
25. *Showalter R.E.* Singular nonlinear evolution equations // Rocky Mt. J. Math. — 1980. — **10**, №3. — P. 499–507.
26. *Tikhonov A., Samarskii A.* Equations of mathematical physics. — Moscow: Nauka, 1972 (in Russian).
27. *Tychonoff A.* Théorèmes d'unicité pour l'équation de la chaleur // Mat. Sb. — 1935. — **42**, №2. — P. 199–216.

28. *Yoshida K.* Functional analysis. — Moscow: Mir, 1967 (in Russian).

Стаття: надійшла до редколегії 02.02.2017
прийнята до друку 13.03.2017

ОПТИМАЛЬНЕ КЕРУВАННЯ В ЗАДАЧАХ БЕЗ ПОЧАТКОВИХ УМОВ ДЛЯ СЛАБКО НЕЛІНІЙНИХ ЕВОЛЮЦІЙНИХ ВАРІАЦІЙНИХ НЕРІВНОСТЕЙ

Микола БОКАЛО, Андрій ЦЕБЕНКО

Львівський національний університет імені Івана Франка,
вул. Університетська, 1, Львів, 79000
e-mail: mm.bokalo@gmail.com, amtseb@gmail.com

Вивчаємо задачу оптимального керування системами, яка описується задачею Фур'є для слабко нелінійних еволюційних варіаційних нерівностей. Керування є коефіцієнтом у нерівності, що описує стан керованої системи. Доведено існування оптимального керування.

Ключові слова: оптимальне керування, задача без початкових умов, варіаційна нерівність.