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THE FOURIER PROBLEM FOR NONLINEAR PARABOLIC EQUATIONS WITH A TIME-DEPENDENT DELAY

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The existence and uniqueness of a weak solution of the Fourier problem for nonlinear parabolic equations with a variable delay are investigated and its a priori estimate is obtained.

Key words: Fourier problem, problem without initial condition, equation with delay, nonlinear parabolic equation.

1. Introduction

The boundary value problems for the nonlinear parabolic equations with a time depended delay are considered. A typical example of the equations being studied here is

$$u_t - \sum_{i,j=1}^{n} \widehat{a}_{ij}(x,t)u_{x_ix_j} + \widehat{a}_0(x,t)u + \int\limits_{t-\tau(t)}^{t} c_0(x,t,s)u(x,s)ds = f(x,t), \tag{1}$$

$(x,t) \in Q := \Omega \times (-\infty, 0]$, where $n \in \mathbb{N}$, Ω is a domain in $\mathbb{R}^n$, $\widehat{a}_{ij} = \widehat{a}_{ji} (i,j = \overline{1,n}), \widehat{a}_0, c_0$ are measurable bounded functions, and there exists $\nu = \mathrm{const} > 0$ such that $\sum_{i,j=1}^{n} \widehat{a}_{ij}(x,t)\xi_i\xi_j \geqslant \nu \sum_{i=1}^{n} \xi_i^2$ for a.e. $(x,t) \in Q$ and for all $(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, $\operatorname*{ess\,inf}_{(x,t)\in Q} \widehat{a}_0(x,t) > 0$, τ is a nonnegative continuous function, f is an integrable function, u is un unknown function.

Fourier problems for evolution equations arise in modeling different nonstationary processes in nature that started a long time ago and initial conditions do not affect on them in the actual time moment, but boundary conditions do affect it. Thus, we can assume that the initial time is $-\infty$, while 0 is the final time, and initial conditions can be replaced with the behaviour of the solution as time variable tends to $-\infty$. The Fourier problem for evolution equations has been widely studied. They appear in modeling in many fields of science such as economics, physics, ecology, cybernetics, etc. (see, e.g.,

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[3], [4], [5], [9], [10], [11], [18], [19], [20], [24], [22], [23]). A lot of information concerning results on problems without initial conditions can be found in [9].

Equations with time delay arise in modelling population dynamics, in non-Newtonian filtration, heat flux, etc. ([13]). The equations of type (1) on finite time interval with constant delay were investigated in [1], [2], [17], [14], [15], etc. Good reference overview on such papers can be found in [17]. We remark that in these papers the semigroup theory is used.

Partial differential equations with a variable delay are less studied, and we known only publications of Rezounenko and Chueshov (in particular, [12], [21]), where equations of type (1) on finite time interval, with $\tau = \tau(u)$, are considered. In [12], a certain abstract parabolic problem with the state dependent delay term of a rather general structure is considered. In [21], the nonlinear partial functional differential equations with main linear elliptic operator and non-local nonlinear term are considered. For proving existence of solutions of problems considered in [12], [21] the Galerkin approximations are used.

Fourier problems for parabolic equations with constant time delay were investigated in [16], [7] (see also references therein).

To the best of our knowledge, the Fourier problems for parabolic equations with time depended delay is an untreated topic in the literature. These problems are considered in our paper. Existence and uniqueness of solution of the problem are proved. The methods of investigation as in [6] are used.

The paper is organized in the following way. In Section 2, the main notations and functional spaces are introduced. The statement of the problem and formulation of the main result are given in Section 3. The main result is proved in Section 4.

2. Notation and auxiliary facts

Let n be a positive integer number, $\mathbb{R}^n$ be the standard linear space of ordered collections $x = (x_1, ..., x_n)$ of real numbers with the norm $|x| := (|x_1|^2 + \ldots + |x_n|^2)^{1/2}$. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with the piecewise smooth boundary $\partial\Omega$. Also, we denote $S := (-\infty, 0]$, $Q := \Omega \times S$, $\overline{Q} := \overline{\Omega} \times S$, $\Sigma := \partial\Omega \times S$.

Let us define some functional spaces. Firstly, denote by $C_c^\infty(\Omega)$ the space of infinite differentiable functions on Ω with compact supports. Denote by $H^1(\Omega) := \{v \in L^2(\Omega) \mid v_{x_i} \in L^2(\Omega)\ (i = \overline{1,n})\}$ the Sobolev space, which is a Hilbert space with the scalar product $(v, w)_{H^1(\Omega)} := \int\limits_\Omega \big\{\nabla v \nabla w + vw\big\}\, dx$, where $\nabla v := (v_{x_1}, \ldots, v_{x_n})$ and the corresponding norm $\|v\|_{H^1(\Omega)} := \big(\int\limits_\Omega \big\{|\nabla v|^2 + |v|^2\big\}\, dx\big)^{1/2}$. By $H_0^1(\Omega)$ we denote the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$.

Let us remind Friedrichs' inequality

$$\int\limits_\Omega |v|^2\, dx \leqslant K_0 \int\limits_\Omega |\nabla v|^2\, dx \quad \forall\ v \in H_0^1(\Omega), \tag{2}$$

where K_0 is a positive constant independent of v. It is known that $1/K_0$ is the first eigenvalue of the problem: $-\Delta v = \lambda v, \quad v|_{\partial\Omega} = 0$.

From Friedrichs' inequality it follows that the norm in $H_0^1(\Omega)$ can also be written as $||v||_{H_0^1(\Omega)} := \int\limits_\Omega |\nabla v|^2 dx$.

For an arbitrary Banach space X by $L^2_{\mathrm{loc}}(S;X)$ we denote the linear space of (classes of) measurable functions defined on S with values in X such that their restrictions on any interval $[a,b]\subset S$ belong to $L^2(a,b;X)$. Denote by $L^p_{\mathrm{loc}}(\overline{Q})$ $(1\leqslant p\leqslant\infty)$ the linear space of (classes of) measurable functions defined on Q such that their restrictions on any bounded measurable set $Q'\subset Q$ belongs to $L^p(Q')$.

Denote by $C^1_c(I)$, where I is an interval, the linear space continuously differentiable finite functions defined on I, moreover, if $I=(t_1,t_2)$, then we will write $C^1_c(t_1,t_2)$ instead of $C^1_c((t_1,t_2))$.

Denote by $F(Q)$ the space of vector-functions $(f_0,f_1,\dots,f_n)$ such that $f_i\in L^2_{\mathrm{loc}}(Q)$ for each $i\in\{0,1,...,n\}$.

Let $\omega\in\mathbb{R}$, X be a Hilbert space with the scalar product $(\cdot,\cdot)_X$ and the corresponding norm $\|\cdot\|_X$. Denote

$$L^2_\omega(S;X):=\Big\{f\in L^2_{\mathrm{loc}}(S;X)\ \Big|\ \int_S e^{2\omega t}\|f(t)\|^2_X dt<\infty\Big\}.$$

$L^2_\omega(S;X)$ is a Hilbert space with the scalar product

$$(f,g)_{L^2_\omega(S;X)}=\int_S e^{2\omega t}(f(t),g(t))_X\,dt$$

and the norm

$$\|f\|_{L^2_\omega(S;X)}:=\Big(\int_S e^{2\omega t}\|f(t)\|^2_X\,dt\Big)^{1/2}. \tag{3}$$

The following auxiliary result, which had been proved in [6], will be used in the sequel.

Lemma 1. *Let $w\in L^2(t_1,t_2;H^1_0(\Omega))$, where $t_1,t_2\in\mathbb{R}$ $(t_1<t_2)$, satisfing the following identity*

$$\int_{t_1}^{t_2}\!\!\int_\Omega\Big\{-wv\varphi'+(g_0v+\sum_{i=1}^n g_iv_{x_i})\varphi\Big\}\,dxdt=0,\quad v\in H^1_0(\Omega),\ \varphi\in C^1_c(t_1,t_2), \tag{4}$$

for some $g_i\in L^2(\Omega\times(t_1,t_2))$ $(i=\overline{0,n})$. Then $w\in C([t_1,t_2];L^2(\Omega))$ and

$$\frac12\theta(t)\int_\Omega|w(x,t)|^2\,dx\Big|_{t=\sigma_1}^{t=\sigma_2}-\frac12\int_{\sigma_1}^{\sigma_2}\!\!\int_\Omega|w|^2\theta'\,dxdt+\int_{\sigma_1}^{\sigma_2}\!\!\int_\Omega\Big\{g_0w+\sum_{i=1}^n g_iw_{x_i}\Big\}\theta\,dxdt=0 \tag{5}$$

for any $\sigma_1,\sigma_2\in[t_1,t_2]$ $(\sigma_1<\sigma_2)$, for every $\theta\in C^1([t_1,t_2])$.

3. Statement of the problem and main result

In this paper we consider weak solutions $u:\overline{Q}\to\mathbb{R}$ of the problem

$$\begin{aligned}u_t-\sum_{i=1}^n\frac{d}{dx_i}\,a_i(x,t,u,\nabla u)+a_0(x,t,u,\nabla u)+\int_{t-\tau(t)}^{t}c(x,t,s,u(x,s))ds=\\=-\sum_{i=1}^n\frac{\partial}{\partial x_i}f_i(x,t)+f_0(x,t),\qquad(x,t)\in Q,\end{aligned} \tag{6}$$

$$u\big|_\Sigma=0, \tag{7}$$

$$\lim_{t\to-\infty} e^{2\omega t}\int_{\Omega} |u(x,t)|^2\,dx = 0, \tag{8}$$

for some $\omega\in\mathbb{R}$. Here $\tau : S\to\mathbb{R}$ is a continuous bounded function such that $\tau(t)\geqslant 0$ for all $t\in S$, and $a_i : Q\times\mathbb{R}^{1+n}\to\mathbb{R}$, $c : Q\times S\times\mathbb{R}\to\mathbb{R}$, $f_i : Q\to\mathbb{R}$ $(i=\overline{0,n})$ are given real-valued functions from the corresponding classes of initial data.

We introduce the following classes of the initial data.

Define $\mathcal{A}$ to be the set of the collections $(a_0,a_1,\dots,a_n)$ of the functions $a_i : Q\times\mathbb{R}^{1+n}\to\mathbb{R}$ $(i\in\{0,1,\dots,n\})$ which satisfy the following conditions:

($\mathcal{A}_1$) for every $i\in\{0,1,\dots,n\}$, a_i is a Caratheodory function (i.e., $a_i(x,t,\cdot,\cdot) : \mathbb{R}^{1+n}\to\mathbb{R}$ is a continuous for a.e. $(x,t)\in Q$, and $a_i(\cdot,\cdot,\rho,\xi) : Q\to\mathbb{R}$ is measurable for every $(\rho,\xi)\in\mathbb{R}^{1+n}$), and $a_i(x,t,0,0)=0$ for a.e. $(x,t)\in Q$;

($\mathcal{A}_2$) for every $i\in\{0,1,\dots,n\}$, for a.e. $(x,t)\in Q$ and for every $(\rho,\xi)\in\mathbb{R}^{1+n}$ the estimate

$$|a_i(x,t,\rho,\xi)|\le C_1\big(|\rho|+\sum_{j=1}^n|\xi_j|\big)+h_i(x,t)$$

is valid, where $C_1>0$ is constant and $h_i\in L^2_{\mathrm{loc}}(Q)$;

($\mathcal{A}_3$) for a.e. $(x,t)\in Q$ and for every (ρ_1,ξ^1), $(\rho_2,\xi^2)\in\mathbb{R}^{1+n}$ the inequality

$$\sum_{i=1}^n\big(a_i(x,t,\rho_1,\xi^1)-a_i(x,t,\rho_2,\xi^2)\big)(\xi_i^1-\xi_i^2)+\big(a_0(x,t,\rho_1,\xi^1)- \\ -a_0(x,t,\rho_2,\xi^2)\big)(\rho_1-\rho_2)\ge K_1\sum_{i=1}^n|\xi_i^1-\xi_i^2|^2+K_2|\rho_1-\rho_2|^2 \tag{9}$$

holds, where $K_1>0, K_2\in\mathbb{R}$ are constants.

Define $\mathcal{C}$ to be the set of the real-value functions $c : Q\times S\times\mathbb{R}\to\mathbb{R}$ which satisfy the following conditions:

($\mathcal{C}_1$) c is a Caratheodory function (i.e., $c(x,t,s,\cdot):\mathbb{R}\to\mathbb{R}$ is a continuous function for a.e. $(x,t,s)\in Q\times S$, and $c(\cdot,\cdot,\cdot,\rho) : Q\times S\to\mathbb{R}$ is a measurable function for every $\rho\in\mathbb{R}$), in addition, $c(x,t,s,0)=0$ for a.e. $(x,t,s)\in Q\times S$;

($\mathcal{C}_2$) there exists a constant $L>0$ such that for a.e. $(x,t,s)\in Q\times S$ and for every ρ_1, $\rho_2\in\mathbb{R}$ the inequality

$$\big|c(x,t,s,\rho_1)-c(x,t,s,\rho_2)\big|\le L|\rho_1-\rho_2| \tag{10}$$

holds.

Remark 1. The condition ($\mathcal{C}_1$) (more precisely, $c(x,t,s,0)=0$) and ($\mathcal{C}_2$) imply that for a.e. $(x,t,s)\in Q\times S$, and for every $\rho\in\mathbb{R}$ the following estimate is valid:

$$|c(x,t,s,\rho)|\le L|\rho|. \tag{11}$$

Now we can give a definition of the weak solution of problem (6)–(8).

Definition 1. *Let* $(a_0,a_1,\dots,a_n)\in\mathcal{A}$, $c\in\mathcal{C}$, $(f_0,f_1,\dots,f_n)\in F(Q)$. *A function* $u\in L^2_{\mathrm{loc}}(S;H^1_0(\Omega))\cap C\big(S;L^2(\Omega)\big)$ *is called a weak solution of problem* (6)–(8) *if it* *satisfies condition* (8)*, and the integral equality*

$$\iint\limits_{Q}\Big\{\sum_{i=1}^{n} a_i(x,t,u,\nabla u)v_{x_i}\varphi + a_0(x,t,u,\nabla u)v\varphi + v\varphi\int\limits_{t-\tau(t)}^{t} c(x,t,s,u(x,s))\,ds \\ - uv\varphi'\Big\}\,dxdt = \iint\limits_{Q}\Big\{\sum_{i=1}^{n} f_i v_{x_i}\varphi + f_0 v\varphi\Big\}\,dxdt \tag{12}$$

holds for every $v \in H_0^1(\Omega)$ *and* $\varphi \in C_c^1(-\infty,0)$.

Denote

$$\tau^+ := \sup_{t\in S}\tau(t), \qquad \chi(\omega) := \begin{cases} \tau^+, & \text{if } \omega = 0, \\ \frac{1}{2\omega}(e^{2\omega\tau^+}-1), & \text{if } \omega \neq 0. \end{cases} \tag{13}$$

We consider the inequality

$$\omega + 2L\sqrt{\tau^+\chi(\omega)} < K_1/K_0 + K_2, \tag{14}$$

where K_2 is from (9).

It is obvious that $\omega + 2L\sqrt{\tau^+\chi(\omega)} \to -\infty$ when $\omega \to -\infty$, because $\chi(\omega) \to 0$ when $\omega \to -\infty$. Hence, inequality (14) has solutions.

Theorem 1. *Let* $(a_0, a_1, \ldots, a_n) \in \mathcal{A}$, $c \in \mathcal{C}$, $(f_0, f_1, \ldots, f_n) \in F(Q)$, *and let* ω *satisfies* (14). *If problem* (6)–(8) *has a solution, then it is unique.*

Theorem 2. *Let the assumptions of Theorem 1 be fulfilled, and* $f_i \in L^2_\omega(S; L^2(\Omega))$ $(i = \overline{0,n})$. *Then there exists a unique solution of problem* (6)–(8)*, and it satisfies the following estimates:*

$$e^{2\omega\sigma}\int\limits_{\Omega}|u(x,\sigma)|^2\,dx \leqslant C_2\int\limits_{-\infty}^{\sigma} e^{2\omega t}||f(\cdot,t)||^2_{L^2(\Omega)}\,dt, \quad \sigma \in S, \tag{15}$$

$$||u||_{L^2_\omega(S;H_0^1(\Omega))} \leqslant C_3\ ||f||_{L^2_\omega(S;L^2(\Omega))}, \tag{16}$$

where C_2, C_3 *are positive constants depending on* $\tau^+, \omega, L, K_0, K_1, K_2$ *only.*

4. Proof of the main results

For a function $w : Q \to \mathbb{R}$ we denote

$$a_j(w)(x,t) := a_j(x,t,w(x,t),\nabla w(x,t)), \quad (x,t) \in Q, \quad j = \overline{0,n},$$

$$c(w)(x,t,s) := c(x,t,s,w(x,s)), \quad (x,t,s) \in Q \times S. \tag{17}$$

Proof of Theorem 1. Suppose the contrary. Let u_1 and u_2 be two distinct weak solutions of the problem. Denote $w := u_1 - u_2$. Considering the difference between (12) for $u = u_2$

and $u = u_1$, we obtain

$$
\begin{aligned}
&-\iint\limits_Q wv\varphi'\,dxdt+\iint\limits_Q\Big[\sum_{i=1}^{n}(a_i(u_1)-a_i(u_2))v_{x_i}+\big(a_0(u_1)-a_0(u_2)\big)v \\
&+v\int\limits_{t-\tau(t)}^{t}\big(c(u_1)-c(u_2)\big)ds\Big]\varphi\,dxdt=0 \qquad \forall v\in H_0^1(\Omega),\ \forall\varphi\in C_c^1(-\infty,0).
\end{aligned}
\tag{18}
$$

It is clear that from (8) for $u = u_2$ and $u = u_1$ we have

$$
e^{2\omega t}\int\limits_\Omega |w(x,t)|^2\,dx \underset{t\to-\infty}{\longrightarrow} 0. \tag{19}
$$

According to Lemma 1, setting $\theta(t) = e^{2\omega t}$, $t \in \mathbb{R}$, from equality (18) we get

$$
\begin{aligned}
&\frac12 e^{2\omega\sigma_2}\int\limits_\Omega |w(x,\sigma_2)|^2\,dx-\frac12 e^{2\omega\sigma_1}\int\limits_\Omega |w(x,\sigma_1)|^2\,dx-\ \omega\int\limits_{\sigma_1}^{\sigma_2}\int\limits_\Omega e^{2\omega t}|w(x,t)|^2\,dxdt+ \\
&+\int\limits_{\sigma_1}^{\sigma_2}\int\limits_\Omega e^{2\omega t}\Big[\sum_{i=1}^{n}(a_i(u_1)-a_i(u_2))(u_{1,x_i}-u_{2,x_i})+(a_0(u_1)-a_0(u_2))(u_1-u_2)+ \\
&\qquad\qquad +w\int\limits_{t-\tau(t)}^{t}\big(c(u_1)-c(u_2)\big)ds\Big]\,dxdt=0,
\end{aligned}
\tag{20}
$$

for arbitrary $\sigma_1, \sigma_2 \in S$ $(\sigma_1 < \sigma_2)$.

From condition $(\mathcal{A}_3)$, for a.e. $(x,t) \in Q$ we have

$$
\begin{aligned}
&\int\limits_{\sigma_1}^{\sigma_2}\int\limits_\Omega e^{2\omega t}\Big[\sum_{i=1}^{n}(a_i(u_1)-a_i(u_2))(u_{1,x_i}-u_{2,x_i})+(a_0(u_1)-a_0(u_2))(u_1-u_2)\Big]\,dxdt\geqslant \\
&\qquad\geqslant\int\limits_{\sigma_1}^{\sigma_2}\int\limits_\Omega e^{2\omega t}\Big[K_1|\nabla w|^2+K_2|w|^2\Big]\,dxdt.
\end{aligned}
\tag{21}
$$

Now, we consider the last term from equality (20). Using condition $(\mathcal{C}_2)$, the Fubini Theorem and the Cauchy–Schwarz inequality, for a.e. $x \in \Omega$ we obtain

$$
\begin{aligned}
&\left|\int\limits_{\sigma_1}^{\sigma_2} e^{2\omega t}w(x,t)\Big(\int\limits_{t-\tau(t)}^{t}\big(c(u_1)(x,t,s)-c(u_2)(x,t,s)\big)ds\Big)dt\right|\leqslant \\
&\qquad\leqslant L\int\limits_{\sigma_1}^{\sigma_2} e^{2\omega t}|w(x,t)|\left(\int\limits_{t-\tau^+}^{t}|w(x,s)|ds\right)dt\leqslant
\end{aligned}
$$

$$\leqslant L\sqrt{\tau^+}\left(\int\limits_{\sigma_1}^{\sigma_2} e^{2\omega t}|w(x,t)|^2dt\right)^{1/2}\left(\int\limits_{\sigma_1}^{\sigma_2} e^{2\omega t}\Big(\int\limits_{t-\tau^+}^{t}|w(x,s)|^2ds\Big)dt\right)^{1/2}. \qquad (22)$$

Changing order of integration and assuming $w(x,t)=0$ for $x\in\Omega,\ t>0$, for a.e. $x\in\Omega$ we have

$$\int\limits_{\sigma_1}^{\sigma_2} e^{2\omega t}\Big(\int\limits_{t-\tau^+}^{t}|w(x,s)|^2ds\Big)dt \leqslant \int\limits_{\sigma_1-\tau^+}^{\sigma_2}|w(x,s)|^2ds\int\limits_{s}^{s+\tau^+}e^{2\omega t}dt =$$

$$= \chi(\omega)\Big(\int\limits_{\sigma_1}^{\sigma_2} e^{2\omega s}|w(x,s)|^2ds + \int\limits_{\sigma_1-\tau^+}^{\sigma_1} e^{2\omega s}|w(x,s)|^2ds\Big), \qquad (23)$$

where $\chi(\omega)$ is defined in (13).

Substituting in (22) the last term from the obtained above chain of relations instead of the first one, and using the inequalities: $\sqrt{ab}\leqslant a+b,\ \sqrt{a+b}\leqslant\sqrt{a}+\sqrt{b}\ (a\geqslant 0, b\geqslant 0)$, we obtain

$$\Big|\int\limits_{\sigma_1}^{\sigma_2} e^{2\omega t}w(x,t)\int\limits_{t-\tau(t)}^{t}\big(c(u_1)(x,t,s)-c(u_2)(x,t,s)\big)dsdt\Big|$$

$$\leqslant L\sqrt{\tau^+\chi(\omega)}\Big(2\int\limits_{\sigma_1}^{\sigma_2} e^{2\omega t}|w(x,t)|^2dt + \int\limits_{\sigma_1-\tau^+}^{\sigma_1} e^{2\omega t}|w(x,t)|^2dt\Big). \qquad (24)$$

Using (21), (24), from (20) we obtain

$$\frac{1}{2}e^{2\omega\sigma_2}\int\limits_{\Omega}|w(x,\sigma_2)|^2\,dx \;-\; \frac{1}{2}e^{2\omega\sigma_1}\int\limits_{\Omega}|w(x,\sigma_1)|^2\,dx \;+K_1\int\limits_{\sigma_1}^{\sigma_2}\int\limits_{\Omega} e^{2\omega t}|\nabla w(x,t)|^2\,dxdt$$

$$+(K_2-2L\sqrt{\tau^+\chi(\omega)}-\omega)\int\limits_{\sigma_1}^{\sigma_2}\int\limits_{\Omega} e^{2\omega t}|w(x,t)|^2\,dxdt - L\sqrt{\tau^+\chi(\omega)}\int\limits_{\sigma_1-\tau^+}^{\sigma_1}\int\limits_{\Omega} e^{2\omega t}|w(x,t)|^2\,dxdt\leqslant 0.$$

From this, using (2), we get

$$\frac{1}{2}e^{2\omega\sigma_2}\int\limits_{\Omega}|w(x,\sigma_2)|^2\,dx \;-\; \frac{1}{2}e^{2\omega\sigma_1}\int\limits_{\Omega}|w(x,\sigma_1)|^2\,dx$$

$$+\;\big(K_1/K_0+K_2-2L\sqrt{\tau^+\chi(\omega)}-\omega\big)\int\limits_{\sigma_1}^{\sigma_2}\int\limits_{\Omega} e^{2\omega t}|w(x,t)|^2\,dxdt$$

$$-L\sqrt{\tau^+\chi(\omega)}\int\limits_{\sigma_1-\tau^+}^{\sigma_1}\int\limits_{\Omega} e^{2\omega t}|w(x,t)|^2\,dxdt\leqslant 0.$$

Since ω is a solution of inequality (14),

$$e^{2\omega\sigma_2}\int\limits_\Omega |w(x,\sigma_2)|^2\,dx \leqslant e^{2\omega\sigma_1}\int\limits_\Omega |w(x,\sigma_1)|^2\,dx + 2L\sqrt{\tau^+\chi(\omega)}\int\limits_{\sigma_1-\tau^+}^{\sigma_1}\int\limits_\Omega e^{2\omega t}|w(x,t)|^2\,dxdt. \tag{25}$$

Let as fix an arbitrary σ_2 in (25), and let σ_1 tends to $-\infty$. According to condition (19), the first term from the right side of inequality (25) tends to 0. Obviously, the second term from the right side of inequality (25) also tends to 0. Indeed,

$$0 \leqslant \int\limits_{\sigma_1-\tau^+}^{\sigma_1}\int\limits_\Omega e^{2\omega t}|w(x,t)|^2\,dxdt \leqslant \tau^+ \max_{t\in[\sigma_1,\sigma_1-\tau^+]}\Big(e^{2\omega t}\int\limits_\Omega |w(x,t)|^2\,dx\Big) \underset{\sigma_1\to-\infty}{\longrightarrow} 0.$$

Thus, we get the equality $e^{2\omega\sigma_2}\int\limits_\Omega |w(x,\sigma_2)|^2\,dx = 0$. Since $\sigma_2 \in S$ is arbitrary, we obtain $w(x,t) = 0$ for a.e. $(x,t) \in Q$, this contradicts our assumption. Therefore, the solution of problem (6)–(8) is unique. □

Proof of Theorem 2. For each $m \in N$ denote $Q_m := \Omega\times(-m,0]$, $\tau_m := \min\limits_{-m\leqslant t\leqslant 0}(t-\tau(t))$. Denote $f_{i,m}(\cdot,t) := f_i(\cdot,t)$ if $-m < t \leqslant 0$, and $f_{i,m}(\cdot,t) := 0$ if $t \leqslant -m$. We consider the problem: to find a function $u_m \in L^2(-m,0;H_0^1(\Omega)) \cap C([-\tau_m,0];L^2(\Omega))$ which satisfies the initial condition

$$u_m(x,t) = 0, \quad (x,t) \in \overline{\Omega}\times[-\tau_m,-m], \tag{26}$$

and equation (6) in Q_m in the sense of integral equality, i.e.,

$$\iint\limits_{Q_m}\Big\{\sum_{i=1}^n a_i(x,t,u_m,\nabla u_m)v_{x_i}\varphi + a_0(x,t,u_m,\nabla u_m)v\varphi + v\varphi\int\limits_{t-\tau(t)}^t c(u_m)(x,t,s)\,ds - u_m v\varphi'\Big\}dxdt$$

$$= \iint\limits_{Q_m}\Big\{\sum_{i=1}^n f_{i,m}v_{x_i}\varphi + f_{0,m}v\varphi\Big\}\,dxdt, \quad v\in H_0^1(\Omega), \quad \varphi\in C_c^1(-m,0). \tag{27}$$

Existence and uniqueness of a solution of this problem follows from the paper [8]. For each $m \in \mathbb{N}$ we extend u_m by 0 onto Q and denote this extension by u_m again.

Now, we shall get estimates of u_m for each $m \in \mathbb{N}$. First, remark that for each $m \in \mathbb{N}$ the function u_m belongs to $L^2(S;H_0^1(\Omega)) \cap C(S;L^2(\Omega))$ and satisfies integral equality (12) with $f_{i,m}$ instead of f_i $(i=\overline{1,n})$, i.e., the following equality holds:

$$\iint\limits_{Q}\Big\{\sum_{i=1}^n a_i(x,t,u_m,\nabla u_m)v_{x_i}\varphi + a_0(x,t,u_m,\nabla u_m)v\varphi +$$

$$+v\varphi\int\limits_{t-\tau(t)}^t c(u_m)(x,t,s)ds - u_m v\varphi'\Big\}dxdt$$

$$= \iint\limits_{Q}\Big\{\sum_{i=1}^n f_{i,m}v_{x_i}\varphi + f_{0,m}v\varphi\Big\}\,dxdt, \quad v\in H_0^1(\Omega), \quad \varphi\in C_c^1(-\infty,0). \tag{28}$$

Applying Lemma 1 with $\theta(t) = 2e^{2\omega t}$, $t \in S$, and $[\sigma_1, \sigma_2] \subset S$, $\sigma_1 < -m$, to equality (28), we obtain

$$e^{2\omega\sigma_2}\int\limits_\Omega |u_m(x,\sigma_2)|^2\,dx - e^{2\omega\sigma_1}\int\limits_\Omega |u_m(x,\sigma_1)|^2\,dx-$$
$$-2\omega\int\limits_{\sigma_1}^{\sigma_2}\int\limits_\Omega e^{2\omega t}|u_m(x,t)|^2\,dxdt + 2\int\limits_{\sigma_1}^{\sigma_2}\int\limits_\Omega e^{2\omega t}\Big[\sum_{i=1}^n a_i(u_m)u_{m,x_i} + a_0(u_m)u_m+$$
$$+u_m\int\limits_{t-\tau(t)}^{t} c(u_m)(x,t,s)ds\Big]\,dxdt = 2\int\limits_{\sigma_1}^{\sigma_2}\int\limits_\Omega e^{2\omega t}\Big\{\sum_{i=1}^n f_{i,m}u_{m,x_i} + f_{0,m}u_m\Big\}\,dxdt. \tag{29}$$

According to the Cauchy inequality for a.e. $t \in S$ we have

$$\int\limits_{\sigma_1}^{\sigma_2}\int\limits_\Omega e^{2\omega t}\Big\{\sum_{i=1}^n f_{i,m}u_{m,x_i} + f_{0,m}u_m\Big\}\,dxdt$$
$$\leqslant \frac{\varepsilon}{2}\int\limits_{\sigma_1}^{\sigma_2}\int\limits_\Omega e^{2\omega t}\Big\{|\nabla u_m|^2 + |u_m|^2\Big\}dxdt + \frac{1}{2\varepsilon}\int\limits_{\sigma_1}^{\sigma_2}\int\limits_\Omega e^{2\omega t}\sum_{i=0}^n |f_{i,m}|^2 dxdt, \tag{30}$$

for arbitrary $\varepsilon > 0$.

Similar to (24), from (11) for a.e. $x \in \Omega$ we can get

$$\Big|\int\limits_{\sigma_1}^{\sigma_2} e^{2\omega t}u_m(x,t)\int\limits_{t-\tau(t)}^{t} c(u_m)(x,t,s)dsdt\Big| \leqslant$$
$$\leqslant L\sqrt{\tau^+\chi(\omega)}\Big(2\int\limits_{\sigma_1}^{\sigma_2} e^{2\omega t}|u_m(x,t)|^2 dt + \int\limits_{\sigma_1-\tau^+}^{\sigma_1} e^{2\omega t}|u_m(x,t)|^2 dt\Big). \tag{31}$$

By $(\mathcal{A}_1)$, $(\mathcal{A}_3)$ and (2) we obtain that

$$\int\limits_{\sigma_1}^{\sigma_2}\int\limits_\Omega e^{2\omega t}\Big\{\sum_{i=1}^n a_i(u_m)u_{m,x_i} + a_0(u_m)u_m\Big\}\,dxdt \geq \int\limits_{\sigma_1}^{\sigma_2}\int\limits_\Omega e^{2\omega t}\Big\{K_1|\nabla u_m|^2 + K_2|u_m|^2\Big\}\,dxdt$$
$$= \int\limits_{\sigma_1}^{\sigma_2}\int\limits_\Omega e^{2\omega t}\Big\{(\delta + 1 - \delta)K_1|\nabla u_m|^2 + K_2|u_m|^2\Big\}\,dxdt$$
$$\geqslant \int\limits_{\sigma_1}^{\sigma_2}\int\limits_\Omega e^{2\omega t}\Big\{(1-\delta)K_1|\nabla u_m|^2 + (\delta K_1/K_0 + K_2)\,|u_m|^2\Big\}\,dxdt, \tag{32}$$

where $\delta > 0$ is a constant close to 1.

By (29) using estimates (30), (31) and (4), and condition (26), and taking $\sigma_1 < -m$ we obtain

$$e^{2\omega\sigma_2}\int\limits_\Omega |u_m(x,\sigma_2)|^2\,dx + (2(1-\delta)K_1-\varepsilon)\int\limits_{-m}^{\sigma_2}\int\limits_\Omega e^{2\omega t}|\nabla u_m(x,t)|^2dxdt$$

$$+\Big(2\big(\delta K_1/K_0+K_2-\omega-2L\sqrt{\tau^+\chi(\omega)}\big)-\varepsilon\Big)\int\limits_{-m}^{\sigma_2}\int\limits_\Omega e^{2\omega t}|u_m(x,t)|^2dxdt$$

$$\leqslant \varepsilon^{-1}\int\limits_{-m}^{\sigma_2}\int\limits_\Omega e^{2\omega t}\sum_{i=0}^n |f_{i,m}(x,t)|^2dxdt. \tag{33}$$

If we take $\varepsilon=\min\{\delta K_1/K_0+K_2-\omega-2L\sqrt{\tau^+\chi(\omega)},(1-\delta)K_1\}$, then

$$e^{2\omega\sigma_2}\int\limits_\Omega |u_m(x,\sigma_2)|^2\,dx + C_4\int\limits_{-m}^{\sigma_2}\int\limits_\Omega e^{2\omega t}|\nabla u_m|^2\,dxdt \leqslant C_5\int\limits_{-m}^{\sigma_2}\int\limits_\Omega e^{2\omega t}\sum_{i=0}^n|f_{i,m}|^2dxdt, \tag{34}$$

where C_4 and C_5 are positive constants depending on K_0,K_1,K_2,L,τ^+ and ω only.

It is clear that u_m belongs to $L^2_\omega(S;H^1_0(\Omega))$. Therefore, from (34) we obtain

$$e^{2\omega\sigma}\int\limits_\Omega |u_m(x,\sigma)|^2\,dx + C_4\int\limits_{-\infty}^{\sigma}\int\limits_\Omega e^{2\omega t}|\nabla u_m|^2\,dxdt \leqslant C_5\int\limits_{-\infty}^{\sigma}\int\limits_\Omega e^{2\omega t}\sum_{i=0}^n|f_{i,m}|^2dxdt, \quad \sigma\in S. \tag{35}$$

By the definition of $f_{i,m}$, from (35) we have

$$e^{2\omega\sigma}||u_m(\cdot,\sigma)||^2_{L^2(\Omega)} \leqslant C_5\int\limits_{-\infty}^{\sigma} e^{2\omega t}\sum_{i=0}^n||f_i(\cdot,t)||^2_{L^2(\Omega)}\,dt, \quad \sigma\in S, \tag{36}$$

$$\|u_m\|_{L^2_\omega(S;H^1_0(\Omega))} \leqslant C_6\sum_{i=0}^n\|f_i\|_{L^2_\omega(S;L^2(\Omega))}, \tag{37}$$

where $C_5>0, C_6>0$ are positive constants depending on $\omega,\tau^+,K_0,K_1,K_2$ and L only.

Let us show that $\{u_m\}$ is a Cauchy sequence. Taking arbitrary $k,l\in\mathbb{N}$ such that $k<l$ and considering difference between u_k and u_l, similarly as estimate (35), for any $\sigma\in S$ such that $-k\leqslant\sigma\leqslant 0$ one can obtain

$$e^{2\omega\sigma}\int\limits_\Omega |u_k(x,\sigma)-u_l(x,\sigma)|^2\,dx + C_7\int\limits_{-l}^{\sigma}\int\limits_\Omega e^{2\omega t}|\nabla(u_k-u_l)|^2\,dxdt$$

$$\leqslant C_8\int\limits_{-l}^{\sigma}\int\limits_\Omega e^{2\omega t}\sum_{i=0}^n|f_{i,k}-f_{i,l}|^2dxdt, \tag{38}$$

where C_7 and C_8 are positive constants independent of k, l. Thus

$$e^{2\omega\sigma}\|u_k(\cdot,\sigma)-u_l(\cdot,\sigma)\|^2_{L^2(\Omega)} \leqslant C_8\int\limits_{-l}^{-k} e^{2\omega t}\sum_{i=0}^{n}\|f_i(\cdot,t)\|^2_{L^2(\Omega)}\,dt, \quad -k\leqslant\sigma\leqslant 0, \tag{39}$$

$$\|u_k-u_l\|_{L^2_\omega(S;H^1_0(\Omega))} \leqslant C_9\int\limits_{-l}^{-k} e^{2\omega t}\sum_{i=0}^{n}\|f_i(\cdot,t)\|^2_{L^2(\Omega)}\,dt. \tag{40}$$

The condition $f_i \in L^2_\omega(S;L^2(\Omega))$ implies that the right-hand sides of inequalities (39) and (40) tend to zero when k and l tend to $+\infty$. This means that the sequence $\{u_m\}_{m=1}^{\infty}$ is a Cauchy sequence in the space $L^2_\omega(S;H^1_0(\Omega))\cap C(S;L^2(\Omega))$. Consequently, we obtain the existence of the function $u\in L^2_\omega(S;H^1_0(\Omega))\cap C(S;L^2(\Omega))$ such that

$$u_m \underset{m\to\infty}{\longrightarrow} u \qquad \text{strongly in} \quad L^2_\omega(S;H^1_0(\Omega))\cap C(S;L^2(\Omega)). \tag{41}$$

Using condition $(\mathcal{C}_2)$, the Cauchy-Schwarz inequality and (41) we get

$$\int\limits_{\sigma_1}^{\sigma_2}\int\limits_{\Omega}\Big|\int\limits_{t-\tau(t)}^{t} c(u_m)(x,t,s)ds-\int\limits_{t-\tau(t)}^{t} c(u)(x,t,s)ds\Big|^2dxdt\leqslant$$

$$\leqslant \tau^+\int\limits_{\sigma_1}^{\sigma_2}\int\limits_{\Omega}\Bigg(\int\limits_{t-\tau^+}^{t}|c(u_m)(x,t,s)-c(u)(x,t,s)|^2ds\Bigg)dxdt\leqslant$$

$$\leqslant L^2\tau^+\int\limits_{\Omega}\int\limits_{\sigma_1}^{\sigma_2}\int\limits_{t-\tau^+}^{t}|u_m(x,s)-u(x,s)|^2ds\,dtdx\leqslant$$

$$\leqslant L^2\tau^+\int\limits_{\Omega}\int\limits_{\sigma_1-\tau^+}^{\sigma_2}\int\limits_{s}^{s+\tau^+}|u_m(x,s)-u(x,s)|^2dt\,dsdx=$$

$$=L^2\tau^{+2}\int\limits_{\sigma_1-\tau^+}^{\sigma_2}\int\limits_{\Omega}|u_m(x,t)-u(x,t)|^2dtdx \underset{m\to\infty}{\longrightarrow} 0.$$

Thus, we obtain

$$\int\limits_{t-\tau(t)}^{t} c(u_m)ds \underset{m\to\infty}{\longrightarrow}\int\limits_{t-\tau(t)}^{t} c(u)ds \quad \text{strongly in} \quad L^2_{\mathrm{loc}}(\overline{Q}). \tag{42}$$

By $(\mathcal{A}_2)$ and estimate (37) we have that for each $\sigma_1,\sigma_2\in S(\sigma_1<\sigma_2)$ the estimate

$$\int\limits_{\sigma_1}^{\sigma_2}\int\limits_{\Omega}|a_i(u_m)|^2\,dxdt\leqslant C_{10}\int\limits_{\sigma_1}^{\sigma_2}\int\limits_{\Omega}\left(|u_m|^2+|\nabla u_m|^2+|h_i|^2\right)dxdt\leqslant C_{11} \tag{43}$$

is correct, where C_{10} and C_{11} are positive constants independent of m.

Hence, from (43) we obtain that the function $a_i(u_m)$ is bounded in $L^2_{\rm loc}(\overline{Q})$. This and (41) yield that there exists a subsequence of $\{u_m\}_{m=1}^{\infty}$ (denoted also by $\{u_m\}_{m=1}^{\infty}$) and functions $\chi_i \in L^2_{\rm loc}(\overline{Q})$ $(i=\overline{0,n})$ such that

$$u_m \underset{m\to\infty}{\longrightarrow} u, \quad u_{m,x_i} \underset{m\to\infty}{\longrightarrow} u_{x_i} \quad \text{a.e. on} \quad Q, \quad i=\overline{0,n}, \tag{44}$$

$$a_i(u_m) \underset{m\to\infty}{\longrightarrow} \chi_i \quad \text{weakly in} \quad L^2_{\rm loc}(\overline{Q}), \quad i=\overline{0,n}, \tag{45}$$

Condition $(\mathcal{A}_1)$ and (44) yield

$$a_i(u_m) \underset{m\to\infty}{\longrightarrow} a_i(u) \quad \text{a.e. on Q}, \quad i=\overline{0,n}. \tag{46}$$

By Lemma 1.3 from [18], (45) and (46) we obtain

$$a_i(u_m) \underset{m\to\infty}{\longrightarrow} a_i(u) \quad \text{weakly in} \quad L^2_{\rm loc}(\overline{Q}), \quad i=\overline{0,n}. \tag{47}$$

Let us show that the function u is a weak solution of problem (6), (7), (8). For this purpose, we tend $m\to\infty$ in identity (27), taking into account (41), (42), (47) and the definition of the function $f_{i,m}$. As a result we obtain identity (12). Now, taking into account (41), we let $m\to+\infty$ in (36). From the resulting inequality and condition $f\in L^2_{\omega}(S;L^2(\Omega))$, we obtain condition (8). Hence, we have proven that u is a weak solution of problem (6), (7), (8).

It is easy to show that inequalities similar to (36), (37), with u instead of u_m hold. Thus, estimates (15), (16) hold. $\square$

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ЗАДАЧА ФУР'Є ДЛЯ МАЙЖЕ ЛІНІЙНИХ ПАРАБОЛІЧНИХ РІВНЯНЬ ЗІ ЗМІННИМ ЗАПІЗНЕННЯМ

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Досліджено існування та єдиність узагальнених розв'язків задачі без початкових умов для нелінійних параболічних рівнянь зі змінним запізненням. Також отримано апріорні оцінки розв'язків розглянутої задачі.

Ключові слова: задача Фур'є, задача без початкових умов, рівняння з запізненням, нелінійне параболічне рівняння.