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THE MIXED PROBLEM FOR A SEMILINEAR HYPERBOLIC EQUATION IN GENERALIZED LEBESGUE SPACES

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The initial boundary value problem for a semilinear hyperbolic equation in a bounded cylindrical domain is considered. Existence conditions for this problem are obtained in generalized Lebesgue spaces.

Key words: hyperbolic equation, mixed problem, generalized Lebesgue spaces.

1. Problems for nonlinear hyperbolic equations were considered by many authors [1-36]. The mixed problem for such equations are well studied in Sobolev spaces specially in case of bounded domains with respect to spacial variables. In particular, semilinear hyperbolic equation of the form

$$u_{tt} - \Delta u + \alpha u_t + \beta |u|^{p-2} u + \gamma |u_t|^{q-2} u_t = f(x,t),$$

where $\alpha \geqslant 0,\ \gamma \geqslant 0,\ \beta \in \mathbb{R}$ and p, q are constants, is the subject of research in [1-22]. Cauchy problems for the previous equation are examined in [1-7] and mixed problems in [8-14] respectively. For these problems the conditions on the coefficients of the equation and nonlinearity exponents are stated which provides the existence, uniqueness or nonexistence of the problems solutions.

Over the last years problems for nonlinear partial differential equations have being actively studied in some special classes of functions namely in generalized Lebesgue and Sobolev spaces. The main properties of these spaces are given in [37]. In this article we consider a mixed problem for certain generalization of the mentioned above equation with $\beta > 0,\ \gamma > 0$, in which nonlinearity exponents depend on spacial variables. Conditions are obtained providing the existence of the solution in generalized Lebesgue spaces.

2. Formulation of the problem. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the regular in Calderon's sense boundary $\partial\Omega$ [38], $Q_\tau = \Omega \times (0,\tau)$, $\tau \in (0,T]$, $Q_{t_1,t_2} = \Omega \times (t_1,t_2)$, $0 \leqslant t_1 < t_2 \leqslant T$, $S_T = \partial\Omega \times (0,T)$.

Consider the following problem in the domain Q_T:

$$u_{tt} - \sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} a_i(x,t)u_{x_i} + a_0(x,t)u_t + c(x,t)u +$$

$$+b_0(x,t)|u_t|^{p_0(x)-2}u_t + b_1(x,t)|u|^{p_1(x)-2}u = f(x,t), \quad (x,t) \in Q_T, \tag{1}$$

$$u|_{S_T} = 0, \tag{2}$$

$$u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), \quad x \in \Omega. \tag{3}$$

We denote by $L^{p_0(x)}(\Omega)$ the class of all measurable functions v, defined on Ω, for which $\int_\Omega |v|^{p_0(x)}\,dx < +\infty$. It is proved in [37] that $L^{p_0(x)}(\Omega)$ is a Banach space with the norm

$$||v; L^{p_0(x)}(\Omega)|| = \inf\left\{\lambda > 0 : \int_\Omega |v/\lambda|^{p_0(x)}\,dx \leqslant 1\right\}.$$

Let

$$V(\Omega) = H_0^1(\Omega) \cap L^{p_0(x)}(\Omega) \cap L^{p_1(x)}(\Omega), \ V(Q_T) = H_0^1(Q_T) \cap L^{p_0(x)}(Q_T) \cap L^{p_1(x)}(Q_T),$$

$$q(x) = \min\{2, p'_0(x)\}, \ b(x) = \min\{p'_0(x), p'_1(x)\}, \ \frac{1}{p_i(x)} + \frac{1}{p'_i(x)} = 1, \ i = 0,1.$$

We assume that for the coefficients of equation (1) the following conditions hold:

(A): $a_{ij} \in L^\infty(\Omega)$, $a_{ij}(x) = a_{ji}(x)$, $i,j = 1,...,n$ for a. e. $x \in \Omega$; $\sum\limits_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant \theta_0 \sum\limits_{i=1}^{n} |\xi_i|^2$, $\theta_0 > 0$ for all $\xi \in \mathbb{R}^n$ and for a. e. $x \in \Omega$; $a_i, c \in L^\infty(Q_T)$, $\quad i = 0,1,...,n$;

(B): $b_0, b_1, b_{1t} \in L^\infty(Q_T)$, $b_0(x,t) \geqslant \beta_0 > 0$, $b_1(x,t) \geqslant \beta_1 > 0$ a. e. in Q_T;

(P): $p_i : \Omega \to (1,+\infty)$, $p_i \in L^\infty(\Omega)$, $1 < \overline{p}_i \leqslant \widehat{p}_i < +\infty$, where $\overline{p}_i = \operatorname{ess\,inf}\limits_{\Omega} p_i(x)$, $\widehat{p}_i = \operatorname{ess\,sup}\limits_{\Omega} p_i(x)$, $i = 0,1$.

Definition. By a weak solution of problem (1) – (3) we understand the function $u \in L^\infty(0,T; H_0^1(\Omega)) \cap L^{p_1(x)}(Q_T) \cap C([0,T]; L^2(\Omega))$, $u_t \in L^2(Q_T) \cap L^{p_0(x)}(Q_T)$ that satisfies the equality

$$\int_{\Omega_T} u_t v\,dx - \int_{\Omega_0} u_1(x)v\,dx + \int_{Q_T} \Bigg[-u_t v_t + \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i}v_{x_j} +$$

$$+\sum_{i=1}^{n} a_i(x,t)u_{x_i}v + a_0(x,t)u_t v + c(x,t)uv + b_0(x,t)|u_t|^{p_0(x)-2}u_t v +$$

$$+b_1(x,t)|u|^{p_1(x)-2}uv - f(x,t)v\Bigg]\,dx\,dt = 0, \quad \forall \ v \in V(Q_T), \ v_t \in L^2(Q_T)$$

and the initial data $u(x,0) = u_0(x)$.

3. Existence of a weak solution. We define a functional $\rho_{p_0}(\cdot,\Omega)$ as follows:

$$\rho_{p_0}(v,\Omega) = \int\limits_{\Omega} |v(x)|^{p_0(x)}\, dx$$

for any functions v for which the right side makes sense. To prove the main result of this paper, we need the auxiliary lemma.

Lemma. *The functional $\rho_{p_0}(\cdot,\Omega)$ is lower semicontinuous.*

Proof. Let us consider some facts from the functional analysis. By Theorem 1.22 [39, p. 173] it is known that if M is a closed linear subset of a Banach space X and $v_0 \in X\backslash M$, then there exists $h \in X^*$ such that $\langle h, v_0\rangle_X = 1$, $\langle h, v\rangle_X = 0$ for all $v \in M$ and, in addition, $||h||_{X^*} = \dfrac{1}{\operatorname{dist}(v_0, M)}$.

We will also use the generalized Hölder's inequality

$$\int\limits_{\Omega} |v(x)w(x)|\, dx \leqslant r_{p_0} ||v; [L^{p_0(x)}(\Omega)]^*||\; ||w; L^{p_0(x)}(\Omega)||,$$

which is true for all $w \in L^{p_0(x)}(\Omega)$, $v \in [L^{p_0(x)}(\Omega)]^*$ and where r_{p_0} is a constant that depends only on p_0 and Ω [37].

Let $v_0 \in L^{p_0(x)}(\Omega)$. Fix an arbitrary $r > 0$. It is clear that there exists a closed linear space M_r (for instance, a hyperline in $L^{p_0(x)}(\Omega)$) such that $\operatorname{dist}(v_0, M_r) = \dfrac{1}{r}$. There also exists an element $h_r \in [L^{p_0(x)}(\Omega)]^*$ such that, in particular, $\langle h_r, v_0\rangle_{L^{p_0(x)}(\Omega)} = 1$ and $||h_r||_{[L^{p_0(x)}(\Omega)]^*} = r$. Hence, the following statement holds: for arbitrary $v \in L^{p_0(x)}(\Omega)$, $\tilde z > 0$, $r > 0$ (let $v_0 = \frac{v}{\tilde z}$) there exists $h \in [L^{p_0(x)}(\Omega)]^*$ such that $\langle h, v\rangle_{L^{p_0(x)}(\Omega)} = \tilde z$ and $||h||_{[L^{p_0(x)}(\Omega)]^*} = r$. It is obvious that h depends on v, $\tilde z$, r.

Now prove the statement of lemma. Let $v_m \to v$ weakly in $L^{p_0(x)}(\Omega)$ as $m \to \infty$. Then there exist such constants $c_1 > 0$, $c_2 > 0$ that $||v; L^{p_0(x)}(\Omega)|| \leqslant c_1$ and $\rho_{p_0}(v_m, \Omega) \leqslant c_2$ for arbitrary $m \in \mathbb{N}$. First of all let us assume that $\varliminf\limits_{m\to\infty} ||v_m; L^{p_0(x)}(\Omega)|| \neq 0$. Then $\varliminf\limits_{m\to\infty} \rho(v_m,\Omega) \neq 0$. Hence, there existnumbers $m_0 \in \mathbb{N}$, $\tilde c_1 > 0$, $\tilde c_2 > 0$ such that for all $m \geqslant m_0$ $\tilde c_1 \leqslant \rho(v_m,\Omega) \leqslant \tilde c_2$, $\tilde c_1 \leqslant ||v_m; L^{p_0(x)}(\Omega)|| \leqslant \tilde c_2$. Let $\tilde z = \rho(v,\Omega)$, $r = \dfrac{\tilde c_1}{r_{p_0}\tilde c_2}$. For these $\tilde z$, r, v choose a corresponding $h \in [L^{p(x)}(\Omega)]^*$. Then

$$\rho(v,\Omega) = \tilde z = \langle h, v\rangle_{L^{p_0(x)}(\Omega)} = \varliminf_{m\to\infty} \langle h, v_m\rangle_{L^{p_0(x)}(\Omega)} \leqslant$$
$$\leqslant \varliminf_{m\to\infty} r_{p_0} ||h; [L^{p_0(x)}(\Omega)]^*||\; ||v_m; L^{p_0(x)}(\Omega)|| \leqslant \varliminf_{m\to\infty} r_{p_0} r \tilde c_2 = \tilde c_1 \leqslant \varliminf_{m\to\infty} \rho(v_m,\Omega).$$

Now let us suppose that $\varliminf\limits_{m\to\infty} ||v_m; L^{p_0(x)}(\Omega)|| = 0$. Reflexivity of the space $L^{p_0(x)}(\Omega)$ gives $||v; L^{p_0(x)}(\Omega)|| \leqslant \varliminf\limits_{m\to\infty} ||v_m; L^{p_0(x)}(\Omega)|| = 0$. This implies $v = 0$. Hence, $\rho(v,\Omega) = 0 \leqslant \varliminf\limits_{m\to\infty} \rho(v_m,\Omega)$. This completes the proof of the lemma.

Theorem. *Suppose that conditions (A), (B), (P) are true and besides that* $f \in L^{q(x)}(Q_T)$, $u_0 \in H_0^1(\Omega) \cap L^{p_1(x)}(\Omega)$, $u_1 \in L^2(\Omega)$. *Then there exists a weak solution* u *of the problem (1) – (3) such that* $u_{tt} \in L^2((0,T); H^{-1}(\Omega)) + L^{b(x)}(Q_T)$.

Proof. We use Faedo-Galerkin's method. As $V(\Omega)$ is a Banach separable space, there exists a linearly independent dense everywhere in $V(\Omega)$ set of functions $\{\varphi^k\}_{k\in\mathbb{N}}$ which is orthonormal in $L^2(\Omega)$. Let us consider a sequence $u^N(x,t) = \sum\limits_{k=1}^{N} C_k^N(t)\varphi^k(x)$, $N \in \mathbb{N}$, where $(C_1^N, \dots C_k^N)$ is a solution of the Cauchy problem

$$\int\limits_\Omega \Big[u_{tt}^N \varphi^k + \sum_{i,j=1}^n a_{ij}(x) u_{x_i}^N \varphi_{x_j}^k + \sum_{i=1}^n a_i(x,t) u_{x_i}^N \varphi^k + a_0(x,t) u_t^N \varphi^k + c(x,t) u^N \varphi^k +$$

$$+ b_0(x,t)|u_t^N|^{p_0(x)-2} u_t^N \varphi^k + b_1(x,t)|u^N|^{p_1(x)-2} u^N \varphi^k - f(x,t)\varphi^k\Big]\, dx = 0, \tag{4}$$

$$C_k^N(0) = u_{0,k}^N, \ C_{kt}^N(0) = u_{1,k}^N, \tag{5}$$

$$u_0^N(x) = \sum_{k=1}^N u_{0,k}^N \varphi^k(x), \quad u_1^N(x) = \sum_{k=1}^N u_{1,k}^N \varphi^k(x),$$

$$\|u_0^N - u_0\|_{H_0^1(\Omega)\cap L^{p_1(x)}(\Omega)} \to 0, \ \|u_1^N - u_1\|_{L^2(\Omega)} \to 0.$$

Substitute $C_k^N(t) = y_k(t)$, $C_{kt}^N(t) = z_k(t)$. Taking into account the orthonormality of $\{\varphi^k\}_{k\in\mathbb{N}}$ in the space $L^2(\Omega)$, (4), (5), we obtain

$$y_k'(t) = z_k(t),$$

$$z_k'(t) = -\int\limits_\Omega \Big[\sum_{i,j=1}^n a_{ij}(x) \sum_{l=1}^N y_l \varphi_{x_i}^l \varphi_{x_j}^k + \sum_{i=1}^n a_i(x,t) \sum_{l=1}^N y_l \varphi_{x_i}^l \varphi^k +$$

$$+ a_0(x,t) \sum_{l=1}^N z_l \varphi^l \varphi^k + c(x,t) \sum_{l=1}^N y_l \varphi^l \varphi^k + b_0(x,t) \Big|\sum_{l=1}^N z_l \varphi^l\Big|^{p_0(x)-2} \sum_{l=1}^N z_l \varphi^l\, \varphi^k +$$

$$+ b_1(x,t) \Big|\sum_{l=1}^N y_l \varphi^l\Big|^{p_1(x)-2} \sum_{l=1}^N y_l \varphi^l\, \varphi^k - f(x,t)\varphi^k\Big]\, dx, \ k = 1, \dots, N, \tag{6}$$

$$y_k(0) = u_{0,k}^N, \ z_k(0) = u_{1,k}^N. \tag{7}$$

Rewrite system (6) in the following way:

$$\begin{cases} y_k' = z_k, \\ z_k' = f_k(y_1, \dots y_N, z_1, \dots z_N). \end{cases}$$

Let $\prod_a = \{(y,x) \in \mathbb{R}^{2N} : |y_i| \leqslant a,\ |z_i| \leqslant a, i = 1, \dots N\}$. The Hölder inequality gives the estimate

$$\Big|\sum_{l=1}^{N} z_l \varphi^l\Big|^{p_0(x)-1} \leqslant \sum_{l=1}^{N} |z_l|^{p_0(x)-1} |\varphi^l|^{p_0(x)-1} N^{\frac{p_0(x)-1}{(p_0(x)-1)'}} \leqslant$$
$$\leqslant N^{\frac{p_0(x)-1}{(p_0(x)-1)'}} a^{p_0(x)-1} \sum_{l=1}^{N} |\varphi^l|^{p_0(x)-1}.$$

Thus, the functions f_k satisfy the conditions of Caratheodory Theorem [40, p. 54]. Then there exists a continuously differentiable solution of problem (6), (7) which is determined in some interval $(0, t_0]$ and has absolutely continuous derivative. From the estimates obtained below it follows that $t_0 = T$.

Multiplying equation (4) by the functions $C_{kt}^N e^{-\eta t}$, $\eta > 0$ respectively, summing by k from 1 to N and integrating along the interval $(0,\tau]$, $\tau \in (0,T]$, we obtain

$$\int\limits_{Q_\tau} \Big[u_{tt}^N u_t^N + \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i}^N u_{x_j t}^N + \sum_{i=1}^{n} a_i(x,t) u_{x_i}^N u_t^N + a_0(x,t)|u_t^N|^2 + c(x,t) u^N u_t^N +$$
$$+ b_0(x,t)|u_t^N|^{p_0(x)} + b_1(x,t)|u^N|^{p_1(x)-2} u^N u_t^N - f(x,t) u_t^N \Big] e^{-\eta t}\, dx\, dt = 0. \tag{8}$$

Estimate the addends in (8), taking into account the conditions of the theorem:

$$I_1 := \int\limits_{Q_\tau} u_{tt}^N u_t^N e^{-\eta t}\, dx\, dt = \frac{1}{2} \int\limits_{\Omega_\tau} |u_t^N|^2 e^{-\eta \tau}\, dx - \frac{1}{2} \int\limits_{\Omega_0} |u_1^N|^2\, dx + \frac{\eta}{2} \int\limits_{Q_\tau} |u_t^N|^2 e^{-\eta t}\, dx\, dt;$$

$$I_2 := \int\limits_{Q_\tau} \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i}^N u_{x_j t}^N e^{-\eta t}\, dx\, dt = \frac{1}{2} \int\limits_{\Omega_\tau} \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i}^N u_{x_j}^N e^{-\eta \tau}\, dx -$$
$$- \frac{1}{2} \int\limits_{\Omega_0} \sum_{i,j=1}^{n} a_{ij}(x) u_{0,x_i}^N u_{0,x_j}^N\, dx + \frac{\eta}{2} \int\limits_{Q_\tau} \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i}^N u_{x_j}^N e^{-\eta t}\, dx\, dt \geqslant$$
$$\geqslant \frac{\theta_0}{2} \int\limits_{\Omega_\tau} |\nabla u^N|^2 e^{-\eta \tau}\, dx - \frac{A_0}{2} \int\limits_{\Omega_0} |\nabla u_0^N|^2\, dx + \frac{\eta \theta_0}{2} \int\limits_{Q_\tau} |\nabla u^N|^2 e^{-\eta t}\, dx\, dt,$$

$$A_0 = n \max_{i,j} \operatorname*{ess\,sup}_{\Omega} |a_{ij}(x)|;$$

$$I_3 := \int\limits_{Q_\tau} \sum_{i=1}^{n} a_i(x,t) u_{x_i}^N u_t^N e^{-\eta t}\, dx\, dt \leqslant \frac{1}{2} \int\limits_{Q_\tau} \Big[A_1 |\nabla u^N|^2 + |u_t^N|^2 \Big] e^{-\eta t}\, dx\, dt,$$

$$A_1 = \operatorname*{ess\,sup}_{Q_T} \sum_{i=1}^{n} a_i^2(x,t);$$

$$I_4 := \int\limits_{Q_\tau} a_0(x,t)|u_t^N|^2 e^{-\eta t}\, dx\, dt \leqslant \frac{A_2}{2} \int\limits_{Q_\tau} |u_t^N|^2 e^{-\eta t}\, dx\, dt,\ A_2 = 2 \operatorname*{ess\,sup}_{Q_T} |a_0(x,t)|;$$

$$I_5 := \int\limits_{Q_\tau} c(x,t)u^N u_t^N e^{-\eta t}\,dx\,dt \leqslant \frac{1}{2}\,C_1 \int\limits_{Q_\tau}\Big[|u^N|^2 + |u_t^N|^2\Big] e^{-\eta t}\,dx\;dt,\; C_1 = \operatorname*{ess\,sup}_{Q_T}|c(x,t)|;$$

$$I_6 := \int\limits_{Q_\tau} b_0(x,t)|u_t^N|^{p_0(x)} e^{-\eta t}\,dx\;dt \geqslant \beta_0 \int\limits_{Q_\tau} |u_t^N|^{p_0(x)} e^{-\eta t}\,dx\,dt;$$

$$I_7 := \int\limits_{Q_\tau} b_1(x,t)|u^N|^{p_1(x)-2}u^N u_t^N e^{-\eta t}\,dx\,dt = \int\limits_{\Omega_\tau} \frac{b_1(x,\tau)}{p_1(x)}|u^N|^{p_1(x)} e^{-\eta\tau}\,dx -$$

$$- \int\limits_{\Omega_0} \frac{b_1(x,0)}{p_1(x)}|u_0^N|^{p_1(x)}\,dx + \int\limits_{Q_\tau} \frac{1}{p_1(x)}(\eta b_1(x,t) - b_{1t}(x,t))|u^N|^{p_1(x)} e^{-\eta t}\,dx\,dt \geqslant$$

$$\geqslant \frac{\beta_1}{\widehat{p}_1}\int\limits_{\Omega_\tau}|u^N|^{p_1(x)}e^{-\eta\tau}\,dx - \frac{\beta_2}{\overline{p}_1}\int\limits_{\Omega_0}|u_0^N|^{p_1(x)}\,dx + \left(\frac{\eta\beta_1}{\widehat{p}_1} - \frac{\beta_3}{\overline{p}_1}\right)\int\limits_{Q_\tau}|u^N|^{p_1(x)}e^{-\eta t}\,dx\,dt,$$

$$\beta_2 = \operatorname*{ess\,sup}_{\Omega}|b_1(x,0)|,\; \beta_3 = \operatorname*{ess\,sup}_{Q_T}|b_{1t}(x,t)|;$$

$$I_8 := \int\limits_{Q_\tau} f(x,t)u_t^N e^{-\eta t}\,dx\,dt \leqslant \int\limits_{Q_\tau}\left[\frac{\delta_0}{q'(x)}|u_t^N|^{q'(x)} + \frac{1}{\delta_0^{\frac{q(x)}{q'(x)}}q(x)}|f(x,t)|^{q(x)}\right]e^{-\eta t}\,dx\,dt \leqslant$$

$$\leqslant \frac{\delta_0}{\overline{q}'}\int\limits_{Q_\tau}|u_t^N|^{q'(x)}e^{-\eta t}\,dx\,dt + \frac{1}{\delta_0^{\gamma_0}\overline{q}}\int\limits_{Q_\tau}|f(x,t)|^{q(x)}e^{-\eta t}\,dx\,dt,\quad \frac{1}{q(x)} + \frac{1}{q'(x)} = 1,$$

$$\overline{q} = \operatorname*{ess\,inf}_{\Omega} q(x),\quad \overline{q}' = \operatorname*{ess\,inf}_{\Omega} q'(x),\quad \gamma_0 = \operatorname*{ess\,sup}_{\Omega}\frac{q(x)}{q'(x)},\quad \delta_0 \in (0,1].$$

Since

$$u^N(x,t) = u^N(x,0) + \int\limits_0^t u_t^N(x,s)\,ds,$$

then

$$\int\limits_{Q_\tau}|u^N|^2 e^{-\eta t}\,dx\,dt \leqslant 2T\bigg(\int\limits_{\Omega_0}|u_0^N|^2\,dx + T\int\limits_{Q_\tau}|u_t^N|^2 e^{-\eta t}\,dx\,dt\bigg). \tag{9}$$

Thus, taking into account the estimates of the integrals $I_1 - I_8$ and (9), from (8) we obtain the inequality

$$\int\limits_{\Omega_\tau}\bigg[|u_t^N|^2 + \theta_0|\nabla u^N|^2 + \frac{2\beta_1}{\widehat{p}_1}|u^N|^{p_1(x)}\bigg]e^{-\eta\tau}\,dx + \int\limits_{Q_\tau}\bigg[\Big(\eta - 1 - A_2 - C_1 -$$

$$-2T^2 - \nu\Big)|u_t^N|^2 + \Big(\theta_0\eta - A_1\Big)|\nabla u^N|^2 + \Big(2\beta_0 - \frac{2\delta_0}{\overline{q}'}(1-\nu)\Big)|u_t^N|^{p_0(x)} +$$

$$+2\Big(\frac{\eta\beta_1}{\widehat{p}_1} - \frac{\beta_3}{\overline{p}_1}\Big)|u^N|^{p_1(x)}\bigg]e^{-\eta t}\,dx\,dt \leqslant \int\limits_{\Omega_0}\bigg[|u_1^N|^2 + \frac{2\beta_2}{\overline{p}_1}|u_0^N|^{p_1(x)} + A_0|\nabla u_0^N|^2 +$$

$$+2TC_1|u_0^N|^2\Big]\,dx+\frac{2}{\delta_0^{\gamma_0}\overline{q}}\int\limits_{Q_\tau}|f(x,t)|^{q(x)}e^{-\eta t}\,dx\,dt, \tag{10}$$

where $\nu=1$ if $q(x)\equiv 2$ and $\nu=0$ if $q(x)\leqslant 2,\ q(x)\not\equiv 2$.

Choose η and δ_0 such that the following conditions hold:

$$\eta-2-A_2-C_1-2T^2\geqslant 1,\ \ \theta_0\eta-A_1\geqslant 1,\ \ \frac{\eta\beta_1}{\widehat{p}_1}-\frac{\beta_3}{\overline{p}_1}\geqslant 1,\ \ \delta_0=\min\{1;\beta_0;\overline{q}\,'\}.$$

Then, considering the convergence of u_0^N to u_0 in the space $H_0^1(\Omega)\cap L^{p_1(x)}(\Omega)$ and the convergence of u_1^N to u_1 in the space $L^2(\Omega)$, from (10) we have the estimates

$$\int\limits_{\Omega_\tau}[|u_t^N|^2+|\nabla u^N|^2+|u^N|^{p_1(x)}]\,dx\leqslant M_1,\ \tau\in(0,T], \tag{11}$$

$$\int\limits_{Q_T}[|u_t^N|^2+|\nabla u^N|^2+|u_t^N|^{p_0(x)}+|u^N|^{p_1(x)}]\,dx\,dt\leqslant M_1, \tag{12}$$

where M_1 does not depend on N.
Besides that

$$\int\limits_{Q_T}||u^N|^{p_1(x)-2}u^N|^{p'_1(x)}\,dx\,dt\leqslant\int\limits_{Q_T}|u^N|^{p_1(x)}\,dx\,dt\leqslant M_2, \tag{13}$$

$$\int\limits_{Q_T}||u_t^N|^{p_0(x)-2}u_t^N|^{p'_0(x)}\,dx\,dt\leqslant\int\limits_{Q_T}|u_t^N|^{p_0(x)}\,dx\,dt\leqslant M_2. \tag{14}$$

From estimates (11) – (14) it follows that

$$\|u^N\|_{L^\infty((0,T);H_0^1(\Omega)\cap L^{p_1(x)}(\Omega))}+\|u_t^N\|_{L^\infty((0,T);L^2(\Omega))}\leqslant M_3, \tag{15}$$

$$\|u^N\|_{L^2((0,T);H_0^1(\Omega))\cap L^{p_1(x)}(Q_T)}\leqslant M_3,\quad \|u_t^N\|_{L^{p_0(x)}(Q_T)}\leqslant M_3, \tag{16}$$

$$\||u^N|^{p_1(x)-2}u^N\|_{L^{p'_1(x)}(Q_T)}\leqslant M_3,\quad \||u_t^N|^{p_0(x)-2}u_t^N\|_{L^{p'_0(x)}(Q_T)}\leqslant M_3, \tag{17}$$

where the constant M_3 does not depend on N.
On the basis of (15) – (17) there exists a subsequence $\{u^{N_k}\}_{N_k\in\mathbb{N}}\subset\{u^N\}_{N\in\mathbb{N}}$ such that

$u^{N_k}\to u\ *-$ weakly in $L^\infty((0,T);H_0^1(\Omega)\cap L^{p_1(x)}(\Omega))$, $u^{N_k}\to u$ weakly in $L^2((0,T);H_0^1(\Omega))\cap L^{p_1(x)}(Q_T)$, $u_t^{N_k}\to u_t\ *-$ weakly in $L^\infty((0,T);L^2(\Omega))$, $u_t^{N_k}\to u_t$ weakly in $L^{p_0(x)}(Q_T)$, $u_t^{N_k}(\cdot,T)\to w$ weakly in $L^2(\Omega)$, $u^{N_k}\to u$ strongly in $L^2(Q_T)$ and a. e. in Q_T, $|u^{N_k}|^{p_1(x)-2}u^{N_k}\to\chi$ weakly in $L^{p'_1(x)}(Q_T)$, $|u_t^{N_k}|^{p_0(x)-2}u_t^{N_k}\to z$ weakly in

$$L^{p'_0(x)}(Q_T) \text{ as } N_k \to \infty.$$

Then from Lemma 2.2 [38, p. 57] it follows that $\chi = |u|^{p_1(x)-2}u$.

Let us consider the set of functions

$$\mathfrak{M} = \bigcup_{N=1}^{\infty} \mathfrak{M}_N, \text{ where } \mathfrak{M}_N = \{v^N : \ v^N(x,t) = \sum_{k=1}^{N} d_k^N(t)\varphi^k(x), \ d_k^N \in C^1([0,T])\}.$$

From (4) we get

$$\int_{Q_T} \Big[-u_t^{N_k} v_t^N + \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i}^{N_k} v_{x_j}^N + \sum_{i=1}^{n} a_i(x,t) u_{x_i}^{N_k} v^N + a_0(x,t) u_t^{N_k} v^N + c(x,t) u^{N_k} v^N +$$
$$+ b_0(x,t)|u_t^{N_k}|^{p_0(x)-2} u_t^{N_k} v^N + b_1(x,t)|u^{N_k}|^{p_1(x)-2} u^{N_k} v^N - f(x,t) v^N \Big]\, dx\, dt =$$
$$= - \int_{\Omega_T} u_t^{N_k} v^N \, dx + \int_{\Omega_0} u_1^{N_k} v^N \, dx, \qquad (18)$$

where N is an arbitrary fixed natural number and $N_k \geqslant N$.
Hence, passing to the limit as $N_k \to \infty$, obtain

$$\int_{Q_T} \Big[-u_t v_t^N + \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} v_{x_j}^N + \sum_{i=1}^{n} a_i(x,t) u_{x_i} v^N + a_0(x,t) u_t v^N + c(x,t) u v^N +$$
$$+ b_0(x,t) z v^N + b_1(x,t)|u|^{p_1(x)-2} u v^N - f(x,t) v^N \Big]\, dx\, dt + \int_{\Omega_T} w v^N \, dx = \int_{\Omega_0} u_1 v^N \, dx. \quad (19)$$

Taking into account the density of the set $\mathfrak{M}$ in the space $H_0^1(Q_T)$ and in the space $L^{p_0(x)}(Q_T) \cap L^{p_1(x)}(Q_T)$ [36], from (19) we get the equality

$$\int_{Q_T} \Big[-u_t v_t + \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} v_{x_j} + \sum_{i=1}^{n} a_i(x,t) u_{x_i} v + a_0(x,t) u_t v + c(x,t) u v +$$
$$+ b_0(x,t) z v + b_1(x,t)|u|^{p_1(x)-2} u v - f(x,t) v \Big]\, dx\, dt + \int_{\Omega_T} w v \, dx = \int_{\Omega_0} u_1 v \, dx, \quad (20)$$

which is correct for all $v \in V(Q_T), \ v_t \in L^2(Q_T)$.
In particular, (20) implies the equality in the sense of distributions

$$u_{tt} = - \sum_{i,j=1}^{n} (a_{ij}(x) u_{x_i})_{x_j} - \sum_{i=1}^{n} a_i(x,t) u_{x_i} - a_0(x,t) u_t - c(x,t) u - b_0(x,t) z -$$
$$- b_1(x,t)|u|^{p_1(x)-2} u + f(x,t). \qquad (21)$$

Then $u_{tt} \in L^2((0,T); H^{-1}(\Omega)) + L^{p'_0(x)}(Q_T) + L^{p'_1(x)}(Q_T) \subset L^2((0,T); H^{-1}(\Omega)) + L^{b(x)}(Q_T)$.

Let $\overline{b} = \operatorname{ess\,inf}_{\Omega} b(x)$ and put $\varkappa_0 = \min\{2, \overline{b}\}$, $\varkappa_1 = \min\{2, \overline{p}_0\}$. Then $u_{tt} \in L^{\varkappa_0}((0,T); H^{-1}(\Omega) + L^{\overline{b}}(\Omega))$. Furthermore, $u_t \in L^2(Q_T) \cap L^{p_0(x)}(Q_T) \subset L^{\varkappa_1}(Q_T)$, $u \in L^2((0,T); H_0^1(\Omega))$. Hence, $u \in C([0,T]; L^2(\Omega))$ [8, p. 20].

Denote by s the smallest positive number for which the embeddings $L^{\varkappa_1}(\Omega) \subset H^{-s}(\Omega)$ and $L^{\overline{b}_0}(\Omega) \subset H^{-s}(\Omega)$ hold. Then $u_{tt} \in L^{\varkappa}((0,T); H^{-s}(\Omega))$ and $u_t \in L^{\varkappa}((0,T); H^{-s}(\Omega))$, where $\varkappa = \min\{\varkappa_0, \varkappa_1\}$. Thus, $u_t \in C([0,T]; H^{-s}(\Omega))$ [8 , p. 20].

Let $v^N|_{t=T} = 0$. Then

$$\int_0^T \int_\Omega u_t^{N_k} v^N \, dx \, dt = -\int_0^T \int_\Omega u^{N_k} v_t^N \, dx \, dt - \int_{\Omega_0} u_0^{N_k} v^N \, dx \underset{N_k \to \infty}{\longrightarrow}$$

$$\underset{N_k \to \infty}{\longrightarrow} -\int_0^T \int_\Omega u v_t^N \, dx \, dt - \int_{\Omega_0} u_0 v^N \, dx \underset{N \to \infty}{\longrightarrow} -\int_{Q_T} u v_t \, dx \, dt - \int_{\Omega_0} u_0 v \, dx.$$

On the other hand,

$$\int_{Q_T} u_t^{N_k} v^N \, dx \, dt \underset{N_k \to \infty}{\longrightarrow} \int_{Q_T} u_t v^N \, dx \, dt = -\int_{Q_T} u v_t^N \, dx \, dt - \int_{\Omega_0} u(x,0) v^N \, dx \underset{N \to \infty}{\longrightarrow}$$

$$\underset{N \to \infty}{\longrightarrow} -\int_{Q_T} u v_t \, dx \, dt - \int_{\Omega_0} u(x,0) v \, dx.$$

Then

$$\int_{\Omega_0} u_0 v \, dx = \int_{\Omega_0} u(x,0) v \, dx.$$

From here

$$u(x,0) = u_0(x).$$

On the basis of (4) we have

$$\langle u_{tt}^{N_k}, w \rangle_{H_0^1(\Omega)} = -\int_{\Omega_t} \Bigg[\sum_{i,j=1}^n a_{ij}(x) u_{x_i}^{N_k} w_{x_j} + \sum_{i=1}^n a_i(x,t) u_{x_i}^{N_k} w + a_0(x,t) u_t^{N_k} w + c(x,t) u^{N_k} w +$$

$$+ b_0(x,t) |u_t^{N_k}|^{p_0(x)-2} u_t^{N_k} w + b_1(x,t) |u^{N_k}|^{p_1(x)-2} u^{N_k} w - f(x,t) w \Bigg] dx,$$

where $N_k \geqslant N$, $w \in \operatorname{Span}\{\varphi^1, \ldots, \varphi^N\}$.
Then

$$\langle u_{tt}^{N_k}, w \rangle_{H_0^1(\Omega)} \underset{N_k \to \infty}{\longrightarrow} -\int_{\Omega_t} \Bigg[\sum_{i,j=1}^n a_{ij}(x) u_{x_i} w_{x_j} + \sum_{i=1}^n a_i(x,t) u_{x_i} w + a_0(x,t) u_t w + c(x,t) u w +$$

$$+b_0(x,t)zw + b_1(x,t)|u|^{p_1(x)-2}uw - f(x,t)w\Bigg] \, dx \quad \text{weakly in } L^{\varkappa_0}(0,T).$$

On the other hand, from (21) we have

$$\frac{d}{dt}\int\limits_{\Omega_t} u_t w \, dx = -\int\limits_{\Omega_t}\Bigg[\sum_{i,j=1}^{n} a_{ij}(x)u_{x_i}w_{x_j} + \sum_{i=1}^{n} a_i(x,t)u_{x_i}w + a_0(x,t)u_t w +$$

$$+c(x,t)uw + b_0(x,t)zw + b_1(x,t)|u|^{p_1(x)-2}uw - f(x,t)w\Bigg] \, dx.$$

Thus,

$$\frac{d}{dt}\int\limits_{\Omega_t} u_t^{N_k} w \, dx \underset{N_k\to\infty}{\longrightarrow} \frac{d}{dt}\int\limits_{\Omega_t} u_t w \, dx \quad \text{weakly in} \quad L^{\varkappa_0}(0,T).$$

Since $\langle u_{tt}^{N_k}, w\rangle_{H_0^1(\Omega)} \in L^{\varkappa_0}(0,T)$, we see that $\int\limits_{\Omega_t} u_t w \, dx \in C([0,T])$. Let $\varphi \in C([0,T])$ and $\varphi(T) = 0$. Then

$$\int\limits_0^T \frac{d}{dt}\Bigg(\int\limits_{\Omega_t} u_t w \, dx\Bigg)\varphi \, dt = -\int\limits_{Q_T} u_t w \varphi' \, dx \, dt - \int\limits_{\Omega_0} u_t(x,0) w \varphi(0) \, dx.$$

On the other hand,

$$\int\limits_0^T \frac{d}{dt}\Bigg(\int\limits_{\Omega_t} u_t^{N_k} w \, dx\Bigg)\varphi \, dt = -\int\limits_{Q_T} u_t^{N_k} w \varphi' \, dx \, dt - \int\limits_{\Omega_0} u_t^{N_k} w \varphi(0) \, dx \underset{N_k\to\infty}{\longrightarrow}$$

$$\underset{N_k\to\infty}{\longrightarrow} -\int\limits_{Q_T} u_t w \varphi' \, dx \, dt - \int\limits_{\Omega_0} u_1 w \varphi(0) \, dx.$$

Thus,

$$\int\limits_{\Omega_0} u_1 w \varphi(0) \, dx = \int\limits_{\Omega_0} u_t(x,0) w \varphi(0) \, dx \ \text{ for all } N \in \mathbb{N},$$

that is,

$$u_t(x,0) = u_1(x).$$

Choosing $\varphi(0) = 0$, we can prove in a similar way that $u_t(x,T) = w(x)$. Then the equality (20) can be written in the form

$$\int\limits_{Q_T}\Bigg[-u_t v_t + \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i}v_{x_j} + \sum_{i=1}^{n} a_i(x,t)u_{x_i}v + a_0(x,t)u_t v + c(x,t)uv + b_0(x,t)zv +$$

$$+b_1(x,t)|u|^{p_1(x)-2}uv - f(x,t)v\Bigg] \, dx \, dt + \int\limits_{\Omega_T} u_t v \, dx = \int\limits_{\Omega_0} u_1(x) v \, dx, \tag{22}$$

which is correct for all $v \in V(Q_T),\ v_t \in L^2(Q_T)$.

Now prove that $z = |u_t|^{p_1(x)-2}u_t$. Let $\tau_0,\ \tau \in (0,T),\ \tau_0 < \tau,\ m \in \mathbb{N},\ \Theta_m$ be a continuous piecewise linear function on the interval $[0,T]$ such that $\Theta_m(t) = 1,\ \tau_0 + \frac{2}{m} < t < \tau - \frac{2}{m},\ \Theta_m(t) = 0,\ t > \tau - \frac{1}{m},\ t < \tau_0 + \frac{1}{m}$. Let ρ_l be a regularizing sequence in $D(\mathbb{R})$, $\rho_l(t) = \rho_l(-t)$,

$$\int\limits_{-\infty}^{+\infty} \rho_l(t)dt = 1, \qquad \text{supp } \rho_l \subset \left[-\frac{1}{l}, \frac{1}{l}\right].$$

Put in (22) $v = ((\Theta_m e^{-\eta t}u_t) * \rho_l * \rho_l)\Theta_m e^{-\eta t}$, where $l > 2m$ and $*$ denotes the convolution by t. As

$$v_t = ((\Theta_m e^{-\eta t}u_t) * \rho_l * \rho_l)_t \Theta_m e^{-\eta t} + ((\Theta_m e^{\eta t}u_t) * \rho_l * \rho_l)\Theta'_m e^{-\eta t} -$$
$$-\eta((\Theta_m e^{-\eta t}u_t) * \rho_l * \rho_l)\Theta_m e^{-\eta t},$$

then the first addend of equality (22) can be presented in the following form:

$$-\int\limits_{Q_T} u_t v_t\, dx\, dt = \int\limits_{Q_T} ((\Theta_m u_t e^{-\eta t})_t * \rho_l)((\Theta_m u_t e^{-\eta t}) * \rho_l)\, dx\, dt -$$
$$-\int\limits_{Q_T} ((\Theta'_m u_t e^{-\eta t}) * \rho_l)((\Theta_m u_t e^{-\eta t}) * \rho_l)\, dx\, dt + \eta \int\limits_{Q_T} ((\Theta_m u_t e^{-\eta t}) * \rho_l)^2\, dx\, dt \underset{l\to\infty}{\longrightarrow}$$
$$\underset{l\to\infty}{\longrightarrow} \eta \int\limits_{Q_T} |u_t|^2 \Theta_m^2 e^{-2\eta t}\, dx\, dt - \int\limits_{Q_T} |u_t|^2 \Theta_m \Theta'_m e^{-2\eta t}\, dx\, dt \underset{m\to\infty}{\longrightarrow}$$
$$\underset{m\to\infty}{\longrightarrow} \eta \int\limits_{Q_{\tau_0,\tau}} |u_t|^2 e^{-2\eta t}\, dx\, dt + \frac{1}{2}\int\limits_{\Omega_\tau} |u_t|^2 e^{-2\eta\tau}\, dx - \frac{1}{2}\int\limits_{\Omega_{\tau_0}} |u_t|^2 e^{-2\eta\tau_0}\, dx.$$

Similarly, for the second term of equality (22) we have

$$\int\limits_{Q_T} \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} v_{x_i}\, dx\, dt = \int\limits_{Q_T} \sum_{i,j=1}^{n} a_{ij}(x)((u_{x_i}\Theta_m e^{-\eta t}) * \rho_l)((u_{x_j}\Theta_m e^{-\eta t})_t * \rho_l)\, dx\, dt +$$
$$+\eta \int\limits_{Q_T} \sum_{i,j=1}^{n} a_{ij}(x)((u_{x_i}\Theta_m e^{-\eta t}) * \rho_l)((u_{x_j}\Theta_m e^{-\eta t}) * \rho_l)\, dx\, dt -$$
$$-\int\limits_{Q_T} \sum_{i,j=1}^{n} a_{ij}(x)((u_{x_i}\Theta_m e^{-\eta t}) * \rho_l)((u_{x_j}\Theta'_m e^{-\eta t}) * \rho_l)\, dx\, dt \underset{l\to\infty}{\longrightarrow}$$
$$\underset{l\to\infty}{\longrightarrow} \eta \int\limits_{Q_T} \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} u_{x_j} \Theta_m^2 e^{-2\eta t}\, dx\, dt - \int\limits_{Q_T} \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} u_{x_j} \Theta_m \Theta'_m e^{-2\eta t}\, dx\, dt \underset{m\to\infty}{\longrightarrow}$$

$$\underset{m\to\infty}{\longrightarrow} \eta \int\limits_{Q_T} \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} u_{x_j} e^{-2\eta t}\, dx\, dt + \frac{1}{2}\int\limits_{\Omega_\tau} \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} u_{x_j} e^{-2\eta\tau}\, dx -$$

$$-\frac{1}{2}\int\limits_{\Omega_{\tau_0}}\sum_{i,j=1}^{n}a_{ij}(x)u_{x_i}u_{x_j}e^{-2\eta\tau_0}\,dx.$$

Moreover,

$$\int\limits_{Q_T}\left[\sum_{i=1}^{n}a_i(x,t)u_{x_i}+a_0(x,t)u_t+c(x,t)u+b_0(x,t)z+b_1(x,t)|u|^{p_1(x)-2}u-\right.$$
$$\left.-f(x,t)\right]v\,dx\,dt\underset{l,m\to\infty}{\longrightarrow}\int\limits_{Q_{\tau_0,\tau}}\left[\sum_{i=1}^{n}a_i(x,t)u_{x_i}+a_0(x,t)u_t+c(x,t)u+\right.$$
$$\left.+b_0(x,t)z+b_1(x,t)|u|^{p_1(x)-2}u-f(x,t)\right]u_te^{-2\eta t}\,dx\,dt.$$

Thus, for a.e. $\tau_0,\ \tau\in(0,T)$ the following equality holds

$$\frac{1}{2}\int\limits_{\Omega_\tau}\left[u_t^2+\sum_{i,j=1}^{n}a_{ij}(x)u_{x_i}u_{x_j}\right]e^{-2\eta\tau}\,dx-\frac{1}{2}\int\limits_{\Omega_{\tau_0}}\left[u_t^2+\sum_{i,j=1}^{n}a_{ij}(x)u_{x_i}u_{x_j}\right]e^{-2\eta\tau_0}\,dx+$$
$$+\int\limits_{Q_{\tau_0,\tau}}\left[(\eta+a_0(x,t))u_t^2+\eta\sum_{i,j=1}^{n}a_{ij}(x)u_{x_i}u_{x_j}+\sum_{i=1}^{n}a_i(x,t)u_{x_i}u_t+c(x,t)uu_t+\right.$$
$$\left.+b_0(x,t)zu_t+b_1(x,t)|u|^{p_1(x)-2}uu_t-f(x,t)u_t\right]e^{-2\eta t}\,dx\,dt=0. \qquad (23)$$

Since $u\in L^\infty((0,T);H_0^1(\Omega))$ and $u_t\in L^\infty((0,T);L^2(\Omega))$, we see that $||u_t(\cdot,t)||_{L^2(\Omega)}\leqslant M_4$, $||u(\cdot,t)||_{H_0^1(\Omega)}\leqslant M_4$ for a.e. $t\in(0,T]$. Suppose that τ is a nonexclusive point of the functions $u(\cdot,t)$, $u_t(\cdot,t)$. Then for the sequences $u^{N_k}(\cdot,\tau)$, $u_t^{N_k}(\cdot,\tau)$ we obtain

$$u^{N_k}(\cdot,\tau)\to\psi_0\quad\text{weakly in}\quad H_0^1(\Omega),\ \ u^{N_k}(\cdot,\tau)\to\psi_0\quad\text{strongly in}\quad L^2(\Omega),$$
$$u^{N_k}(\cdot,\tau)\to\psi_0\quad\text{a. e. in}\quad\Omega,\quad u_t^{N_k}(\cdot,\tau)\to\psi_1\quad\text{weakly in}\quad L^2(\Omega)\quad\text{as}\ N_K\to\infty. \qquad (24)$$

This implies

$$|u^{N_k}(\cdot,\tau)|^{p_1(x)-2}u^{N_k}(\cdot,\tau)\to|\psi_0|^{p_1(x)-2}\psi_0\quad\text{weakly in}\quad L^{p'_1(x)}(\Omega).$$

Let $\{\tau_k\}_{k\in\mathbb{N}}\subset(0,T)$ be a sequence of nonexclusive values for $||u_t(\cdot,t)||_{L^2(\Omega)}$ and $||u(\cdot,t)||_{H_0^1(\Omega)}$ such that $\lim\limits_{k\to\infty}\tau_k=\tau,\ \tau\in[0,T]$. Then from the estimates

$$||u(\cdot,\tau_k)||_{H_0^1(\Omega)}\leqslant M_4,\quad ||u_t(\cdot,\tau_k)||_{L^2(\Omega)}\leqslant M_4,$$

we obtain the existence of a subsequence of the sequence $\{\tau_k\}_{k\in\mathbb{N}}$ (let it again be $\{\tau_k\}_{k\in\mathbb{N}}$) such that

$$u(\cdot,\tau_k)\to\psi_0\ \text{weakly in}\ H_0^1(\Omega),\quad u_t(\cdot,\tau_k)\to\psi_1\ \text{weakly in}\ L^2(\Omega)\ \text{as}\ \tau_k\to\tau.$$

On the other hand, we have the convergence

$$u(\cdot,\tau_k)\to u(\cdot,\tau) \quad \text{in} \quad L^2(\Omega), \quad u_t(\cdot,\tau_k)\to u_t(\cdot,\tau) \ \text{weakly in} \quad H^{-s}(\Omega) \ \text{as} \ \tau_k\to\tau.$$

Hence, $\psi_1(x)=u_t(x,\tau)$ and $\psi_0(x)=u(x,\tau)$.

Now choose $\{\tau_0^k\}_{k\in\mathbb{N}}$ a sequence of nonexclusive values for $||u_t(\cdot,t)||_{L^2(\Omega)}$ and $||u(\cdot,t)||_{H_0^1(\Omega)}$ such that $\lim\limits_{k\to\infty}\tau_0^k=0$. Consider (23) in which τ_0 is substituted by the elements of the constructed sequence $\{\tau_0^k\}_{k\in\mathbb{N}}$. Further passage to the limit as $k\to\infty$ and Lemma 5.3 [38, p. 20] provide the fulfillment of the inequality

$$\frac{1}{2}\int\limits_{\Omega_\tau}\left[u_t^2+\sum_{i,j=1}^n a_{ij}(x)u_{x_i}u_{x_j}\right]e^{-2\eta\tau}\,dx+\int\limits_{Q_\tau}\left[(\eta+a_0(x,t))u_t^2+\right.$$
$$+\eta\sum_{i,j=1}^n a_{ij}(x)u_{x_i}u_{x_j}+\sum_{i=1}^n a_i(x,t)u_{x_i}u_t+c(x,t)uu_t+b_0(x,t)zu_t+$$
$$\left.+\frac{1}{p_1(x)}(2\eta b_1(x,t)-b_{1t}(x,t))|u|^{p_1(x)}-f(x,t)u_t\right]e^{-2\eta t}\,dx\,dt+$$
$$+\int\limits_{\Omega_\tau}\frac{1}{p_1(x)}|u|^{p_1(x)}e^{-2\eta\tau}\,dx\geqslant\frac{1}{2}\int\limits_{\Omega_0}\left[u_t^2+\sum_{i,j=1}^n a_{ij}(x)u_{0,x_i}u_{0,x_j}\right]dx+$$
$$+\int\limits_{\Omega_0}\frac{1}{p_1(x)}|u|^{p_1(x)}\,dx. \tag{25}$$

If $u_0=0$ and $u_1=0$, then (25) transforms into an equality.

Let us consider the sequence

$$0\leqslant y_k=\int\limits_{Q_\tau}b_0(x,t)(|u_t^{N_k}|^{p_0(x)-2}u_t^{N_k}-|v|^{p_0(x)-2}v)(u_t^{N_k}-v)e^{-2\eta t}\,dx\,dt=$$
$$=\int\limits_{Q_\tau}\left[f(x,t)u_t^{N_K}-u_{tt}^{N_k}u_t^{N_K}-\sum_{i,j=1}^n a_{ij}(x)u_{x_i}^{N_k}u_{x_jt}^{N_k}-\sum_{i=1}^n a_i(x,t)u_{x_i}^{N_k}u_t^{N_k}-\right.$$
$$\left.-a_0(x,t)|u_t^{N_k}|^2-c(x,t)u^{N_k}u_t^{N_k}-b_1(x,t)|u_t^{N_k}|^{p_1(x)-2}u^{N_k}u_t^{N_k}\right]e^{-2\eta t}\,dx\,dt-$$
$$-\int\limits_{Q_\tau}b_0(x,t)[|u_t^{N_k}|^{p_0(x)-2}u_t^{N_k}v+|v|^{p_0(x)-2}v(u_t^{N_k}-v)]e^{-2\eta t}\,dx\,dt=$$

$$=\int\limits_{Q_\tau}\left[f(x,t)u_t^{N_K}-(\eta+a_0(x,t))|u_t^{N_k}|^2-2\eta\sum_{i,j=1}^n a_{ij}(x)u_{x_i}^{N_k}u_{x_j}^{N_k}-\sum_{i=1}^n a_i(x,t)u_{x_i}^{N_k}u_t^{N_k}-\right.$$

$$-c(x,t)u^{N_k}u_t^{N_k} - \frac{1}{p_1(x)}(2\eta b_1(x,t) - b_{1t}(x,t))|u^{N_k}|^{p_1(x)}\Bigg] e^{-2\eta t}\,dx\,dt -$$

$$-\int\limits_{\Omega_\tau}\left[\frac{1}{2}|u_t^{N_k}|^2 + \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i}^{N_k}u_{x_j}^{N_k} + \frac{1}{p_1(x)}b_1(x,\tau)|u^{N_k}|^{p_1(x)}\right]e^{-2\eta\tau}\,dx +$$

$$+\int\limits_{\Omega_0}\left[\frac{1}{2}|u_1^{N_k}|^2 + \sum_{i,j=1}^{n} a_{ij}(x)u_{0,x_i}^{N_k}u_{0,x_j}^{N_k} + \frac{1}{p_1(x)}b_1(x,0)|u^{N_k}|^{p_1(x)}\right]dx -$$

$$-\int\limits_{Q_\tau} b_0(x,t)[|u_t^{N_k}|^{p_0(x)-2}u_t^{N_k}v + |v|^{p_0(x)-2}v(u_t^{N_k} - v)]e^{-2\eta t}\,dx\,dt. \qquad (26)$$

It is easy to show that for sufficiently large η the functional

$$J_1(u^{N_k}) = \Bigg(\int\limits_{Q_\tau}\Bigg[(\eta + a_0(x,t))|u_t^{N_k}|^2 + 2\eta\sum_{i,j=1}^{n} a_{ij}(x)u_{x_i}^{N_k}u_{x_j}^{N_k} +$$

$$+\sum_{i=1}^{n} a_i(x,t)u_{x_i}^{N_k}u_t^{N_k} + c(x,t)u^{N_k}u_t^{N_k}\Bigg]e^{-2\eta t}\,dx\,dt\Bigg)^{\frac{1}{2}}, \quad \tau \in (0,T]$$

is equivalent to the norm $||u^{N_k}||_{H_0^1(Q_\tau)}$ and the functional

$$J_2(u^{N_k}) = \int\limits_{Q_\tau}\frac{1}{p_1(x)}(2\eta b_1(x,t) - b_{1t}(x,t))|u^{N_k}|^{p_1(x)}e^{-2\eta t}\,dx\,dt, \quad \tau \in (0,T]$$

specifies a norm in the space $L^{p_1(x)}(Q_\tau)$ which is equivalent to the norm $||u^{N_k}||_{L^{p_1(x)}(Q_\tau)}$. Taking into account (26), (24), Lemma at the beginning of this section and Lemma 5.3 [38, p. 20], we obtain

$$0 \leqslant y_k = \int\limits_{Q_\tau}\Bigg[f(x,t)u_t - (\eta + a_0(x,t))|u_t|^2 - 2\eta\sum_{i,j=1}^{n} a_{ij}(x)u_{x_i}u_{x_j} -$$

$$-\sum_{i=1}^{n} a_i(x,t)u_{x_i}u_t - c(x,t)uu_t - \frac{1}{p_1(x)}(2\eta b_1(x,t) - b_{1t}(x,t))|u|^{p_1(x)}\Bigg]e^{-2\eta t}\,dx\,dt -$$

$$-\int\limits_{\Omega_\tau}\left[\frac{1}{2}|u_t|^2 + \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i}u_{x_j} + \frac{1}{p_1(x)}b_1(x,\tau)|u|^{p_1(x)}\right]e^{-2\eta\tau}\,dx +$$

$$+\int\limits_{\Omega_0}\left[\frac{1}{2}|u_1|^2 + \sum_{i,j=1}^{n} a_{ij}(x)u_{0,x_i}u_{0,x_j} + \frac{1}{p_1(x)}b_1(x,0)|u|^{p_1(x)}\right]dx -$$

$$-\int\limits_{Q_\tau} b_0(x,t)[zv+|v|^{p_0(x)-2}v(u_t-v)]e^{-2\eta t}\,dx\,dt, \quad \forall\ v\in L^{p_0(x)}(Q_\tau). \qquad (27)$$

By adding (25) and (27), we get

$$0\leqslant y_k=\int\limits_{Q_\tau} b_0(x,t)(u_t-v)(z-|v|^{p_0(x)-2}v)e^{-2\eta t}\,dx\,dt. \qquad (28)$$

Suppose that $\{\tau_k\}_{k\in\mathbb{N}}$ is a sequence of nonexclusive points of the function $u_t(\cdot,t)$ such that $\lim\limits_{k\to\infty}\tau_k=T$. Then from (28) for all $v\in L^{p_0(x)}(Q_T)$ we obtain the estimate

$$0\leqslant y_k=\int\limits_{Q_T} b_0(x,t)(u_t-v)(z-|v|^{p_0(x)-2}v)e^{-2\eta t}\,dx\,dt. \qquad (29)$$

Let us consider the functional

$$J(v)=\int\limits_{Q_T}\frac{1}{p_0(x)}|v|^{p_0(x)}\,dx\,dt, \quad v\in L^{p_0(x)}(Q_T).$$

Since $J(u)$ is convex and its derivative in Gateaux's sense equals

$$J'(v)=\int\limits_{Q_T}|v|^{p_0(x)-2}v\ dx\,dt,$$

then according to [8, p. 169] the operator $A:L^{p_0(x)}(Q_T)\to L^{p_0'(x)}(Q_T)$ which is defined by the formula

$$\langle Av,u\rangle_1=\int\limits_{Q_T}|v|^{p_0(x)-2}vu\ dx\,dt,$$

where $\langle\ ,\ \rangle_1$ denotes the pairing between the spaces $L^{p'_0(x)}(Q_T)$, $L^{p_0(x)}(Q_T)$, is semicontinuous. Taking into account (29), we have the inequality

$$\langle z-Av,u_t-v\rangle_1\geqslant 0, \quad \forall\ u_t,\ v\in L^{p_0(x)}(Q_T). \qquad (30)$$

Let us take in (30) $v=u_t-\lambda\omega$, where $\lambda>0$, $\omega\in L^{p_0(x)}(Q_T)$ are arbitrary. Passing to the limit as $\lambda\to 0$ we get

$$\langle z-Au_t,\omega\rangle_1\geqslant 0 \quad \text{for all}\ \ \omega\in L^{p_0(x)}(Q_T).$$

Let h be an arbitrary element from $L^{p_0(x)}(Q_T)$. By setting in the last inequality first $\omega=h$ and then $\omega=-h$, we obtain $z=|u_t|^{p_0(x)-2}u_t$. Thus, u is a weak solution of (1) – (3). The proof of the theorem is completed.

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МІШАНА ЗАДАЧА ДЛЯ СЛАБКО НЕЛІНІЙНОГО ГІПЕРБОЛІЧНОГО РІВНЯННЯ В УЗАГАЛЬНЕНИХ ПРОСТОРАХ ЛЕБЕГА

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В обмеженій циліндричній області розглянуто мішану задачу для слабко нелінійного гіперболічного рівняння. Умови існування розв'язку такої задачі отримано в узагальнених просторах Лебега.

Ключові слова: гіперболічне рівняння, мішана задача, узагальнені простори Лебега.

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