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# OPEN-MULTICOMMUTATIVITY OF THE FUNCTOR OF UPPER-CONTINUOUS CAPACITIES

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Notion of open-multicommutativity, introduced by Kozhan and Zarichnyi [5], is investigated. Weakly normal covariant functor of upper-continuous capacities is considered. The main result of the paper is that this functor open-multicommutative.

*Key words:* capacity, covariant functor, open-multicommutativity.

**1.** The impact of the non-additive probability theory in the modern economic theory and finance increased significantly during the last decades. This theory is based on the notion of the capacity which was first introduced by Choquet [2]. By the 80's the number of authors (Quiggin [7], Yaari [10], Schmeidler [8]) presented an axiomatization of individual's preferences and developed non-expected utility theory which based on the notion of the Choquet integral.

From the topological point of view capacities were considered by Zhou [12]. He investigated the structure of the space of upper-continuous capacities and an integral representation of continuous comonotonically additive functionals with respect to them.

Here we study the space of upper-continuous capacities from the viewpoint of the categorical topology. We prove an analogical result which was investigated in the case of the probability measures space. The notion of open-multicommutativity which combines properties of a covariant functor to be open and bicommutative has been introduced in [5]. The main result of their paper is that the functor of probability measures is open-multicommutative in the category of compact Hausdorff spaces. Here we extend an area of this investigation and consider the functor of upper-continuous capacities. Although, this functor turns out to be weakly-normal, it satisfies the open-multicommutativity property.

The paper is organized as follows. In Section 2 we remind some definitions which we use below. Section 3 contains a proof of the finite open-multicommutativity. The main result is given at the end of this Section.

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## 2. Notations and Definitions.

**2.1. Functor of upper-continuous capacities.** Let $X$ be a compact Hausdorff space and $\mathcal{F}$ a $\sigma$-algebra of its Borel subsets.

**Definition 1.** *A real-valued set function $\mu$ on $\mathcal{F}$ is called a* capacity *if $\mu(\emptyset) = 0$, $\mu(X) = 1$ and $\mu(A) \leqslant \mu(B)$ for all $A \subseteq B$, $A, B \in \mathcal{F}$.*

**Definition 2.** *A capacity $\mu$ is* upper-continuous *if $\lim\limits_{n\to\infty} \mu(A_n) = \mu(\bigcap\limits_{n=1}^{\infty} A_n)$ for any monotonic sequence of sets $A_1 \supseteq A_2 \supseteq A_3 \supseteq ...$ with $A_n \in \mathcal{F}$, $n \in \mathbb{N}$.*

We denote a set of all upper-continuous capacities on $X$ as $M(X)$. Due to Zhou [12] we can identify the set $M(X)$ with the set of all comonotonically additive, monotonic and continuous functional on $C(X)$ by the formula

$$\mu(f) = \int_0^{\infty} \mu(f \geqslant t)dt + \int_{-\infty}^{0} (\mu(f \geqslant t) - 1)dt.$$

The above integral is called the Choquet integral.

Let us endow the set $M(X)$ with the weak-* topology. The base of this topology consists of the set of the form

$$O(\mu_0, f_1, ..., f_n, \varepsilon) = \{\mu \in M(X) \colon |\mu_0(f_i) - \mu(f_i)| < \varepsilon, i = 1, ..., n\},$$

where $\mu_0 \in M(X)$, $f_1, ..., f_n \in C(X)$ and $\varepsilon > 0$.

We can consider $M \colon \mathrm{Comp} \to \mathrm{Comp}$ as a covariant functor in the category Comp and as it is shown in [6] that it is also weakly normal. Another important property of this functor is that it is open and bicommutative.

**Proposition 1.** *Functor $M$ is bicommutative.*

*Proof.* Let us consider an arbitrary bicommutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow[s]{} & T \end{array} \tag{1}$$

in the category Comp. In order to prove that $M$ is a bicommutative functor it is sufficient to show that for every $\mu \in M(X)$ and $\nu \in M(Y)$ such that $Mh(\mu) = Ms(\nu) = \tau \in M(T)$ there exists a capacity $\lambda \in M(Z)$ with

$$Mf(\lambda) = \mu \text{ and } Mg(\lambda) = \nu. \tag{2}$$

Due to condition (2) for every $A \in \mathcal{F}_X$ and $B \in \mathcal{F}_Y$ it must hold

$$\lambda(f^{-1}(A)) = \mu(A) \text{ and } \lambda(g^{-1}(B)) = \nu(B).$$

Denote $\mathcal{S} = \{f^{-1}(A), g^{-1}(B)\colon A \in \mathcal{F}_X, B \in \mathcal{F}_Y\}$. Let $\lambda$ be an inner measure defined as

$$\lambda(D) = \sup\{\mu(f(C)), \nu(g(C))\colon C \subseteq D, C \in \mathcal{S}\}$$

for every $D \in \mathcal{F}_Z$. Defined in such way set function $\lambda$ is an upper-continuous capacity (see, for instance, [11]). Let us show that condition (2) is satisfied. Let $A \in \mathcal{F}_X$ and $A' = f^{-1}(A) \subset Z$. Obviously that $\sup\{\mu(f(C))\colon C \subseteq A', C \in \mathcal{S}\} = \mu(A)$. We assume that there exists a subset $B \subset Y$ such that $B' = g^{-1}(B) \subseteq A'$ and $\nu(B) > \mu(A)$. Note that the set $\tilde{A} = h^{-1}(s(B)) \subseteq A$. Indeed, due to the definition of this set for every point $a \in \tilde{A}$ we can find $b \in B$ such that $h(a) = s(b)$. Because of the bicommutativity of diagram (1) there exists point $z \in Z$ satisfying $f(z) = a$ and $g(z) = b$. Since $B'$ is a full preimage of the set $B$ it is necessary that $b \in B' \subseteq A'$. This implies $a \in A$. Due to the condition (2) we have

$$\nu(B) \leqslant \tau(s(B)) = \mu(h^{-1}(s(B))) = \mu(\tilde{A}) \leqslant \mu(A),$$

which contradicts our assumption. Thus, $\lambda(f^{-1}(A)) = \mu(A)$ for every $A \in \mathcal{F}_X$. Analogically we can prove that $\lambda(g^{-1}(B)) = \nu(B)$ for each $B \in \mathcal{F}_Y$. Therefore, condition (2) is satisfied.

**Proposition 2.** *Functor $M$ is open.*

*Proof.* This proposition is proved in [6].

**2.2. Open-multicommutative functors and characteristic map.** Let us recall the notion of the multi-commutativity of a weakly-normal functor which is introduced in [5].

Suppose that $\mathcal{G}$ is a finite partially ordered set and we also regard it as a finite directed graph. Denote by $\mathcal{V}\mathcal{G}$ the class of all vertices of graph $\mathcal{G}$ and by $\mathcal{E}\mathcal{G}$ the set of its edges. A functor $\mathcal{O}\colon \mathcal{G} \to \mathrm{Comp}$ is called a *diagram*. A *cone* over $\mathcal{O}$ consists of a space $X \in |\mathrm{Comp}|$ and a family of maps $\{X \to \mathcal{O}(o)\}_{o \in \mathcal{V}\mathcal{G}}$ that satisfy obvious commutativity conditions. Given such a cone, $\mathcal{C} = (\{X \to \mathcal{O}(o)\}_{o \in \mathcal{V}\mathcal{G}})$, we denote by $\chi_{\mathcal{C}}\colon X \to \lim \mathcal{O}$ its characteristic map.

We say that the cone $\mathcal{C}$ is *open-multicommutative* if its characteristic map is an open onto map.

**Definition 3.** *A normal functor $F$ in* Comp *is called* open-multicommutative *(*finite open-multicommutative*) if it preserves the class of open-multicommutative diagrams (which consist of finite spaces).*

The following result can be found in [4].

**Proposition 3.** *For a weakly normal open bicommutative functor $F$ the following properties are equivalent:*

*$F$ is open-multicommutative;*

*$F$ is finite open-multicommutative.*

**3. Case of discrete spaces.** In this section we assume that all spaces $\mathcal{O}(o)$, $o \in \mathcal{VG}$ are finite and discrete. According to Proposition 3. for the open-multicommutativity of the functor $M$ it is sufficient to show that it is finite open-multicommutative, i.e. the characteristic map $\chi\colon M(\lim\mathcal{O}) \to \lim M(\mathcal{O})$ is open and surjective. Let us assume also that $\mathcal{VG}$ is finite.

Let us also recall that $\lim\mathcal{O}$ can be defined in terms of *threads*. We say that the point $x = (x_o)_{o\in\mathcal{VG}} \in \prod\limits_{o\in\mathcal{VG}} \mathcal{O}(o)$ is a *thread* of the diagram $\mathcal{O}$ if for every $o_1, o_2 \in \mathcal{VG}$ with $o_1 \leqslant o_2$ it holds $\mathrm{pr}_{o_1}(x) = \varphi_{o_1o_2}\mathrm{pr}_{o2}(x)$. It is well known that $\lim\mathcal{O} \subseteq \prod\limits_{o\in\mathcal{VG}} \mathcal{O}(o)$ and since all $\mathcal{O}(o)$ are discrete, the limit of the diagram is also discrete space.

Let $\lambda^0 \in M(\lim\mathcal{O})$ be a capacity on the space $\lim\mathcal{O}$ and $\mu^0_o \in M(\mathcal{O}(o))$ be its marginals for $o \in \mathcal{VG}$. Let $O(\lambda^0, f_1, ..., f_n, \varepsilon)$ be an arbitrary weak-* neighborhood of the point $\lambda^0$.

In order to prove the openness of the characteristic map it is sufficient to find a neighborhood of the point $(\mu^0_o)_{o\in\mathcal{VG}}$ such that every point from this neighborhood can be covered by some capacity from $O(\lambda^0, f_1, ..., f_n, \varepsilon)$.

**Lemma 1.** *Let $X$ be a discrete compactum. The base of the weak-* topology on $M(X)$ consists of the sets of the form*

$$O(\mu, F_1, ..., F_n, \varepsilon) = \{\nu \in M(X)\colon |\nu_0(F_i) - \mu(F_i)| < \varepsilon, i = 1, ..., n\},$$

*$\mu_0 \in M(X)$ and $F_i \subset X$, $i = 1, ..., n$.*

*Proof.* Let us show first that for every set of the form $O(\mu, F_1, ..., F_n, \varepsilon)$ we can find a basis neighborhood $O(\mu, f_1, ..., f_k, \delta)$ for some $f_i \in C(X)$ and $\delta > 0$, $i = 1, ..., k$. Indeed, if we set $f_i = \mathbf{1}_{F_i}$ and $\delta = \varepsilon$ we get

$$O(\mu, F_1, ..., F_n) = O(\mu, f_1, ..., f_n, \varepsilon).$$

Conversely, consider an element of the sub-base $O(\mu, f, \varepsilon)$. Since the space $X$ is discrete we can represent $f = \sum\limits_{i=1}^{k} \alpha_i \mathbf{1}_{F_i}$ such that $F_1 \subset F_2 \subset ... \subset F_k$. It is clear (see [1]) that for every capacity $\nu \in M(X)$ holds $\nu(f) = \sum\limits_{i=1}^{k} \alpha_i \nu(F_i)$. Let us consider a set $O(\mu, F_1, .., F_k, \frac{\varepsilon}{k\alpha})$, where $\alpha = \max\{|\alpha_1|, ..., |\alpha_k|\}$. Comonotonicity of functions $\mathbf{1}_{F_i}$ implies that for every capacity $\nu \in O(\mu, F_1, .., F_k, \frac{\varepsilon}{k\alpha})$ we have that

$$|\mu(f) - \nu(f)| = |\sum_{i=1}^{k} \alpha_i(\mu(F_i) - \nu(F_i))| \leqslant \sum_{i=1}^{k} |\alpha_i||\mu(F_i) - \nu(F_i)| < \sum_{i=1}^{k} \frac{\varepsilon\alpha_i}{k\alpha} < \varepsilon.$$

Hence, $O(\mu, F_1, .., F_k, \frac{\varepsilon}{k\alpha}) \subset O(\mu, f, \varepsilon)$.

According to Lemma 1. we can assume without loss of generality that functions $f_1, ..., f_n$ are of the form $f_i(x) = \begin{cases} 1, & x \in F_i, \\ 0, & x \notin F_i \end{cases}$ for every $x \in \lim D$ with $F_1, ..., F_n$ are subsets of $\lim\mathcal{O}$.

We consider a neighborhood

$$U = O(\mu_1^0, \{x_1^1\}, ..., \{x_{m_1}^1\}, \delta) \times ... \times O(\mu_k^0, \{x_1^k\}, ..., \{x_{m_k}^k\}, \delta),$$

where $X_i = \{x_1^i, ..., x_{m_i}^i\}$. Let $(\mu_1, ..., \mu_k)$ be an arbitrary point in $U$. Let us define a capacity $\lambda$ on $\lim \mathcal{O}$.

For every subset $A \subset \lim \mathcal{O}$ we define

$$l_A = \max_{o' \in \mathcal{VG}} \{\max\{\mu_{o'}(W) \colon W \subseteq \mathcal{O}(o'), (\prod_{o \in \mathcal{VG} \setminus \{o'\}} \mathcal{O}(o) \times W) \cap \lim \mathcal{O} \subseteq A\}\}.$$

Analogically,

$$u_A = \min_{o' \in \mathcal{VG}} \{\min\{\mu_{o'}(W) \colon W \subseteq \mathcal{O}(o'), (\prod_{o \in \mathcal{VG} \setminus \{o'\}} \mathcal{O}(o) \times W) \cap \lim \mathcal{O} \supseteq A\}\}.$$

Note that the interval $[l_A, u_A]$ is not empty and in order $\lambda$ to be well defined it should satisfies inequalities

$$l_A \leqslant \lambda(A) \leqslant u_A$$

for every subset $A \subset \lim \mathcal{O}$. Recall also that $(\mu_1, ..., \mu_k) \in U$ and this implies that for every $A \subset \lim \mathcal{O}$ we have $l_A - \delta < \lambda^0(A) < u_A + \delta$.

**Lemma 2.** *If $A \subseteq B$ then $l_A \leqslant l_b$ and $u_A \leqslant u_B$.*

*Proof.* It is clear that for every $W \subseteq \mathcal{O}(o')$ such that $(\prod_{o \in \mathcal{VG} \setminus \{o'\}} \mathcal{O}(o) \times W) \cap \lim \mathcal{O} \subseteq A$ we have that $(\prod_{o \in \mathcal{VG} \setminus \{o'\}} \mathcal{O}(o) \times W) \cap \lim \mathcal{O} \subseteq B$ for every $j = 1, ..., k$. This implies that $l_A \leqslant l_B$.

The analogical result for the upper bounds can be derived from the statement

$$(\prod_{o \in \mathcal{VG} \setminus \{o'\}} \mathcal{O}(o) \times W) \cap \lim \mathcal{O} \supseteq A \supseteq B.$$

For a Borel set $A \subset \lim \mathcal{O}$ we define a capacity $\lambda$ as

$$\lambda(A) = \max\{l_A, \min\{u_A, \lambda_0(A)\}\}.$$

**Lemma 3.** *The set function $\lambda$ is a well-defined capacity.*

*Proof.* First of all, $l_{(\lim \mathcal{O})} = u_{(\lim \mathcal{O})} = 1$ this implies that $\lambda(\lim \mathcal{O}) = 1$.

$l_\emptyset = u_\emptyset = 0$ this implies that $\lambda(\emptyset) = 0$.

Let us check now a monotonicity of $\lambda$. We suppose that $A \subset B \subset \lim \mathcal{O}$. Consider three cases:

1). $\lambda_0(A) \in [l_A, u_A]$. In this case

$$l_A < \lambda(A) = \lambda_0(A) \leqslant \min\{u_A, \lambda_0(B)\} \leqslant \min\{u_B, \lambda_0(B)\} = \lambda(B).$$

2). $\lambda_0(A) > u_A$. We have

$$l_A < \lambda(A) = u_A \leqslant \min\{u_B, \lambda_0(A)\} \leqslant \min\{u_B, \lambda_0(B)\} = \lambda(B).$$

3). $\lambda_0(A) < l_A$. This condition implies that

$$\lambda(A) = l_A \leqslant l_B \leqslant \lambda(B).$$

Let us set now $\delta = \varepsilon$. In this case we obtain for every $i = 1, ..., n$ that

$$|\lambda(F_i) - \lambda_0(F_i)| < \delta = \varepsilon.$$

This leads to $\lambda \in O(\lambda^0, f_1, ..., f_n, \varepsilon)$.

Due to the definition of $l_A$ and $u_A$ it is easy to check that

$$l_{((\prod\limits_{o \in \mathcal{VG} \setminus \{o'\}} \mathcal{O}(o) \times W) \cap \lim \mathcal{O})} = \mu_{o'}(W)$$

and

$$u_{((\prod\limits_{o \in \mathcal{VG} \setminus \{o'\}} \mathcal{O}(o) \times W) \cap \lim \mathcal{O})} = \mu_{o'}(W)$$

for every $o' \in \mathcal{VG}$ and $W \subset X_{o'}$. This implies that $\lambda((\prod\limits_{o \in \mathcal{VG} \setminus \{o'\}} \mathcal{O}(o) \times W) \cap \lim \mathcal{O}) = \mu_{o'}(W)$ and hence $M\mathrm{pr}_{o'}(\lambda) = \mu_{o'}$ for all $o' \in \mathcal{VG}$.

Hence we proved that the inverse to the correspondence map is open in the case of discrete $\mathcal{O}(o)$, $o \in \mathcal{VG}$. Thus, applying this fact to Proposition 3. we obtain

**Theorem 1.** *The correspondence map $\chi$ of the diagram $\mathcal{O}$ is open and surjective for every $\mathcal{O}(o) \in |\mathrm{Comp}|$, $o \in \mathcal{VG}$.*

A special case of open-multicommutativity was considered by Eifler [3]. One can get this case setting the set $\mathcal{EG} = \emptyset$. Eifler proved that the functor of the probability measures preserves surjectivity and openness of the characteristic maps of such kind of diagrams. Thus, the result of Theorem 1. is an extension of Eifler's theorem on the case of non-additive measures.


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## ВІДКРИТА МУЛЬТИКОМУТАТИВНІСТЬ ФУНКТОРА НЕПЕРЕРВНИХ ЗВЕРХУ ЄМНОСТЕЙ

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Досліджуємо поняття відкритої мультикомутативності, вперше введене Кожаном та Зарічним [5]. Розглянуто слабко-нормальний функтор неперервних зверху ємностей. Основним результатом статті є доведення факту відкритої мультикомутативності цього функтора.

*Ключові слова:* ємність, коваріантний функтор, відкрита мультикомутативність.

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