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CHARACTERIZATION OF G-SYMMETRIC POWER FUNCTORS IN THE COARSE CATEGORY

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It is proved that a normal functor of finite degree acting in the coarse category admits an extension onto the Kleisli category of the hyperspace monad if and only if this functor is isomorphic to the symmetric power functor.

Key words: Coarse category, G-symmetric power functor, hyperspace monad.

1. The coarse category (i.e. the category of coarse spaces and coarse maps) was introduced by Roe in [4]. This theory turned out to be an appropriate universe for studying asymptotic properties of structures more general then metric spaces. Some results in the direction of asymptotic algebra (i.e. those concerning algebraic properties of coarse structures) are obtained [1],[8].

In particular, in [1] the hyperspace functor acting in the category of coarse topological spaces was considered. It was proved in [1] that the hyperspace functor determines a monad in the coarse category.

In [8] the author considered the notion of normal functor in the coarse category and established some properties of the normal functors. The aim of this note is to characterize the class of G-symmetric power functor in the coarse category by means of their extension onto the Kleisli category of the hyperspace monad. The main result is a counterpart of the characterization theorem proved in [7].

2. Preliminaries. We briefly recall some necessary definitions and results concerning the functors in the coarse category and also the Kleisli categories of monads.

2.1. Functors in the coarse category. For the convenience of reader we recall some definitions of the coarse topology; see, e.g. [4], [2] for details.

Let X be a set and $M, N \subset X \times X$. The *composition* of M and N is the set

$$MN = \{(x,y) \in X \times X \mid \text{ there exists } z \in X \text{ such that } (x,z) \in M,\ (z,y) \in N\},$$

the *inverse* of M is the set $M^{-1} = \{(x,y) \in X \times X \mid (y,x) \in M\}$.

A *coarse stucture* on a set X is a family $\mathcal{E}$ of subsets, which are called the *entourages*, in the product $X \times X$ that satisfies the following properties:

1. any finite union of entourages is contained in an entourage;
2. for every entourage M, its inverse M^{-1} is contained in an entourage;
3. for every entourages M, N their composition MN is contained in an entourage;
4. $\cup\mathcal{E} = X \times X$.

A *coarse space* is a pair $(X, \mathcal{E})$, where $\mathcal{E}$ is a coarse structure on a set X.

Let (X, d) be a metric space. The family

$$\mathcal{E}_d = \{\{(x,y) \in X \times X \mid d(x,y) < n\} \mid n \in \mathbb{N}\}$$

forms a *metric coarse structure* on X.

Given $M \in \mathcal{E}$ and $A \subset X$ we define the *M-neighborhood* $M(A)$ of A as follows: $M(A) = \{x \in X \mid (a,x) \in M \text{ for sone } a \in A\}$. We use the notation $M(\{a\})$ instead of $M(a)$. A set $A \subset X$ is *bounded* if there exists $x \in X$ such that $A \subset M(x)$.

Let $(X_i, \mathcal{E}_i)$, $i = 1, 2$, be coarse spaces. A map $f\colon X_1 \to X_2$ is called *coarse*, if the following two conditions hold:

1. for every $M \in \mathcal{E}_1$ there exists $N \in \mathcal{E}_2$ such that $(f \times f)(M) \subset N$;
2. for any bounded subset A of X_2 the set $f^{-1}(A)$ is bounded.

Let $f, g\colon X_1 \to X_2$ be coarse maps. If there exists $U \in \mathcal{E}_2$ (here $\mathcal{E}_2$ is the coarse structure on X_2) such that $(f(x), g(x)) \in U$ for every $x \in X_1$ then the maps f, g are said to be *U-close*. Define the relation $\sim$ on the set of all coarse maps as follows: $f \sim g$ if and only if f and g are U-close, for some U. It is easy to see that $\sim$ is an equivalence relation on the set of coarse maps from $X - 1$ to X_2. We denote by $[f]$ the equivalence class of $\sim$ which contains f.

The composition of the equivalence classes of the maps in the next way: $[gf] = [g][f]$

It is easy to see that the coarse spaces and coarse maps form a category. We denote it by CS and by CS/$_\sim$ we denote the category whose objects are coarse spaces and whose morphisms are the equivalence classes of the morphisms of the category CS.

We briefly recall some notions from the theory of normal functors in the category **Comp** of compact Hausdorff spaces; see, e.g., [9] for details. An endofunctor F in **Comp** is called *normal* if F is continuous , monomorphic, epimorphic, preserves weight of infinite compacta, intersections, preimages, singletons and empty set. A normal functor is called *finitary* if it preserves the class of finite sets.

Now let F be finitary normal functor of degree $n \geq 1$, $(X, \mathcal{E})$ a coarse space. For any $U \in \mathcal{E}$ define

$$\begin{aligned}\hat{U} = &\{(a,b) \in FX \times FX \mid \text{ there exist } W_1, \dots, W_k \in \mathcal{E},\\ &f_1, \dots, f_{2k} \in C(n, X),\ c_1, \dots, c_k \in Fn \text{ such that}\\ &W_1 \dots W_k \subset U, \text{ are } f_{2i-1}, f_{2i}\ U\text{-close},\ i = 1, \dots, k,\\ &\text{i } Ff_1(c_1) = a,\ Ff_{2k}(c_k) = b,\\ &Ff_{2j}(c_j) = Ff_{2j+1}(c_{j+1}),\ j = 1, \dots, k-1\}.\end{aligned}$$

Note that here we consider the set X as a discrete topological space, that is why it is possible to consider the discrete space FX, which is identified with the underlying set.

In [?] it is proved that the family $\{\hat{U} | U \in \mathcal{E}\}$ forms the coarse structure on FX.

See [8] for the proof of the following result.

Lemma 1. *Let $f, g\colon (X_1, \mathcal{E}_1) \to (X_2, \mathcal{E}_2)$. If $f \sim g$ then $F(f) \sim F(g)$.*

This lemma allows us to consider a functor F in the category $\mathrm{CS}/_\sim$ because of the equality $F[f] = [Ff]$.

Definition 1. *A functor $F\colon \mathrm{CS} \to \mathrm{CS}$ is* normal *in* CS *if:*

1) F preserves weight;
2) F is monomorphic;
3) F is epimorphic;
4) F preserves preimages;
5) F preserves $\emptyset$ (i.e. bounded coarse spaces).

The corresponding functor in the category $\mathrm{CS}/\sim$ is also called normal.

2.2. Kleisli category of the hyperspace monad. If T is an endofunctor in a category $\mathcal{C}$ and $\eta\colon 1_{\mathcal{C}} \to T$ and $\mu\colon T^2 \equiv TT \to T$ are natural transformations, then $\mathbb{T} = (T, \eta, \mu)$ is called a *monad* if and only if the following diagrams commute:

$$\begin{array}{ccc} T & \xrightarrow{\eta T} & T^2 \\ {\scriptstyle T\eta}\downarrow & \searrow^{1_T} & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow{\mu} & T \end{array} \qquad \begin{array}{ccc} T^3 & \xrightarrow{\mu T} & T^2 \\ {\scriptstyle T\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow{\mu} & T. \end{array}$$

See [1] for the definition of the hyperspace monad in the coarse category.

The *Kleisli category of* $\mathbb{T}$ is the category $\mathcal{C}_{\mathbb{T}}$ defined as follows: $|\mathcal{C}_{\mathbb{T}}| = |\mathcal{C}|$, $\mathcal{C}_{\mathbb{T}}(X, Y) = \mathcal{C}(X, TY)$, and the composition $g * f$ of morphisms $f \in \mathcal{C}_{\mathbb{T}}(X, Y)$, $g \in \mathcal{C}_{\mathbb{T}}(Y, Z)$ is given by $g * f = \mu Z \circ Tg \circ f$.

Define the functor $I\colon \mathcal{C} \to \mathcal{C}_{\mathbb{T}}$ by $F_{\mathbb{T}}X = X$, $X \in |\mathcal{C}|$ and $If = \eta Y \circ f$ for $f \in \mathcal{C}(X, Y)$.

A functor $\overline{F}\colon \mathcal{C}_{\mathbb{T}} \to \mathcal{C}_{\mathbb{T}}$ called an *extension of the functor $F\colon \mathcal{C} \to \mathcal{C}$ on the Kleisli category* $\mathcal{C}_{\mathbb{T}}$ if $IF = \overline{F}I$.

In the sequel we will need the following result.

Theorem 1. *There exists a bijective correspondence between extensions of functor F onto the Kleisli category $\mathcal{C}_{\mathbb{T}}$ of monad $\mathbb{T}$ and natural transformations $\xi\colon FT \to TF$ satisfying*

1. $\xi \circ F\eta = \eta F$;

2. $\mu F \circ T\xi \circ \xi T = \xi \circ F\mu$.

3. Characterization theorem.

Theorem 2. *A normal functor F of degree $n \geq 1$ in the category $\mathrm{CS}/\sim$ can be extended onto the category $(\mathrm{CS}/\sim)_{\mathbb{H}}$ if and only if $F \simeq SP^n_G$, for some subgroup G of S_n.*

Proof. For every coarse space X, define $\xi_X \colon SP^n_G(\exp X) \to \exp SP^n_G(X)$ by the formula:

$$\xi_X([A_1, \dots, A_n]_G) = \{[a_1, \dots, a_n]_G \mid a_i \in A_i \text{ for all } i \leq n\}).$$

That the natural transformation ξ satisfies the the properties of Theorem 1., is remarked in [1] (the corresponding natural transformation is denoted by d herein). We supplement the proof from [1] by explicit proof that ξ_X is a coarse map. Recall that, given a coarse structure $\mathcal{E}$ on X, we define a coarse structure $\tilde{\mathcal{E}}$ on $SP^n_G(X)$ as follows: $\tilde{\mathcal{E}} = \{\tilde{U} \mid U \in \mathcal{E}\}$, where $([a_1, \dots, a_n]_G, [b_1, \dots, b_n]_G) \in \tilde{U}$ if and only if there is a permutation $\sigma \in G$ such that $(a_i, b_{\sigma(i)}) \in U$, for every $i \leq n$.

Recall also that we consider the Hausdorff coarse structure $\hat{\mathcal{E}}$ on $\exp X$: given $U \in \mathcal{E}$, we define

$$\hat{U} = \{(A, B) \in \exp X \times \exp X \mid A \subset U(B),\ B \subset U(A)\}$$

and let $\hat{\mathcal{E}} = \{\hat{U} \mid U \in \mathcal{E}\}$.

Now, let $\tilde{\hat{U}} \in \tilde{\hat{\mathcal{E}}}$ and $([A_1, \dots, A_n]_G, [B_1, \dots, B_n]_G) \in \tilde{\hat{U}}$. Then there is a permutation $\sigma \in G$ such that $(A_i, B_{\sigma(i)}) \in \hat{U}$, for every $i \leq n$. For any $[a_1, \dots, a_n]_G \in \xi_X([A_1, \dots, A_n]_G)$ and any $i \leq n$, one can find a point, which we denote by $b_{\sigma(i)}$, such that $(a_i, b_{\sigma(i)}) \in U$. We conclude that $\xi_X(\tilde{\hat{U}}) \subset \hat{\tilde{U}}$ and therefore the map ξ_X is coarse uniform. One can easily see that the map ξ_X is coarsely proper.

Now assume that there exists a natural transformation $\xi = (\xi_X) \colon SP^n_G \exp \to \exp SP^n_G$ satisfies the conditions of Theorem 1.. For every object A of the category $\mathcal{K}_n$, let $\mathcal{S}(A) = A \times \mathbb{N} \times \mathbb{N}$ and define a metric d on $\mathcal{S}(A)$ as follows:

$$d((x_1, m_1, n_1), (x_2, m_2, n_2)) = |m_1^{n_1} - m_2^{n_2}| + \max\{m_1, m_2\}\varrho(x, y),$$

where ϱ denotes the discrete metric on A. That d is a metric on $\mathcal{S}(A)$ can be easily verified and we leave this to the reader. Given a map $f \colon A \to B$ in $\mathcal{K}_n$, denote by $\mathcal{S}(f) \colon \mathcal{S}(A) \to \mathcal{S}(B)$ the map defined as follows: $\mathcal{S}(f)(x, m, l) = (f(x), m, l)$. Clearly, we obtain a covariant functor $\mathcal{S} \colon \mathcal{K}_n \to \mathrm{CS}$.

For any A in $\mathcal{K}_n$, write $\xi_{\mathcal{S}(A)} = [\psi_A]$, where $\psi_A \colon SP^n_G \exp A \to \exp SP^n_G A$ is a map.

Since ψ_A is a coarse map, for any $m \in \mathbb{N}$ there exists $l(m) \in \mathbb{N}$ such that $\psi_A(A \times \{m\} \times \{l\}) \subset A \times \{m\} \times \{l\}$, for all $n \geq l(m)$.

Since all the spaces in $\mathcal{K}_n$ are finite and $\mathcal{K}_n$ is a finite category, the fact that the distances between the distinct points in $B \times \{m\} \times \{l\}$ (and consequently in $F'(B \times \{m\} \times \{l\})$, for any B in $\mathcal{K}_n$ and any finitary normal functor F') are $\geq m$ implies the following: there exist $m, n \in \mathbb{N}$ such that, for any map $f\colon A \to B$ in $\mathcal{K}_n$ the diagram

$$\begin{array}{ccc} F(\exp A \times \{m\} \times \{l\}) & \xrightarrow{\psi_A|\dots} & \exp F(A \times \{m\} \times \{l\}) \\ {\scriptstyle F(\exp \mathcal{S}(f))|\dots}\downarrow & & \downarrow{\scriptstyle \exp F(\mathcal{S}(f))|\dots} \\ F(\exp B \times \{m\} \times \{l\}) & \xrightarrow{\psi_B|\dots} & \exp F(B \times \{m\} \times \{l\}) \end{array}$$

is commutative (for brevity, we drop the explicit indication of spaces onto which the restriction is considered). Note that m, n can be chosen as large as we wish.

This allows us to define a natural transformation $\psi'\colon F\exp \to \exp F$ in $\mathcal{K}_n$ by the condition $\psi_A(x, m, l) = (\psi'_A(x), m, l)$.

If m, n are large enough, the natural transformation ψ' satisfies the conditions of Theorem 1. (with ξ replaced by ψ') in $\mathcal{K}_n$. It follows from the results of [7] that F is isomorphic to SP^n_G for some subgroup G of the symmetric group S_n.

4. Remarks. In [6] the symmetric power functors are also characterized as those having an extension onto the Kleisli category of the probability measure monad. We leave as an open question that of finding a counterpart of this result in the coarse category.

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Характеризація функторів G-симетричного степеня в грубій категорії

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Доведено, що нормальний функтор скінченного степеня, що діє в грубій категорії, має продовження на категорію Клейслі монади гіперпростору, якщо і тільки якщо цей функтор ізоморфний функторові симетричного степеня.

Ключові слова: груба категорія, функтор G-симетричного степеня, монада гіперпростору.

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