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OPENNESS POINTS OF THE PROJECTION MAP OF CONVEX BODIES OF CONSTANT WIDTH

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It is proved that the set of points at which the projection map of the hyperspace of compact convex bodies of constant width in $\mathbb{R}^3$ onto the corresponding hyperspace in $\mathbb{R}^2$ is not open, is dense in the hyperspace. A similar result can be proven for the projection of $\mathbb{R}^n$ onto $\mathbb{R}^2$, where $n > 2$.

Key words: Constant width, hyperspace, open map

1. Let $\mathrm{cc}(\mathbb{R}^n)$ denote the set of all compact convex subsets in $\mathbb{R}^n$ endowed with the Hausdorff metric. For $m \le n$, we assume that $\mathbb{R}^m$ is embedded in $\mathbb{R}^n$ as the set $(x_1, \dots, x_m, 0 \dots, 0)$. For $m \le n$, the projection map $\mathrm{pr}\colon \mathbb{R}^n \to \mathbb{R}^m$ induces the natural map $A \mapsto \mathrm{pr}(A)\colon \mathrm{cc}(\mathbb{R}^n) \to \mathrm{cc}(\mathbb{R}^m)$. It is well-known that the induced map (we preserve the notation pr for it) is open. Moreover, this map is even soft.

A closed convex body C in $\mathbb{R}^n$ is of *constant width* $d > 0$ if

$$C - C = \{x - y \mid x, y \in C\} = B^n_d(0)$$

(the closed ball in $\mathbb{R}^n$ of radius $d > 0$ centered at the origin). This is equivalent to the following: the distance between the two supporting planes to the body in given direction is independent of direction and equals d.

Let $\mathrm{cw}_d(\mathbb{R}^n)$ denote the set of all convex bodies of constant width $d > 0$ in $\mathbb{R}^n$. We endow this set with the Hausdorff metric. This metric, d_H, is in fact defined on the family $\exp \mathbb{R}^n$ of all nonempty compact subsets in $\mathbb{R}^n$:

$$d_H(A, B) = \inf\{\varepsilon > 0 \mid A \subset O_\varepsilon(B),\ B \subset O_\varepsilon(A)\},\ A, B \in \exp X.$$

It is proved in [2] (see also [1] for an alternative proof) that, for $n \geq 2$, the space $\mathrm{cw}_d(\mathbb{R}^n)$ is a manifold modeled on the Hilbert cube Q (a Q-manifold). This result corresponds to the well-known result due to Nadler, Quinn and Stavrokas [7] that $\mathrm{cc}(\mathbb{R}^n)$ is a Q-manifold if $n \geq 2$. However, there is no complete analogy between the case of compact convex sets and that of compact convex sets of constant width. Namely, it is proved in [1] that the induced projection map $\mathrm{pr}\colon \mathrm{cw}_d(\mathbb{R}^3) \to \mathrm{cw}_d(\mathbb{R}^2)$ is not open.

Recall that a map of topological spaces is *open* if the image of every open set is open. We say that a map is *open at a point* if the image of any neighborhood of this point is a (not necessarily open) neighborhood of the image of the point. If a surjective map $f\colon X \to Y$ of metric spaces is open at a point $x \in X$, then for every sequence (y_i) in Y converging to $f(x)$ there exists a sequence (x_i) in X converging to x such that $y_i = f(x_i)$, $i = 1, 2, \dots$.

A map $f\colon X \to Y$ is *soft* if for every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \psi\ } & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[\ \varphi\ ]{} & Y, \end{array}$$

where $i\colon A \to Z$ is a closed embedding into a paracompact space Z, there exists a map $\Phi\colon Z \to X$ such that $\Phi|A = \psi$ and $f\Phi = \varphi$. The notion of soft map was introduced by E.V. Shchepin [10].

The aim of this note is to show that the set of points at which the map $\mathrm{pr}\colon \mathrm{cw}_d(\mathbb{R}^3) \to \mathrm{cw}_d(\mathbb{R}^2)$ is not open is dense in $\mathrm{cw}_d(\mathbb{R}^3)$.

By ∂A we denote the boundary of A. If A is a convex body in $\mathbb{R}^n$ of constant width $d > 0$ then any chord $[v, w]$ in A with $d(v, w) = \|v - w\| = d$ is said to be a *diameter* of A.

2. **Result.** We will need the following geometric statement.

Proposition 1. *Let $A \in \mathrm{cw}_d(\mathbb{R}^n)$. For every $\varepsilon > 0$, there exists $\delta > 0$ which satisfies the following property. For every compact convex B with $\mathrm{diam} B \leq d$ and $d_H(A, B) < \delta$, and every $B' \in \mathrm{cw}_d(\mathbb{R}^n)$ with $A' \supset B$, we have $d_H(A', A) < \varepsilon$.*

Proof. It is sufficient to prove that $B' \subset O_{c\varepsilon}(A)$. In turn, it is sufficient to prove that $\partial B' \subset O_{c\varepsilon}(A)$.

Let $x \in \partial B'$. There exists $y \in B'$ such that $\|x - y\| = d$. There exists a diameter $[a, b]$ in A parallel to $[x, y]$. Moreover, we assume that $y - x = b - a$. Then there exist $a_1, b_1 \in B$ such that $\|a_1 - a\| < \delta$, $\|b_1 - b\| < \delta$. Let $b_2 = b + (a_1 - a)$, then the linear segments $[x, y]$ and $[a, b_2]$ are parallel. Note that $|\|a_1 - b_2\| - d| < 2\delta$.

There exists $b_3 \in \mathbb{R}^n$ such that the linear segments $[x, y]$ and $[a, b_3]$ are parallel and $\|a - b_3\| = \|x - y\| = d$. Denote by h the height of the parallelogram P with vertices x, y, a, b_3, i.e. the distance between the lines containing $[x, y]$ and $[a, b_3]$ respectively. Denote by C the maximal length of the diagonal of P. Then $C \geq \sqrt{d^2 + h^2} \geq d + h$. On the other hand, $C \leq d + 5\varepsilon$, whence $h \leq 5\varepsilon$. Let c be a point on the line containing $[a, b_3]$ such that the segments $[x, c]$ and $[a, b_3]$ are orthogonal. Since $C \leq d + 5\varepsilon$, we conclude that $\|a-c\| < 5\varepsilon$. Then

$$\|x-a\| \le \|x-c\| + \|c-a\| \le 5\varepsilon + 5\varepsilon = 10\varepsilon$$

and we are done.

Let C be a set of constant width in $\mathbb{R}^2$. A subset V of ∂C (the boundary of C) is said to be a *pinching* set of C if every diameter (a maximal chord) of C is incident with at least one point of V. We say that a set of constant width is a *Reuleaux polygon* if it has a finite pinching set. It is well-known (see, e.g. [4]) that the family of all Reuleaux polygons of width d is dense (with respect to the Hausdorff distance) in $\mathrm{cw}_d(\mathbb{R}^2)$. We consider the projection $\mathrm{pr}\colon \mathbb{R}^3 \to \mathbb{R}^2$.

It is well-known (and easy to prove) that, for every $A \in \mathrm{cw}_d(\mathbb{R}^3)$, we have $\mathrm{pr}(A) \in \mathrm{cw}_d(\mathbb{R}^2)$. We denote by $e(A)$ the set $A \cap (\mathrm{pr}^{-1}(\partial \mathrm{pr}(A)))$. It is easy to see that the map $\mathrm{pr}|e(A)\colon e(A) \to \partial \mathrm{pr}(A)$ is a homeomorphism.

Lemma 1. *The map $A \mapsto e(A)$ is continuous with respect to the Hausdorff metric as a map from $\mathrm{cw}_d(\mathbb{R}^3)$ to the set $\exp \mathbb{R}^3$.*

Proof. Let (A_i) be a sequence in $\mathrm{cw}_d(\mathbb{R}^3)$ that converges to A. It is well-known that the sequence (∂A_i) converges to ∂A and, similarly, the sequence $(\partial \mathrm{pr}(A_i))$ converges to $\partial \mathrm{pr}(A)$.

Suppose that a sequence (x_i) is such that $x_i \in e(A_i)$, for every i, and $x_i \to x$ as $i \to \infty$. Then $x \in A$. Obviously, $\mathrm{pr}(x_i) \to \mathrm{pr}(x)$ and, since $(\partial \mathrm{pr}(A_i)) \to \partial \mathrm{pr}(A)$, $i \to \infty$, we see that $\mathrm{pr}(x) \in \partial A$.

On the other hand, suppose that $x \in e(A)$. Then $y = \mathrm{pr}(x) \in \partial(\mathrm{pr}(A))$ and, since $(\partial \mathrm{pr}(A_i)) \to \partial \mathrm{pr}(A)$, $i \to \infty$, there exists a sequence (y_i) in $\mathbb{R}^2$ such that $y_i \in \partial \mathrm{pr}(A_i)$, $i \in \mathbb{N}$, and $y_i \to y$ as $i \to \infty$.

Lemma 2. *Let A be a Reuleaux polygon in $\mathbb{R}^2$. Then, for any $C \in \mathrm{cw}_d(\mathbb{R}^3)$ with $\mathrm{pr}(C) = A$, the set $e(C)$ is planar (i.e. is located in a plane in $\mathbb{R}^3$).*

Proof. Let V be the set of vertices of ∂A (i.e. the minimal pinching set). Given $v \in V$, denote by A_v the set of endpoints (distinct of v) of the diameters with endpoint v. For any $w \in A_v$, we have $d(v,w) = d$ and therefore, $d(v',w') = d$ for any $v', w' \in C$ with $\mathrm{pr}(v') = v$, $\mathrm{pr}(w') = w$. Thus, v', w' are located on a plane parallel to $\mathbb{R}^2$. This implies that A_v is located on a plane parallel to $\mathbb{R}^2$ and therefore $e(C)$ is a planar set.

Theorem 1. *Given $d > 0$, let*

$$N = \{A \in \mathrm{cw}_d(\mathbb{R}^3) \mid \mathrm{pr} \textit{ is not open at } A\}.$$

Then the set N is dense in $\mathrm{cw}_d(\mathbb{R}^3)$.

Proof. Let $A \in \mathrm{cw}_d(\mathbb{R}^3)$. Suppose first that the set $e(A)$ is not planar. We are going to show that pr is not open at A. Indeed, suppose the contrary. Consider a sequence $(B_i)_{i=0}^{\infty}$ of Reuleaux polygons that converges to $\partial \mathrm{pr} A$. Since pr is open at A, it follows from well-known properties of open maps that there exists a sequence $(A_i)_{i=1}^{\infty}$ that converges to A and such that $\mathrm{pr}(A_i) = B_i$, $i = 1, 2, \ldots$. By Lemma 2., every set A_i is planar and, by Lemma 1., so is the set $e(A)$, which gives a contradiction.

Now we consider the case when the set $e(A)$ is planar. Passing, if necessary, to a closed neighborhood of a suitable homothetic copy of A, one may assume that the boundary ∂A is of the class C^1. Then the closed curve $e(A)$ is also of the class C^1.

There exist diameters $[v_i, w_i]$, $i = 1, \dots, k$, of the set $A \cap \pi$, which is obviously of constant width in π, such that the set $\{v_1, \dots, v_k, w_1, \dots, w_k\}$ is close enough to $\partial_\pi(A \cap \pi)$.

As usual, by $\mathbf{k}$ we denote the unit vector in the direction of the z-axis. Let A' be an affine copy of A with center at an interior point of A and coefficient $c < 1$ sufficiently close to 1. There exists $\eta > 0$ such that the set

$$A'' = A' \cup \{v_2, \dots, v_k, w_2, \dots, w_k\} \cup \{v_1 + \eta\mathbf{k}, w_1 + \eta\mathbf{k}\}$$

is of diameter d. It follows from the results of [6] that there exists a convex body $\tilde{A}$ of constant width d that contains A''. Simple geometric arguments show that one can make A'' as close to A as we wish by choosing $c < 1$ close enough to 1 and $\eta > 0$ close enough to 0. Then

$$e(A'') \supset \{v_2, \dots, v_k, w_2, \dots, w_k\} \cup \{v_1 + \eta\mathbf{k}, w_1 + \eta\mathbf{k}\}$$

and therefore $e(A'')$ is not planar. As we have proven above, A'' is not a point of openness of the map pr. Therefore, the set N is dense in $\mathrm{cw}_d(\mathbb{R}^3)$.

One can similarly prove the following result.

Theorem 2. *If $n \geq 3$, then the set*

$$\{A \in \mathrm{cw}(\mathbb{R}^n) \mid \mathrm{pr} \text{ is not open at } A\}$$

is dense in $\mathrm{cw}(\mathbb{R}^n)$.

3. **Remarks.** The notion of convex body of constant width can be defined also in any Minkowski space (i.e. any finite-dimensional normed space). It was remarked in [1] that the considered projection map of the hyperspaces of compact convex bodies of constant width can be open for some choice of norms.

Question 1. Suppose that the unit balls of Minkowski spaces are strictly convex (i.e., the unit spheres do not contain linear segments). Suppose also that the projection map preserves the constant width property. Is the projection map of the corresponding hyperspaces of compact convex sets of constant open?

The notion of closed convex body of constant width can be defined in any normed space. We conjecture that the AR-properties of the hyperspaces of closed convex sets in normed spaces established in [8] (see also [9]) have their counterparts for the hyperspaces of closed convex bodies of constant width.

One can formulate the question of openness of the projection map also in the case of pairs. Recall that a pair (A, B) of compact convex bodies in $\mathbb{R}^n$ is said to be of *constant relative width* [5] if $A - B = B_r(x)$ (the closed ball of radius r centered at a point $x \in \mathbb{R}^n$).

One can conjecture that the property of planarity of the set $e(A)$ characterizes the sets $A \in \mathrm{cw}(\mathbb{R}^3)$ at which the map pr is not open.

In connection to Theorem 1. the following general question arises.

Question 2. Let $n > m \geq 2$. Is the set of points in $\mathrm{cw}_d(\mathbb{R}^n)$ at which the projection map $\mathrm{pr}\colon \mathrm{cw}_d(\mathbb{R}^n) \to \mathrm{cw}(\mathbb{R}^m)$ is not open, dense in $\mathrm{cw}_d(\mathbb{R}^n)$?

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Точки відкритості відображення проектування опуклих тіл сталої ширини

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Доведено, що множина точок, у яких відображення проектування гіперпростору компактних опуклих тіл сталої ширини в $\mathbb{R}^3$ на відповідний гіперпростір в $\mathbb{R}^2$ не відкрите, є всюди щільною в цьому гіперпросторі. Подібний результат можна довести для проектування $\mathbb{R}^n$ на $\mathbb{R}^2$, де $n > 2$.

Ключові слова: стала ширина, гіперпростір, відкрите відображення

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