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# BEZOUT MORPHIC RINGS

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In this paper we established that the finite homomorphic images of a commutative Bezout domain are morphic rings. We described the commutative Bezout domains, whose finite homomorphic images are the Kasch rings. Moreover, we presented an example of a commutative morphic ring which is not a clean ring.

*Key words:* morphic ring, Bezout ring, Kasch ring.

The concept of morphic ring was introduced by Nicholson and Sanchez Campos in [11]. This class of rings is rather important and they are of particular interest in modern research. It is known that the commutative morphic rings are Bezout rings.

In this paper we will prove that finite homomorphic images of a commutative Bezout domain are morphic rings. We will describe the commutative Bezout domains whose finite homomorphic images are Kasch rings. In addition, we construct an example of a commutative morphic ring which is not a clean ring.

We will recall all the necessary definitions and facts. We denote $U = U(R)$ for the group of units of R, denote left and right annihilators of a subset $X \subseteq R$ by $l(X)$ and $r(X)$ respectively, and we write $\mathbb{Z}$ for the ring of integers and $\mathbb{Z}_n$ for the ring of integers modulo $n$.

If $R$ is a commutative ring then $l(a) = r(a)$ for all elements $a$ in $R$, where $l(a)$ and $r(a)$ are left and right annihilators of an element $a$ respectively, and in this case we write $\mathrm{Ann}(a) = l(a) = r(a)$.

All necessary definitions and facts can be found in [1], [2], [11]-[15].

**Definition 1.** *An element $a$ in a ring $R$ is called left morphic if $R/Ra \cong l(a)$ as left $R$-modules. The ring $R$ is called left morphic if every its element is left morphic. The right morphic rings are defined analogously. The ring $R$ is called morphic if it is left and right morphic* [2], [11]-[15].

**Lemma 1.** [11] *The following statements are equivalent for an element $a$ in a ring $R$:*

1) *An element $a$ is left morphic, i. e. $R/Ra \cong l(a)$.*
2) *There exists an element $b$ in a ring $R$ such that $Ra = l(b)$ and $l(a) = Rb$.*

3) *There exists an element $b$ in a ring $R$ such that $Ra = l(b)$ and $l(a) \cong Rb$.*

**Definition 2.** *A ring $R$ is called uniquely morphic if for any element $a$ in the ring $R$ there exists a unique element $b$ in the ring $R$ such that $Ra = l(b)$ and $l(a) = Ra$* [15].

**Definition 3.** *A commutative ring $R$ is called $P$-injective if for any element $a$ in the ring $R$ we have that* $\mathrm{Ann}(\mathrm{Ann}(aR)) = aR$ [14].

**Definition 4.** *A ring $R$ is called a Bezout ring if all finitely generated ideals are principal* [16].

**Theorem 1.** [2, 11] *Let $R$ be a commutative morphic ring. Then:*

1) *for any element $a$ in a ring $R$ we have* $\mathrm{Ann}(\mathrm{Ann}(aR)) = aR$ *(i. e., a ring $R$ is a $P$-injective ring);*
2) *for any finite set of elements $a_1, a_2, \ldots, a_n$ in a ring $R$ there exists an element $b$ in $R$ such that $a_1R \cap a_2R \cap \ldots \cap a_nR = bR$ (that is finite intersection of principal ideals is again principal one);*
3) *for any finite set of elements $a_1, a_2, \ldots, a_n$ in a ring $R$ there exists an element $b$ in $R$ such that $a_1R + a_2R + \ldots + a_nR = bR$ (that is $R$ is Bezout ring).*

**Definition 5.** *A commutative ring $R$ is said to be coherent if*

1) *the annihilator of any element $a$ in $R$ is a finitely generated ideal, and*
2) *finite intersection of any finitely generated ideals is again finitely generated* [14].

Thus from previous theorem we obtain that morphic rings are coherent.

**Definition 6.** *A commutative ring $R$ is called almost Baer if for any element $x$ there exists an element $y$ such that* $\mathrm{Ann}(xR) = yR$ [16].

Gathering known facts about Bezout rings we have:

**Theorem 2.** *Let $R$ be a commutative Bezout domain. Then for any nonzero element $a \in R$ we have:*

1) *$R/aR$ is a coherent ring;*
2) *$R/aR$ is a $P$-injective ring;*
3) *$R/aR$ is a morphic ring.*

*Proof.* 1) From [16] we know that ring $R/aR$ is almost Baer ring and using Theorem 1 we obtain that $R$ is a coherent ring.

2) This is proved in [14].

3) Let $\bar{b}$ be an element in the ring $\overline{R} = R/aR$. Then we have $\mathrm{Ann}(\overline{bR}) = \overline{cR}$, because $\overline{R}$ is an almost Baer ring. As $\overline{R}$ is a $P$-injective ring, then we have $\mathrm{Ann}(\mathrm{Ann}(\overline{bR})) = \overline{bR}$ and finally $\mathrm{Ann}(\mathrm{Ann}(\overline{bR})) = \mathrm{Ann}(\overline{cR})$. Hence we have $\overline{bR} = \mathrm{Ann}(\overline{cR})$.

Therefore for any element $\bar{b}$ there exists $\bar{c}$ such that $\mathrm{Ann}(\bar{b}) = \overline{cR}$ and $\mathrm{Ann}(\bar{c}) = \overline{bR}$ that is $\overline{R}$ is a morphic ring (according to Lemma 1).

**Definition 7.** *An element $a$ in a ring $R$ is called clean if it is the sum of an idempotent and a unit in $R$. If all elements in $R$ are clean then $R$ called clean* [2, 8, 10].

Note that the clean rings are $PM$-rings (that means that any its prime ideal belongs to a unique maximal ideal) [4, 10].

As a consequence of this fact we can give an example of a commutative morphic ring that is not clean. It is a negative answer to a question in the article [11].

Let $R$ be Henriksen ring [9], namely $R = \{z_0 + a_1x + a_1x^2 + \ldots | z_0 \in \mathbb{Z}, a_i \in \mathbb{Q}, i = 1, 2, \ldots\}$. It is known that $R$ is a commutative Bezout domain [9]. The factor ring $R/xR$ according to Theorem 2 is a morphic ring but it is not clean since any homomorphic image of the ideal $N = \{a_1x + a_1x^2 + \ldots | a_i \in \mathbb{Q}, i = 1, 2, \ldots\}$ is an ideal $N/xR$ that is prime, but belongs to all maximal ideals in the factor ring $R/xR$. That is why $R/xR$ is not clean because any clean ring has to be a $PM$-ring. Note that $xR \neq N$ whereas $1/2 \in N$ but $1/2 \notin xR$.

So we have a negative answer to the problem in [11].

**Definition 8.** *A ring $R$ is called a left Kasch ring if every simple left $R$-module embeds in ${}_RR$ that is $r(L) \neq 0$ for every (maximal) left ideal $L$ in a ring $R$* [14].

**Theorem 3.** *Let $R$ be a commutative Bezout domain $R$ and $a$ is some nonzero element in $R$. The following statements are equivalent:*

1) *$R/aR$ is a Kasch ring;*
2) *Any maximal ideal $M$ that contains the element $a$ is principal.*

*Proof.* (1) $\Rightarrow$ (2). Let us consider a Kasch ring $R/aR$ and let $\overline{M}$ be a maximal ideal in this ring. We can write $\mathrm{Ann}(\overline{M}) = \overline{H}$ where $\overline{H}$ is an ideal in $R/aR$ and $\overline{H} \neq \{\overline{0}\}$. Since $\overline{H}$ annihilates the maximal ideal $\overline{M}$, we can write $\overline{HM} = \{\overline{0}\}$. Since the maximal ideal $\overline{M}$ belongs to $\mathrm{Ann}(\overline{H})$, by maximality of $\overline{M}$, we have that $\overline{M} = \mathrm{Ann}(\overline{H}) \neq \overline{R}$.

Since $\overline{M}$ is a maximal ideal, then for every element $\overline{d} \neq \overline{0}$ which belongs to the ideal $\overline{H}$ we have the equality $\overline{dM} = \{\overline{0}\}$. Thus we obtain that the maximal ideal $\overline{M}$ belongs to $\mathrm{Ann}(\overline{d})$, where $\overline{d}$ is a nonunit.

Hence $\overline{M} = \mathrm{Ann}(\overline{d}) = \overline{bR}$ because $R/aR$ is a morphic ring. Therefore, $\overline{M} = \overline{bR}$ and $M = bR + aR = cR$, because $R$ is a commutative Bezout domain for some $c \in R$.

Hence $M$ is a maximal ideal which is a principal ideal.

(2) $\Rightarrow$ (1). Suppose that any maximal ideal $M$ that contains an element $a$, is a principal one. Considering its homomorphic image we have $\overline{M} = \overline{m}\overline{R} = \mathrm{Ann}(\overline{n}\overline{R})$ because $R/aR$ is a morphic ring. Since $\overline{m} \notin U(\overline{R})$, we have $\mathrm{Ann}(\overline{n}\overline{R}) \neq \overline{R}$ and then $\overline{n}\overline{R} \neq \{\overline{0}\}$.

Then $\mathrm{Ann}(\overline{M}) = \mathrm{Ann}(\mathrm{Ann}(\overline{n}\overline{R})) = \overline{n}\overline{R} \neq \{\overline{0}\}$, therefore $\mathrm{Ann}(\overline{M})$ is a nonzero principal ideal, and this proves that $R/aR$ is a Kasch ring.

**Corollary 1.** *If $R$ is a commutative principal ideal domain, then $R/aR$ is a Kasch ring, for any nonzero $a \in R$.*

**Theorem 4.** [15] *Any uniquely morphic ring $R$ is one of the following types:*

1) *$R$ is a division ring;*
2) *$R$ is a boolean ring;*
3) *$R \cong \mathbb{Z}_2[x]/(x^2)$;*
4) *$R \cong \mathbb{Z}_4$;*
5) *$R \cong \mathbb{M}_2(\mathbb{Z}_2)$.*

**Definition 9.** *A ring $R$ is called a ring of stable range 1 if for any pair of elements $a, b \in R$ such that $aR + bR = R$ there is an element $t \in R$ such that $a + bt \in U(R)$* [1, 8].

**Theorem 5.** *Any uniquely morphic ring has stable range 1.*

*Proof.* According to Theorem 4 we may assume that a ring $R$ is one of the five mentioned types.

We are going to prove that the stable range of each of these rings is equal to 1.

Firstly we consider case (2). A ring $R$ is a boolean ring, which means that $x^2 = x$ for any $x \in R$. Then for any $a \in R$ we have $a \cdot 1 \cdot a = a$, that is $R$ is a unit-regular ring. According to [5] the stable range of unit-regular ring equals 1.

Since $R \cong \mathbb{Z}_2[x]/(x^2) = \{\overline{0}, \overline{1}, \overline{x}, \overline{x+1}\}$ is a semilocal ring and $\mathbb{Z}_4 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$ is a local ring with maximal ideal $M = (\overline{2})$. All the rings of type (1), (3) and (4) have stable range 1, due to [1].

In case when $R \cong \mathbb{M}_2(\mathbb{Z}_2)$ we have a finite ring, so it is a ring with stable range 1.

Theorem is proved.

**Definition 10.** *We say that a ring $R$ is an elementary divisor ring if any matrix $A$ over $R$ admits a diagonal reduction, that is for the matrix $A$ there exist invertible matrices $P$ and $Q$ of appropriate sizes such that $PAQ = D = (d_i)$ is diagonal matrix such that $Rd_{i+1}R \subseteq d_iR \cap Rd_i$* [7].

*If only $1 \times 2$ $(2 \times 1)$ matrices over a ring $R$ admit a diagonal reduction then $R$ is said to be a right (left) Hermite ring. An Hermite ring is a ring which is both right and left Hermite ring* [7, 17].

**Theorem 6.** [17] *A right Bezout ring of stable range 1 is a right Hermite ring.*

Since uniquely morphic rings are morphic and they are Bezout rings, then according to Theorem 5 we can conclude that uniquely morphic rings are Bezout rings of stable range 1. Finally we have obtained next result.

**Theorem 7.** *Any uniquely morphic ring is an Hermite ring.*

**Theorem 8.** *Any uniquely morphic ring is an elementary divisor ring.*

*Proof.* Let $R$ be a uniquely morphic ring. Note that if $R$ is either a boolean ring, $\mathbb{Z}_2[x]/(x^2)$ or $\mathbb{Z}_4$ then $R$ is commutative Bezout ring of stable range 1, since, according to [16] is an elementary divisor ring.

The case when $R$ is a division ring is obvious.

As the field $\mathbb{Z}_2$ is an elementary divisor ring and any matrix ring over an elementary divisor ring is again an elementary divisor ring then in the case of $\mathbb{M}_2(\mathbb{Z}_2)$ we are done.

Theorem is proved.

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## МОРФІЧНІ КІЛЬЦЯ БЕЗУ

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Доведено, що скінченний гомоморфний образ комутативної області Безу є морфічним кільцем. Описано комутативні області Безу, скінченні гомоморфні образи яких є кільцями Каша. Наведено приклад комутативного морфічного кільця, яке не є чистим.

*Ключові слова:* морфічне кільце, кільце Безу, кільце Каша.

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# МОРФИЧЕСКИЕ КОЛЬЦА БЕЗУ

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Доказано, что конечный гомоморфный образ комутативной области Безу является морфичным кольцом. Описано комутативные области Безу, конечные гомоморфные образы которых являются кольцами Каша. Наведено пример комутативного морфического кольца, которое не является чистым.

*Ключевые слова:* морфическое кольцо, кольцо Безу, кольцо Каша.