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IDEMPOTENT ULTRAMETRIC FRACTALS

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The notion of invariant measure is defined for the idempotent measures (Maslov measures) in the ultrametric setting. We prove that the ultrametric space of the idempotent measures on a complete ultrametric space is also complete and use this fact to prove the existence of the invariant idempotent measure for the IFSs. We also discuss the case of the upper-semicontinuous capacities, of the max-min measures, and also of idempotent measures on metric spaces.

Key words: idempotent measure, Maslov measure, ultrametric space.

1. Introduction. The invariant probability measures for the iterated function systems (IFS) were first defined by Hutchinson [5]. They found various applications in mathematics, quantum mechanics, image processing etc.

A Maslov measure (an idempotent measure) is a measure m on X defined as follows: $m(A) = \sup_{x\in A}\psi(x)$, where $\psi\colon X \to \mathbb{R}$ is a function. The notion of idempotent measure belongs to the so-called Idempotent Mathematics, i.e., a part of mathematics in which the usual arithmetical operations are replaced by idempotent ones (like $x \oplus y = \max\{x, y\}$). The informal Correspondence principle asserts that to every meaningful and interesting notion of ordinary mathematics there corresponds a meaningful and interesting notion of the Idempotent Mathematics.

Recall that a metric d on a set X is called an ultrametric (a non-Archimedean metric) if it satisfies the following strong triangle inequality:

$$d(x, y) \le \max\{d(x, z), d(z, y)\}, \quad x, y, z \in X.$$

The aim of this note is to define a counterpart of the invariant measures [5] for the idempotent measures and for the ultrametric spaces. We prove the existence of the invariant idempotent measures and consider an example of such a measure on an ultrametric Cantor set. Since the idempotent measures are special examples of non-additive measures, we discuss a possibility to define invariant objects in another classes of measures. Actually, we focus on the class of the upper semi-continuous capacities and the max-min measures in the ultrametric setting. We also discuss some metrizations of the idempotent measures for all metric spaces.

2. Idempotent measures. By $\exp X$ we denote the set of all nonempty compact subsets in a topological space X. If X is a metric space, we endow $\exp X$ with the Hausdorff metric.

Denote by $C(X)$ the set of continuous functions on a compact Hausdorff space X. Given $c \in \mathbb{R}$, we denote by $c_X \in C(X)$ the constant function which takes the value c on X. Let $c \odot \varphi$ denote the function $c_X + \varphi$ and let $\varphi \oplus \psi$ denote the function $\max\{\varphi, \psi\}$. Also, $\odot$ and $\oplus$ mean the addition and max in the set of reals $\mathbb{R}$ respectively.

Definition 1. *Let X be a compact Hausdorff space. A functional $\mu\colon C(X) \to \mathbb{R}$ is called an* idempotent measure *if it satisfies the following properties:*

1) $\mu(c_X) = c$;
2) $\mu(c \odot \varphi) = c \odot \mu(\varphi)$;
3) $\mu(\varphi \oplus \psi) = \mu(\varphi) \oplus \mu(\psi)$.

By $I(X)$ we denote the set of all idempotent measures on X. The following is an example of an idempotent measure. Let $x_1, \dots, x_n \in X$ and let $\alpha_1, \dots, \alpha_n \in [-\infty, 0]$ be such that $\oplus_{i=1}^n \alpha_i = 0$; then define $\mu = \oplus_{i=1}^n \alpha_i \odot \delta_{x_i} \in I(X)$ as follows: $\mu(\varphi) = \oplus_{i=1}^n \alpha_i \odot \varphi(x_i)$.

Every continuous map $f\colon X \to Y$ of compact Hausdorff spaces induces a map $I(f)\colon I(X) \to I(Y)$ by the formula $I(f)(\mu)(\varphi) = \mu(\varphi f)$. We obtain a functor acting from the category of compact Hausdorff spaces and a known procedure by Chigogidze [4] allows us to extend this functor onto the category of Tychonov spaces and continuous maps. We keep the notation I for this extension. Note that there is a natural definition of support for the functor I.

Thus, for a Tychonov space X, the set $I(X)$ consists of the idempotent measures on X with compact supports.

Given an ultrametric space (X, d), for every $r > 0$, we denote by $\mathcal{F}_r(X)$ the set of real-valued functions on X which are constant onto the balls of radius r. We endow $I(X)$ with the following metric $\hat{d}$:

$$\hat{d}(\mu, \nu) = \inf\{r > 0 \quad | \quad \mu(\varphi) = \nu(\varphi) \text{ for all } \varphi \in \mathcal{F}_r\}$$

(see [1] for details).

Theorem 1. *Let (X, d) be a complete ultrametric space. Then $I(X)$ is also a complete ultrametric space.*

Proof. Let (μ_i) be a Cauchy sequence in the space $I(X)$. Then $(A_i = \operatorname{supp}(\mu_i))$ is a Cauchy sequence in the space $\exp X$ (see [1]) and there exists the limit $A = \lim_{i\to\infty} A_i$. Without loss of generality, one may assume that $X = A \cup \bigcup_{i=1}^{\infty} A_i$.

Let $\varphi \in C(X)$. We are going to show that $(\mu_i(\varphi))$ is a Cauchy sequence. Let $\varepsilon > 0$. Since the function φ is uniformly continuous, there exists $\delta > 0$ such that $|\varphi(x) - \varphi(y)| < \delta$, whenever $d(x, y) < \varepsilon$. Then there exists $N \in \mathbb{N}$ such that $\hat{d}(\mu_i, \mu_j) < \delta$ for all $i, j \geq N$. Denote by $\mathcal{D}_\delta = \{B_\delta(x_k)\}$ the decomposition of X into the balls of radius δ. Then

$\varphi = \oplus_k \varphi_k$, where $\varphi_k | B_\delta(x_k) = \varphi | B_\delta(x_k)$ and $\varphi_k | (X \setminus B_\delta(x_k)) = c_k$, for small enough c_k. Then

$$|\mu_i(\varphi) - \mu_j(\varphi)| = | \oplus_k \mu_i(\varphi_k) - \oplus_k \mu_j(\varphi_k)| = |\mu_i(\oplus_k \varphi_k) - \mu_j(\oplus_k \varphi_k)| \le \delta$$

as $a_k \le \varphi_k \le a_k + \delta$, for some constant a_k.

Thus, the sequence $(\mu_i(\varphi))$ is a Cauchy sequence and we denote its limit by $\mu(\varphi)$. We are going to show that $\mu\colon C(X) \to \mathbb{R}$ is an element of $I(X)$. Note first that $\mu(c_X) = c_X$.

Note also that

$$\mu(\varphi \oplus \psi) = \lim_{i\to\infty} \mu_i(\varphi \oplus \psi) = \lim_{i\to\infty} \mu_i(\varphi) \oplus \lim_{i\to\infty} \mu_i(\psi) = \mu(\varphi) \oplus \mu(\psi).$$

Therefore, $\mu \in I(X)$.

In order to show that $\mu = \lim_{i\to\infty} \mu_i$ let $\varepsilon > 0$. Since (μ_i) is a Cauchy sequence, there is $N \in \mathbb{N}$ such that, for every $i, j \ge N$, $\mu_i(\varphi) = \mu_j(\varphi)$, for every $\varphi \in \mathcal{F}_\varepsilon$. Then $\mu(\varphi) = \mu_i(\varphi)$, for every $\varphi \in \mathcal{F}_\varepsilon$ and $i \ge N$.

Thus, $I(X)$ is complete.

2.1. IFSs and invariant idempotent measures. Recall that a map $f\colon X \to Y$ of metric spaces (X, d) and (Y, ϱ) is called a contraction if there is $\lambda \in (0, 1)$ (called a contraction coefficient) such that $\varrho(f(x), f(y)) \le \lambda d(x, y)$, for all $x, y \in X$.

Let X be a complete ultrametric space and $f_1, \dots, f_n\colon X \to X$ a family of contractions (we call it an iterated function system (IFS)). Let also $a = \oplus_{i=1}^n \alpha_i \odot \delta_i \in I(\{1, \dots, n\})$, where $\alpha_i \le 0$, $i = 1, \dots, n$, and $\oplus_{i=1}^n \alpha_i = 0$. Define the map $\Phi\colon I(X) \to I(X)$ as follows: $\Phi(\mu) = \oplus_{i=1}^n \alpha_i \odot I(f_i)(\mu)$. Note that, clearly, $\Phi(\mu) \in I(X)$.

Proposition 1. *The map Φ is a contraction.*

Proof. Let $\lambda \in (0, 1)$ be a contraction coefficient for the IFS $f_1, \dots, f_n$ (e.g., the maximal of the contraction coefficients for f_i, $i = 1, \dots, n$).

Given $\mu, \nu \in I(X)$ with $\hat{d}(\mu, \nu) < c$, we obtain $\hat{d}(I(f)(\mu), I(f)(\nu)) < \lambda c$. Then, for any $x \in X$, we have

$$I(f_i)(\mu)(B_{\lambda a}(x)) = I(f_i)(\nu)(B_{\lambda a}(x)), \quad i = 1, \dots, n.$$

Therefore $\oplus_{i=1}^n \alpha_i \odot I(f_i)(\mu) = \oplus_{i=1}^n \alpha_i \odot I(f_i)(\nu)$ and we conclude that Φ is a contraction.

Since the metric space $(I(X), \hat{d})$ is complete, there exists a unique fixed point of the map Φ. We call this fixed point the *invariant idempotent measure* of the ISF $f_1, \dots, f_n$ and $a \in I(\{1, \dots, n\})$.

2.2. Example. Let $C = 2^\omega$ be the Cantor set. We consider the following metric d on C:

$$d((x_i), (y_i)) = \inf\{1/k \mid x_i = y_i \text{ for all } i < k\}.$$

Clearly, d is an ultrametric on C. In the sequel, we identify every $(x_1, \dots, x_n) \in 2^n$ with $(x_1, \dots, x_n, 0, 0, \dots) \in 2^\omega = C$.

Consider the IFS $f_1, f_2\colon X \to X$ defined by

$$f_1(x_1, x_2, \dots) = (0, x_1, x_2, \dots), \quad f_2(x_1, x_2, \dots) = (1, x_1, x_2, \dots).$$

Let $a = 0 \odot \delta_1 \oplus (-1) \odot \delta_2 \in I(\{1, 2\})$.

Consider $\mu_0 = 0 \odot \delta_{(0,0,\dots)} \in I(C)$. Then, for every natural n, we obtain

$$\Phi^n(\mu_0) = \bigoplus \left\{ -\left(\sum_{i=1}^{n} x_i\right) \odot \delta((x_i)) \mid (x_i) \in 2^n \subset 2^\omega \right\}.$$

Then the unique fixed point of the map Φ is

$$\mu = \bigoplus \left\{ \left(-\sum_{i=1}^{\infty} x_i\right) \odot \delta((x_i)) \mid \sum_{i=1}^{\infty} x_i < \infty \right\}.$$

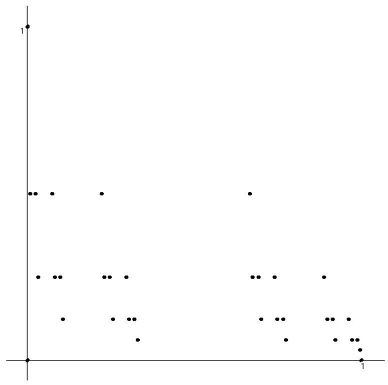
Indeed, it is enough to verify that $\hat{d}(\Phi(\mu_0), \mu) < 1/n$. Note that $\mathcal{F}_{1/n}$ consists of the functions which are constant on the sets of the form

$$K_{(x_1,\dots,x_n)} = \{(y_i)_{i=1}^{\infty} \mid y_i = x_i \text{ for every } i = 1, \dots, n\},$$

where $(x_1, \dots, x_n) \in 2^n$. Let $\varphi \in \mathcal{F}_{1/n}$. Let $B \subset X$ be an open ball of radius $1/n$. Then there exists $(x_1, \dots, x_n) \in 2^n \subset 2^\omega = C$ such that $B = B_{1/n}((x_1, \dots, x_n))$. Then, clearly, for any $m \geq n$,

$$\Phi^m(\mu_0)(\varphi) = \bigoplus \left\{ \varphi(x_1, \dots, x_n) - \sum_{i=1}^{n} x_i \in 2^n \subset 2^\omega \right\},$$

whence $\hat{d}(\Phi^n(\mu_0), \Phi^m(\mu_0)) < 1/n$ for all $m \geq n$.



Pic. 1. Visualization of the measure $\Phi^n(\mu_0)$.

In the picture, the measure $\Phi^n(\mu_0)$ is visualized as follows. First, we represent C as the middle-third Cantor set. Actually, we plot the graph of the (partial) function $y_i \mapsto 2^{\alpha_i}$ in order to represent $\mu = \oplus \alpha_i \odot \delta_{y_i}$.

2.3. Remark. A metric in the spaces of idempotent measures of compact metric spaces is defined in [8]. One can formulate the problem of existence of invariant idempotent measures for this metric.

3. Discussion. Here we discuss a possibility to extend the results of the previous section onto another classes of non-additive measures as well as onto the case of metric (not necessarily ultrametric) spaces.

3.1. Capacities. We first consider the case of the upper-semicontinuous capacities.

An upper-semicontinuous capacity of a compact Hausdorff space X is a function c defined on the closed subsets of X and satisfying the properties:

1) $c(\emptyset) = 0$, $c(X) = 1$;
2) $c(A) \leq c(B)$, whenever $A \subset B$;
3) if $c(A) < a$, then there is a neighborhood U of A such that $c(B) < a$, for every $B \subset U$.

The set of all upper-semicontinuous capacities on X is denoted by $M(X)$. It is known (see, e.g., [3]) that M is a functor on the category of compact Hausdorff spaces. Similarly as above, one can define the ultrametric space of upper-semicontinuous capacities with compact supports on an ultrametric space X.

There are metrizations of the space $M(X)$ which are counterparts of the Hutchinson and Prohorov metric respectively. In [6], an ultrametrization of $M(X)$, for an ultrametric X, is defined. Given an ultrametric space (X, d), for every $r > 0$, we denote by $\mathcal{F}_r(X)$ the set of real-valued functions on X which are constant onto the balls of radius r. We endow $M(X)$ with the following metric $\hat{d}$:

$$\hat{d}(c_1, c_2) = \inf\left\{ r > 0 \ \mid \ \int\limits_X \varphi dc_1 = \int\limits_X \varphi dc_2 \text{ for all } \varphi \in \mathcal{F}_r \right\},$$

where $\int_X \varphi dc$ is the Choquet integral defined as follows:

$$\int\limits_X \varphi dc = \int\limits_0^\infty c(\varphi \geq t)dt + \int\limits_{-\infty}^0 (c(\varphi \geq t) - 1)dt,$$

where $(\varphi \geq t)$ stands for the set $\{x \in X \mid \varphi(x) \geq t\}$.

However, one cannot proceed as in the previous section in order to define the invariant measure, because the obtained ultrametric space $(M(X), \tilde{d})$, in general, is not complete (see [6] for an example).

In [6], the following ultrametric on $M(X)$ is considered:

$$\tilde{d}(c_1, c_2) = \max\{\hat{d}(c_1, c_2), \ \ d_H(\mathrm{supp}(c_1), \mathrm{supp}(c_2))\}.$$

Clearly, the map (supp: $M(X) \to \exp X$ is nonexpanding. In [6], it is proved that the ultrametric space $(M(X), \tilde{d})$ is complete if so is (X, d). However, this construction does not satisfy the following property: if $f\colon X \to Y$ is a nonexpanding map of ultrametric spaces, then so is the map $M(f)\colon M(X) \to M(Y)$.

Indeed, consider a set $X = \{x, y, z, w\}$ endowed with the metric d:

$$d(x, y) = d(y, z) = d(x, z) = 1, \quad d(x, w) = d(y, w) = d(z, w) = 2.$$

Clearly, d is an ultrametric. Let $Y = \{x, y, w\}$ be endowed with the subspace metric. Denote by $f\colon X \to Y$ a retraction sending z to x. The map f is nonexpanding.

Let $c_1, c_2\colon \{\emptyset\} \cup \exp X \to [0, 1]$ be defined as follows:

$$c_1(A) = \begin{cases} 1, & \text{if } |A \cap \{x, y, w\}| \geq 2, \\ 0, & \text{otherwise,} \end{cases} \qquad c_2(A) = \begin{cases} 1, & \text{if } |A \cap \{x, z, w\}| \geq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Then $M(f)(c_1)(A)$ equals 1, whenever $|A| \geq 2$ and 0 otherwise. It is easy to see that $\mathrm{supp}(M(f)(c_2)) = \{x\}$. This implies

$$\tilde{d}(c_1, c_2) = 1, \quad \tilde{d}(M(f)(c_1), M(f)(c_2)) = 2$$

and therefore the map $M(f)$ is not nonexpanding.

We see that the reason of lack of the non-expanding property is connected with the property of preservation of preimages. One can speculate whether a kind of the Open Set Condition can repair the situation. We leave this as an open problem.

3.2. Max-min measures. The results of this note can be extended on the case of the so-called max-min measures on ultrametric spaces (see [7]). Every max-min measure of finite support is of the form $\oplus_{i=1}^n \alpha_i \otimes \delta_{x_i}$, where $\otimes$ stand for the min operation, $\alpha_i \in [-\infty, \infty]$, for all $i = 1, \dots, n$, and $\oplus_{i=1}^n \alpha_i = \infty$. In [7], the max-min measures on the complete ultrametric spaces are defined as the elements of the completion of the space of the max-min measures of finite supports with respect to the ultrametric which is a counterpart of that used above for the idempotent measures.

3.3. Idempotent measures on metric spaces. Let (X, d) be a compact metric space. By $I(X)$ we denote the set of all idempotent measures of compact support on X.

By $\mathrm{LIP}_n = \mathrm{LIP}_n(X, d)$ we denote the set of Lipschitz functions with the Lipschitz constant $\leq n$ from $C(X)$.

Fix $n \in \mathbb{N}$. For every $\mu, \nu \in I(X)$, let

$$\hat{d}_n(\mu, \nu) = \sup\{|\mu(\varphi) - \nu(\varphi)| \quad | \quad \varphi \in \mathrm{LIP}_n\}.$$

It is proved in [8] that the function $\hat{d}_n$ is a continuous pseudometric on $I(X)$. We let $\tilde{d}_n = (1/n)\hat{d}_n$. In [8], the following metric was defined on the set $I(X)$:

$$\tilde{d}(\mu, \nu) = \sum_{i=1}^{\infty} \frac{\tilde{d}_i(\mu, \nu)}{2^i}. \tag{1}$$

One can also show that the following is a metric on $I(X)$:

$$\check{d}(\mu, \nu) = \bigoplus_{i=1}^{\infty} \frac{\tilde{d}_i(\mu, \nu)}{2^i}. \tag{2}$$

One can easily prove the following fact for the metric $\check{d}$.

Proposition 2. *Let $\alpha_1, \dots, \alpha_n \in (-\infty, 0]$ be such that $\oplus_{i=1}^n \alpha_i = 0$. Let $\mu_i, \nu_i \in I(X)$, $i = 1, \dots, n$, be such that $\hat{d}(\mu_i, \nu_i) \leq K$, for all $i = 1, \dots, n$. Then the map*

$$(\mu_1, \dots, \mu_n) \mapsto \bigoplus_{i=1}^{n} \alpha_i \odot \mu_i \colon I(X)^n \to I(X)$$

(we consider the max-metric on the product) is nonexpanding.

Now, in order to prove that the map Φ defined as above for an IFS is a contraction, we have to show that the functor I preserver the class of contractions, i.e. that $I(f)$ is a contraction, whenever so is f. However, this is not the case, as the following example demonstrates.

Let $X = \{a, b\}$ with $d(a, b) = K > 0$. Let

$$\mu = 0 \odot \delta_a \oplus \alpha \odot \delta_b, \quad \nu = 0 \odot \delta_a \oplus \beta \odot \delta_b \in I(X).$$

Without loss of generality, one may assume that $\varphi(a) = 0$ for all Lipschitz functions φ. Then, if $\alpha, \beta > -K$, then $\check{d}(\mu, \nu) = |\alpha - \beta|$, i.e., $\check{d}(\mu, \nu)$ does not depend on K. This easily implies that the map $I(f)$, where f the identity map of (X, d) onto (X, ϱ) with $\varrho(a, b) = K/2$ is not a contraction.

We conclude that the map Φ is not a contraction and one should apply methods other than Banach's contracting principle in order to examine the question of existence and uniqueness of invariant measure.

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ІДЕМПОТЕНТНІ УЛЬТРАМЕТРИЧНІ ФРАКТАЛИ

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Для ідемпотентних мір (мір Маслова) означено поняття інваріантної міри для ітерованої системи функцій на ультраметричному просторі. Доводимо, що ультраметричний простір ідемпотентних мір на повному ультраметричному просторі є також повним і використовуємо цей факт для доведення існування інваріантної ідемпотентної міри для ітерованих систем функцій. Також обговорюється випадок напівнеперервних згори ємностей.

Ключові слова: ідемпотентна міра, міра Маслова, ультраметричний простір.

ИДЕМПОТЕНТНЫЕ УЛЬТРАМЕТРИЧЕСКИЕ ФРАКТАЛЫ

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Для идемпотентных мер (мер Маслова) определено понятие инвариантной меры для итерованной системы функций на ультраметрическом пространстве. Доказано, что ультраметрическое пространство идемпотентных мер на полном ультраметрическом пространстве полно и этот факт использован для доказательства существования инвариантной идемпотентной меры для итерированной системы функций. Также рассматривается случай полунепрерывной свёртки ёмкостей.

Ключевые слова: идемпотентная мера, мера Маслова, ультраметрическое пространство.